Decision Problems

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1/11/2013

- Decisions $d \in \mathcal{D}$
- Experiments with outcomes in Ω.
- Reward $r \in \mathcal{R}$ depending on experiment and outcome.
- Utility $U : \mathcal{R} \to \mathbb{R}$.

Example 1 (Taking the umbrella)

- There is some probability of rain.
- We don't like carrying an umbrella.
- We really don't like getting wet.

- Random outcome $\omega \sim P$.
- Decision $d \in D$

Definition 2 (Reward function)

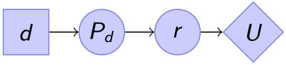
When we take decision d, then ω is randomly chosen, and we obtain a reward:

$$r = \rho(\omega, d). \tag{2.1}$$

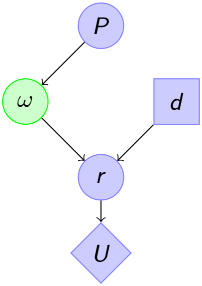
For every $d \in \mathcal{D}$, the function $\rho : \Omega \times D \to \mathcal{R}$ induces a probability distribution P_d on $\mathcal{R}$.

$$P_d(B) \triangleq P(\{\omega \mid \rho(\omega, d) \in B\}). \tag{2.2}$$

Thus, instead of directly choosing some distribution of rewards, we choose a decision d, which corresponds to a particular distribution P_d.



(a) The combined decision problem



(b) The separated decision problem

Expected utility

$$\mathbb{E}_{P_d}(U) = \sum_{r \in \mathcal{R}} U(r) P_d(r) = \sum_{\omega \in \Omega} U[\rho(\omega, d)] P(\omega). \tag{2.3}$$

Example 3

You are going to work, and it might rain. The forecast said that the probability of rain (ω_1) was 20%. What do you do?

- d_1: Take the umbrella.
- d_2: Risk it!

$\rho(\omega, d)$	d_1	d_2
ω_1	dry, carrying umbrella	wet
ω_2	dry, carrying umbrella	dry
$U[\rho(\omega, d)]$	d_1	d_2
ω_1	0	-10
ω_2	0	1
$\mathbb{E}_P(U \mid d)$	0	-1.2

Table : Rewards, utilities, expected utility for 20% probability of rain.

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Application to statistical estimation

Example 4 (Voting)

Let us say for example that you wish to estimate the number of votes for different candidates in an election. The *unknown parameters* of the problem mainly include: the percentage of likely voters in the population, the probability that a likely voter is going to vote for each candidate. One simple way to estimate this is by polling.

- The unknown outcome of the experiment ω is called a parameter.

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Definition 4 (Simplified expected utility of a given decision)

$$U(P, d) \triangleq \sum_{\omega \in \Omega} U[\rho(\omega, d)] P(\omega). \tag{2.4}$$

Definition 5 (Bayes-optimal utility)

$$U^*(P) \triangleq \max_d U(P, d) \tag{2.5}$$

Voting example

- Consider a nation with k political parties.
- Let $\omega = (\omega_1, \ldots, \omega_k) \in [0, 1]^k$ be the voting percentages for each party.
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Squared error

We can set $\rho(\omega, d) = (\omega_1 - d_1, \ldots, \omega_k - d_k)$, our error vector $r \in [0, 1]^k$. Then we set $U(r) \triangleq -\|r\|^2$, where $\|r\|^2 = \sum_i |x_i|^2$.

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Predicting the winner

In that case $\rho(\omega, d) = 1$ if $\arg\max_i \omega_i = \arg\max_i d_i$ and 0 otherwise, and $U(r) = r$.

Example 6 (Squared error)

Consider the case $\Omega = D = \mathbb{R}$. Our problem is:

$$\max_d U(P, d), \qquad U(P, d) \triangleq -\int_{\mathbb{R}} |\omega - d|^2 \, \mathrm{d}P(\omega).$$

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Under some technical assumptions, we can write

$$\frac{\partial}{\partial d} \int_{\mathbb{R}} |\omega - d|^2 \,\mathrm{d}P(\omega) = \int_{\mathbb{R}} \frac{\partial}{\partial d} |\omega - d|^2 \,\mathrm{d}P(\omega) \tag{2.9}$$

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$$= 2 \int_{\mathbb{R}} d \, \mathrm{d}P(\omega) - 2 \int_{\mathbb{R}} \omega \, \mathrm{d}P(\omega) \tag{2.8}$$

$$= 2d - 2\,\mathbb{E}(\omega), \tag{2.9}$$

so the optimal decision is $d = \mathbb{E}(\omega)$.

