## What is a monotone dependently typed function?

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PhD from EPFL Lausanne in November 2017

Higher-Order Subtyping with Type Intervals

Partial formalization of Scala's type system (on paper and in Agda).

- Bounded type operators.
- Lower and upper bounds (hence Type Intervals).
- Not yet: variances.
- Other research interests:
  - Category theory.
  - Graph rewriting.
  - Systems biology.
  - Domain specific languages (esp. probablistic and stochastic systems).
- Is looking for a PostDoc position at Chalmers! https://sstucki.github.io/



## Upper and lower bounds

From the Java 9 standard library:

```
interface Map<K,V> {
   V computeIfAbsent
      ( K key
      , Function<? super K, ? extends V> mappingFunction
      );
}
```

Presentation in System F with bounded quantification:

```
\forall K V (K' \geq K) (V' \leq V) \rightarrow Map K V \rightarrow K \rightarrow Function K' V' \rightarrow V
```

• With *type intervals* (\* =  $\bot$ .. $\top$ ):

```
(K : *) (V : *) (K' : K..\top) (V' : \bot..V) \rightarrow Map K V \rightarrow K \rightarrow Function K' V' \rightarrow V
```



#### Variance

Scala has variance annotations:

Function<-K,+V> 
$$\cong$$
 K  $\to$  V K'  $\geq$  K, V'  $\leq$  V  $\vdash$  Function K' V'  $\leq$  Function K V

• With variances, no bounds needed:

$$\forall$$
 K V  $\rightarrow$  Map K V  $\rightarrow$  Function K V  $\rightarrow$  K  $\rightarrow$  V

## Variance and bounds

Maps need comparable keys:

Precise kinding of identity:

$$Id : (A : *) \rightarrow A..A$$

Identity should be covariant (monotone):

$$\begin{array}{l} \texttt{Id} \; : \; (\texttt{+A} \; : \; \texttt{*}) \; \rightarrow \; \texttt{A..A} \\ \texttt{A} \; < \; \texttt{B} \; \vdash \; \texttt{Id} \; \; \texttt{A} \; < \; \texttt{Id} \; \; \texttt{B} \end{array}$$

Incomparable kinds!

Id A : A..A 
$$B..A \le A..A \le A..B$$
  
Id B : B..B  $B..A \le B..B \le A..B$ 

# Bounded higher-order variant subtyping

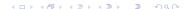
• Judgement  $\Gamma \vdash F \leq G : K$   $K \leq K' \leq Comparable$ ,  $V' \leq V \vdash$   $Map \leq Map : (-K : \bot..Comparable) (+V : *) \rightarrow *$   $Map K' \leq Map K : (+V : *) \rightarrow *$   $Map K' V' \leq Map K V : *$ 

• What about dependencies?

```
A : *, B : A..\top \vdash Id \leq Id : (+X : *) \rightarrow X..X Id A \leq Id B : ??
```

• We wish for:

```
A : *, B : A..\top \vdash Id \leq Id : (+X : *) \rightarrow X..X Id A \leq Id B : A..B
```



# Variance in Type Theory

Lowering everything one level:

```
\begin{array}{lll} {\sf Kinds} & \to & {\sf Types} \\ {\sf Types} & \to & {\sf Terms} \\ {\sf Type \ operators} & \to & {\sf Functions} \\ {\sf Subtyping} & \to & {\sf Partial \ order} \end{array}
```

Examples:

```
id : (+ n : \mathbb{N}) \rightarrow n..n minus : (+ n : \mathbb{N}) \rightarrow (- i : 0..n) \rightarrow 0..n
```

# A falling paradigm

Type theory paradigm:

Types and terms share a typing context.

```
If \Gamma \vdash t : T then \Gamma \vdash T : \mathsf{Type}.
```

Cannot work for variance:

```
+n : \mathbb{N} ⊢ minus n : (- i : 0..n) → 0..n
```

but not

+n : 
$$\mathbb{N}$$
  $\vdash$  (- i : 0..n)  $\rightarrow$  0..n : Type

Term is covariant in n, but its type mixed-variant!

## Paradigm has fallen already

Linearity [McBride 2016: I got plenty of nutting]

While I can drink my beer only once (resource, linear), I can contemplate it over and over (mental object, unrestricted).

```
(\lambda x \rightarrow x) : (1 x : \mathbb{N}) \rightarrow x..x
```

• Erasure [Barras/Bernardo, Sheard/Mishram-Linger 2008]: Irrelevant in the term, but relevant in the type.

```
(\lambda \ n \ x \ xs \rightarrow x :: xs) : (0 \ n : \mathbb{N}) \ (x : A) \ (xs : Vec A n) \rightarrow Vec A (1 + n)
```

## Idea for dependent variance

# On the type side, distinguish positive and negative occurrences of a variable.

Substitute a type-side variable by two values!

```
\vdash id : (+ n : N) → n..n

\vdash 3 ≤ 5 : N

\vdash id 3 ≤ id 5 : (n..n)[n:=(3,5)]

\vdash id 3 ≤ id 5 : n[n:=(5,3)] .. n[n:=(3,5)]

\vdash id 3 < id 5 : 3..5
```

• Same when *going types*: If  $+n : \mathbb{N} \vdash t : T$  (or  $-n : \mathbb{N} \vdash t : T$ ) then

$$-n^-: \mathbb{N}, +n^+: \mathbb{N} \vdash T[n := (n^-, n^+)]: \mathsf{Type}$$



#### Related idea

- Decompose mixed variance into co- and contravariance.
- Forces monotonicity.
- Example: negative data types.

data 
$$D = Lam (D \rightarrow D)$$

$$D = \mu(X^-, X^+). \ (X^- \to X^+)$$



# Applications of dependent variance

- Subtyping dependent types.
- Sized types Nat :  $(+i : Size) \rightarrow Type$ .
- Explaining Agda's positivity checker.



## Future work

- Does it work?
- Work out the details.
- Complete Agda formalization.
- Write a paper.

#### Related work

- Andreas Nuyts et al.: Modal models of type theory: Parametricity, irrelevance.
- Models types as categories instead of groupoids.

