# Semi-continuous Sized Types and Termination

## Termination Checking via Type Systems

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# Theorem Provers for Constructive Logic

Theorem Provers built on Dependent Type Theory:

- Coq (INRIA, France)
- Epigram (Nottingham, UK)
- Agda (Chalmers, Sweden)

Their soundness is based on *termination*.

# Constructive Logics

The Curry-Howard Isomorphism:

| Proposition | Type |
|---|---|
| $A$ implies $B$ | $A \to B$ |
| Proof | Purely Functional Program |
| Valid Proof | Terminating Program |

Non-terminating programs lead to inconsistency:

$$f : (0 = 0) \to (0 = 1)$$
$$f(p) = f(p)$$

# Type-Based Termination, Informally

**Recipe**

Step 1 In the type system, attach sizes to data structures.

Step 2 Using type-checking, ensure that recursive calls use only arguments with decreased size.

# Step 1: Sized Binary Trees

- Let $\mathsf{BTree}^{\imath}$ denote trees of height $< \imath$.
- The empty tree has height 0, hence $\mathsf{leaf} : \mathsf{BTree}^{1}$, but also $\mathsf{leaf} : \mathsf{BTree}^{2}$, $\mathsf{leaf} : \mathsf{BTree}^{3}$, ...
- In general $\mathsf{leaf} : \mathsf{BTree}^{\imath+1}$ for all $\imath$.

$$\begin{array}{lcl} \mathsf{leaf} & : & \forall \imath.\, \mathsf{BTree}^{\imath+1} \\ \mathsf{node} & : & \forall \imath.\, \mathsf{Int} \times (\mathsf{BTree}^{\imath} \times \mathsf{BTree}^{\imath}) \to \mathsf{BTree}^{\imath+1} \end{array}$$

- $\mathsf{BTree}^{\infty}$ contains all binary trees.
- Subtyping: $\mathsf{BTree}^{\imath} \subseteq \mathsf{BTree}^{\imath+1} \subseteq \cdots \subseteq \mathsf{BTree}^{\infty}$.

# Step 2: Equality Test for Sized Binary Trees

- Code annotated with sizes:

$$\mathsf{eq} : \forall \imath.\, \mathsf{BTree}^{\imath} \rightarrow \mathsf{BTree}^{\imath} \rightarrow \mathsf{Bool}$$

$$\begin{array}{l} \mathsf{eq}\ \mathsf{leaf}\ \mathsf{leaf} = \mathsf{true} \\ \mathsf{eq}\ \mathsf{node}(i_1, (l_1, r_1))^{\imath+1}\ \mathsf{node}(i_2, (l_2, r_2))^{\imath+1} = (i_1 == i_2)\ \ \&\& \\ \quad\quad \mathsf{eq}\ {l_1}^{\imath}\ {l_2}^{\imath}\ \ \&\&\ \ \mathsf{eq}\ {r_1}^{\imath}\ {r_2}^{\imath} \\ \mathsf{eq}\ \_\ \_ = \mathsf{false} \end{array}$$

- Input arguments assumed to be of size $\imath + 1$.
- Recursive arguments inferred to be of size $\imath$.
- Descend in size, hence, termination.

# Abstracting the Branching Type

- Generalize to $F$-Branching $\mathsf{Int}$-labelled trees $\mathsf{Tree}^{\imath}\, F$
- Constructors:

$$\begin{array}{lcl} \mathsf{leaf} & : & \forall F \forall \imath.\, \mathsf{Tree}^{\imath+1}\, F \\ \mathsf{node} & : & \forall F \forall \imath.\, \mathsf{Int} \times F\, (\mathsf{Tree}^{\imath}\, F) \to \mathsf{Tree}^{\imath+1}\, F \end{array}$$

- Valid instances

| | |
|---|---|
| binary trees | $F\ T = T \times T$ |
| lists | $F\ T = T$ |
| finitely branching trees | $F\ T = \mathsf{List}^{\infty}\ T$ |
| infinitely branching trees | $F\ T = \mathsf{Nat}^{\infty} \to T$ |

- Invalid instance ($F$ not monotone), e.g., $F\ T = T \to \mathsf{Bool}$

# Equality of $F$-Branching Trees

- Generalize equality test to $F$-branching trees:
- Termination not inferable with untyped methods.