The utility for quadratic loss

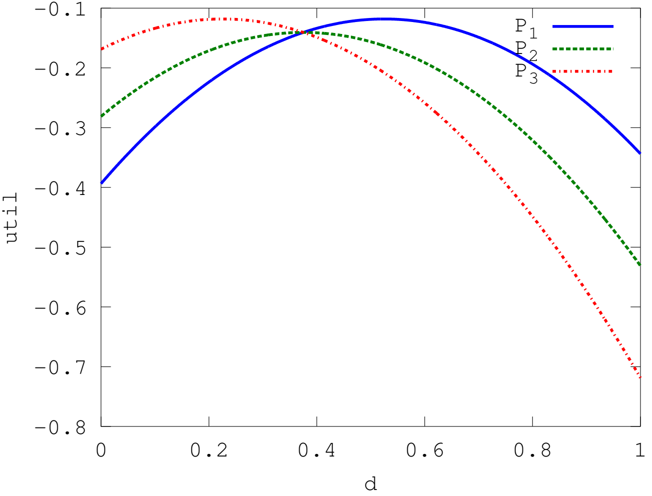

Figure : Fixed distribution, varying decision. The decision utility under three different distributions.

A mixture of distributions

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Consider two probability measures P, Q on Ω.
These define two alternative distributions for ω. For any P, Q and $\alpha \in [0, 1]$, we define

$$Z_\alpha = \alpha P + (1 - \alpha) Q$$

to mean the probability measure such that

$$Z_\alpha(A) = \alpha P(A) + (1 - \alpha) Q(A)$$

for any $A \in \mathfrak{F}_\Omega$.

Convexity of the Bayes-optimal utility

Theorem 7

For probability measures P, Q on Ω and any $\alpha \in [0, 1]$,

$$U^*[Z_\alpha] \leq \alpha U^*(P) + (1 - \alpha) U^*(Q), \tag{2.10}$$

where $Z_\alpha = \alpha P + (1 - \alpha) Q$.

Proof.

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Proof.

From the definition of the expected utility (2.4), for any decision $d \in \mathcal{D}$,

$$U(Z_\alpha, d) = \alpha U(P, d) + (1 - \alpha) U(Q, d).$$

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Hence, by definition (2.5) of the Bayes-optimal utility:

$$U^*(Z_\alpha) = \max_{d \in D} [\alpha U(P, d) + (1 - \alpha) U(Q, d)].$$

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Proof.

$$U^*(Z_\alpha) = \max_{d \in D} [\alpha U(P, d) + (1 - \alpha) U(Q, d)].$$

Use $\max_x [f(x) + g(x)] \leq \max_x f(x) + \max_x g(x)$ to bound r.h.s:

$$U^*[Z_\alpha] \leq \alpha \max_{d \in D} U(P, d) + (1 - \alpha) \max_{d \in D} U(Q, d)$$

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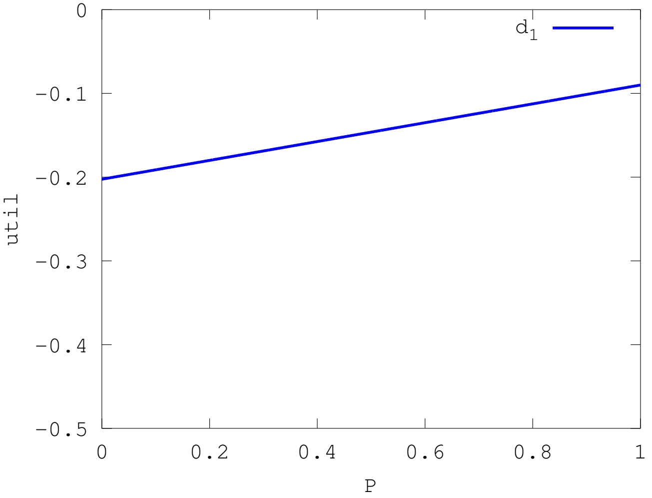

Figure : Fixed decision, varying distribution. The util of a fixed decision is a linear function of P

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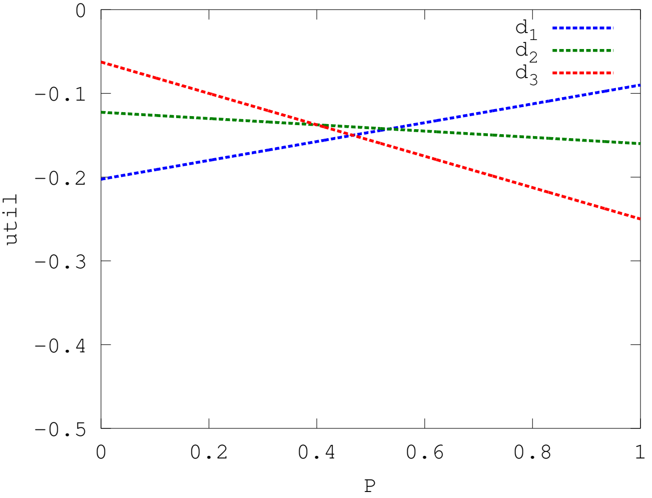

Figure : The util of a few decisions as P varies. Each decision corresponds to one of these lines.

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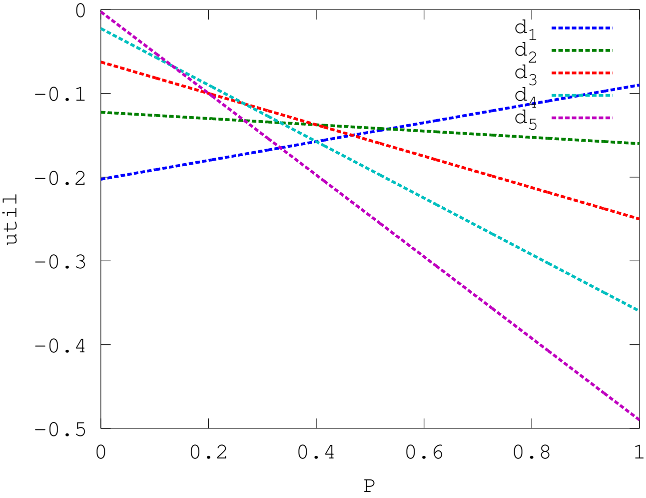

Figure : For each P, there is at least one decision maximising the util.

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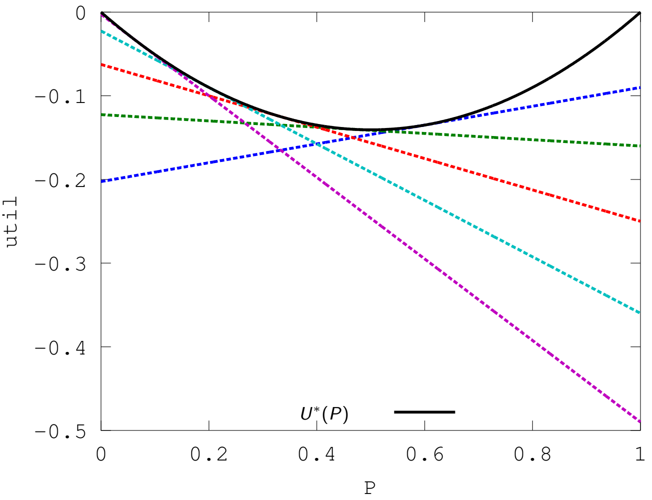

Figure : The Bayes util is convex and the maximising decision is tangent to it.

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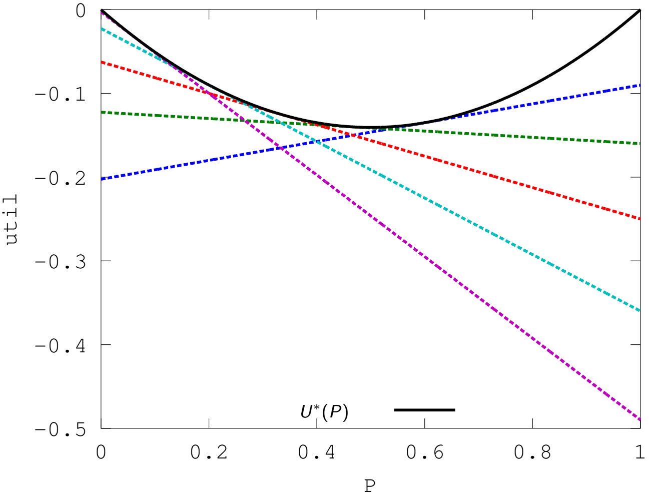

Figure : If we are not very wrong about P, then we are not far from optimal.