# Termination and Polymorphism

$$\mathsf{Eq}\ T = T \to T \to \mathsf{Bool}$$

$$\mathsf{eq} : (\forall T. \mathsf{Eq}\ T \to \mathsf{Eq}\,(F\ T)) \to \forall\imath.\, \mathsf{Eq}\,(\mathsf{Tree}^{\imath} F)$$

$$\mathsf{eq}\ \mathit{eqF}\ \mathsf{leaf}\ \mathsf{leaf} = \mathsf{true}$$

$$\mathsf{eq}\ \mathit{eqF}\ \mathsf{node}(i_1, \mathit{ft}_1)\ \overbrace{\mathsf{node}(i_2, \underbrace{\mathit{ft}_2}_{F\,(\mathsf{Tree}^{\imath}\,F)})}^{\mathsf{Tree}^{\imath+1}\,F} = (i_1 == i_2)\ \ \&\&$$

$$\mathit{eqF}\quad \overbrace{(\mathsf{eq}\ \mathit{eqF})}^{\mathsf{Eq}\ T = \mathsf{Eq}\,(\mathsf{Tree}^{\imath}\,F)}\quad \mathit{ft}_1\ \mathit{ft}_2$$

$$\mathsf{eq}\ \_\ \_\ \_ = \mathsf{false}$$

Observe the role reversal: The recursive function $(\mathsf{eq}\ \mathit{eqF})$ becomes an argument to its own argument $\mathit{eqF}$!

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# Evaluation

No untyped formalism can handle this example:

- In the untyped setting, eq diverges, e.g., define

$$eqF\ eqT\ ft_1\ ft_2 = eqT\ \mathsf{node}(0, ft_1)\ \mathsf{node}(0, ft_2)$$

- and execute the function clause

$$\begin{array}{l}\mathsf{eq}\ eqF\ \mathsf{node}(i_1, ft_1)\ \mathsf{node}(i_2, ft_2) = \ldots \\ \quad eqF\ (\mathsf{eq}\ eqF)\ ft_1\ ft_2\end{array}$$

A typed formalism such as TBT uses the information that

$$eqF : \forall T.\mathsf{Eq}\ T \to \mathsf{Eq}\,(F\ T)$$

is polymorphic (hence, the above instance of $eqF$ is ill-typed).

# Type-Based Termination, Formally

**Theorem**

$f = s(f) : \forall\imath.\, A(\imath)$ *is well-defined if*

1. *(bottom check)* $A(0)$ *contains all programs, e.g.,* $A(\imath) = \mathsf{BTree}^{\imath} \to C$.
2. *(descent)* $f : A(\imath)$ *implies* $s(f) : A(\imath + 1)$.
3. *(admissibility)* $\bigcap_{\alpha<\lambda} A(\alpha) \subseteq A(\lambda)$ *for all limit ordinals* $\lambda \neq 0$.

**Proof.**

By transfinite induction on $\imath$.

1. (base) $f : A(0)$ trivial.
2. (step) ind.hyp. $f : A(\alpha)$ implies $s(f) = f : A(\alpha + 1)$.
3. (limit) $f : \bigcap_{\alpha<\lambda} A(\alpha)$ by ind.hyp., hence $f : A(\lambda)$.

□

# Operational Semantics

- Terms:

$$\begin{array}{lcl} r,s,t & ::= & c \mid x \mid \lambda x t \mid r\, s \\ c & ::= & \mathsf{fix}^{\mu} \mid \mathsf{fix}^{\nu} \mid \mathsf{inl} \mid \mathsf{inr} \mid \mathsf{case} \ldots \end{array}$$

- Addition $\mathsf{add} = \lambda x.\, \mathsf{fix}^{\mu}(\lambda add \lambda y.\, \mathsf{case}\, y\, (\lambda\_.\, x)\, (\lambda y'.\, \mathsf{inr}\, (add\, y')))$.
- Stream $\mathsf{repeat} = \lambda x.\, \mathsf{fix}^{\nu}(\lambda repeat.\, (x, repeat))$.
- Reduction of open terms:

$$\begin{array}{lcl} (\lambda x t)\, s & \longrightarrow & [s/x]t \\ \mathsf{case}\, (\mathsf{inl}\, r) & \longrightarrow & \lambda x \lambda y.\, x\, r \\ \ldots & & \end{array}$$

- Naive fixed-point reduction $\mathsf{fix}^{\mu} s \longrightarrow s\, (\mathsf{fix}^{\mu} a)$ diverges.

# Fixed-Point Unfolding

- Only reduce recursive functions applied to a value.

$$\begin{array}{rcl} v & ::= & \lambda x t \mid \mathsf{inl}\, t \mid \mathsf{inr}\, t \mid \mathsf{fix}^{\nu} s \mid \ldots \\ \mathsf{fix}^{\mu} s\, v & \longrightarrow & s\,(\mathsf{fix}^{\mu} s)\, v \end{array}$$

- Only reduce corecursion on demand.

$$\begin{array}{rcl} e(\bullet) & ::= & \bullet\, s \mid \mathsf{case}\, \bullet \mid \ldots \\ e(\mathsf{fix}^{\nu} s) & \longrightarrow & e(s\,(\mathsf{fix}^{\nu} s)) \end{array}$$

- Goal: Well-typed programs are strongly normalizing.