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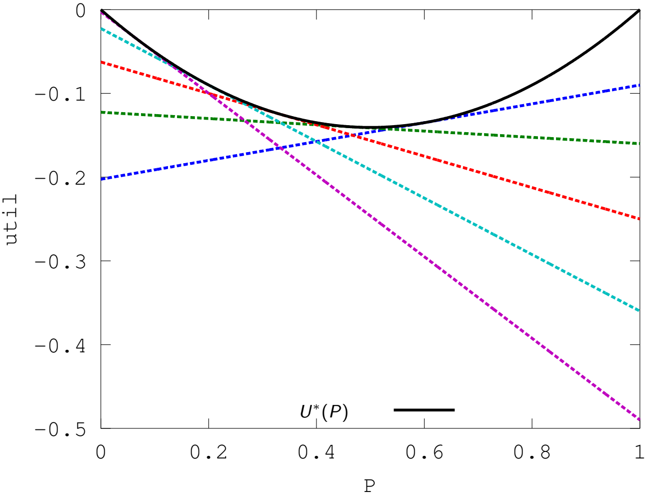

Figure : We can approximate the Bayes util by taking the maximum of a finite number of decisions.

Only prior information

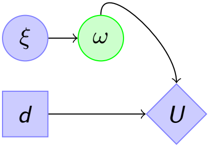

Figure : Statistical decision problem without observations

1. There is an unknown parameter $\omega \in \Omega$ with $\omega \sim \xi$.

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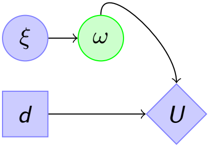

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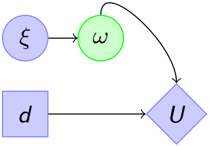

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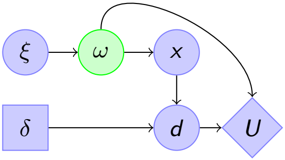

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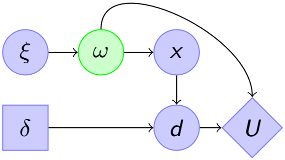

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2. Now consider a family of conditional probabilities measures on the observation set $\mathcal{S}$:

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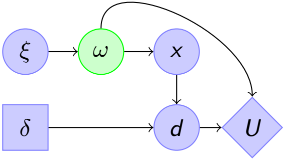

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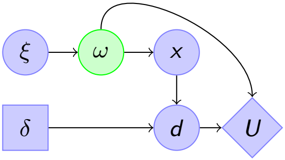

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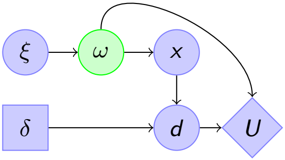

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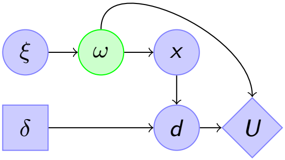

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6. We want to find a decision function $\delta : \mathcal{S} \to D$ maximising expected utility

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$$\mathbb{E}_\xi(U \mid d, x) = \sum_{\omega \in \Omega} U(\omega, d)\xi(\omega \mid x)$$

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- Bayes decision rule:

$$\delta^*(x) = \arg\max_{d \in D} \mathbb{E}_\xi(U \mid d, x).$$

Exercise

Abdul Alhazred claims that he is psychic and can always predict a coin toss. Let $P(A) = 2^{-16}$ be your prior belief that AA is a psychic.

- Abdul bets you 100 CU that he can predict the next four coin tosses. How much are you willing to bet against that (assuming that you are using a fair coin).
- You throw the coin 4 times, and AA guesses correctly all four times. Abdul now bets you another 100 CU that he can predict the next four coin tosses. Up to how much would you bet now?

Assumption 1

- *You use a fair coin, such that the probability of it coming heads is* $1/2$.
- *Your utility for money is linear, i.e.* $U(x) = x$ *for any amount of money* x.

Quick summary

- We want to make a decision against an unknown parameter ω.
- The Bayes utility is the maximum expected utility, and it is convex with respect to the distribution of ω.
- Our decisions can depend on observations, via a decision function.
- We can construct a complete decision function by computing $U(\xi, \delta)$ for all decision functions (normal form).
- We can instead wait until we observe x and compute $U[\xi(\cdot \mid x), d]$ for all decisions (extensive form).