# Typing

- Types:

$$\begin{array}{lcll} a & ::= & \imath \mid a+1 \mid \infty & \text{size expressions} \\ F & ::= & C \mid X \mid \lambda XF \mid F\,G & \text{type constructors} \\ C & ::= & \to \mid \forall \mid \mu \mid \nu \mid + \mid \times \mid \dots & \text{type constants} \end{array}$$

- Equalities:

$$\begin{array}{lcll} (\lambda XF)\,G & = & [G/X]F & \\ \mu^{a+1} F & = & F\,(\mu^{a} \mathcal{F}) & \text{same for } \nu \\ \infty & = & \infty + 1 & \end{array}$$

- Recursion typing (analogous for $\nu$).

$$\frac{s : \forall \imath.\, A\,\imath \to A\,(\imath + 1)}{\mathsf{fix}^{\mu} s : \forall \imath.\, A\,\imath}\ A\ \mathsf{fix}^{\mu}\text{-adm}$$

# Term model

- Each type $A$ denotes a saturated set $\mathcal{A}$ of s.n. terms.
- Each constructor $\mathcal{F}$ denotes an operator on sat. sets.

$$\mathcal{A} \to \mathcal{B} = \{r \mid r\, s \in \mathcal{B} \text{ for all } s \in \mathcal{A}\}$$

$$\begin{aligned}
\mu^0 \mathcal{F} &= \bot \qquad \text{least sat. set} \\
\mu^{\alpha+1} \mathcal{F} &= \mathcal{F}(\mu^\alpha \mathcal{F}) \\
\mu^\lambda \mathcal{F} &= \bigcup_{\alpha<\lambda} \mu^\alpha \mathcal{F}
\end{aligned}$$

$$\begin{aligned}
\nu^0 \mathcal{F} &= \mathsf{SN} \qquad \text{greatest sat. set} \\
\nu^{\alpha+1} \mathcal{F} &= \mathcal{F}(\nu^\alpha \mathcal{F}) \\
\nu^\lambda \mathcal{F} &= \bigcap_{\alpha<\lambda} \nu^\alpha \mathcal{F}
\end{aligned}$$

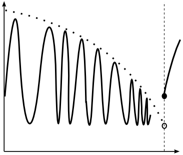
# Upper Semi-Continuous Types

**Definition (upper semi-continuous)**

A semantical type $\mathcal{A} : \mathsf{On} \to \mathcal{P}(\mathsf{SN})$ is upper semi-continuous (*usc*) if for all limits $\lambda \neq 0$

$$\limsup_{\alpha \to \lambda} \mathcal{A}(\alpha) := \left( \bigcap_{\alpha_0 < \lambda} \bigcup_{\alpha_0 \leq \alpha < \lambda} \mathcal{A}(\alpha) \right) \subseteq \mathcal{A}(\lambda)$$

An *usc* type fulfills $\bigcap_{\alpha < \lambda} \mathcal{A}(\alpha) \subseteq \mathcal{A}(\lambda)$, hence, is admissible.

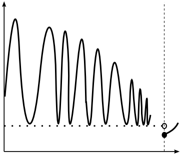


# Lower Semi-Continuous Types

**Definition (upper semi-continuous)**

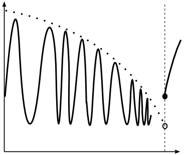
A semantical type $\mathcal{A} : \mathsf{On} \to \mathcal{P}(\mathsf{SN})$ is lower semi-continuous ($usc$) if for all limits $\lambda \neq 0$

$$\mathcal{A}(\lambda) \subseteq \liminf_{\alpha \to \lambda} \mathcal{A}(\alpha) := \bigcup_{\alpha_0 < \lambda} \bigcap_{\alpha_0 \leq \alpha < \lambda} \mathcal{A}(\alpha)$$



# Continuous Types

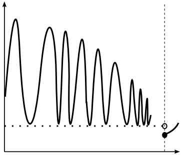
*usc*



$\limsup_{\alpha \to \lambda} \mathcal{A}(\alpha) \subseteq \mathcal{A}(\lambda)$

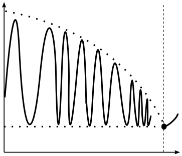
+

*lsc*



$\mathcal{A}(\lambda) \subseteq \liminf_{\alpha \to \lambda} \mathcal{A}(\alpha)$

continuous



$\lim_{\alpha \to \lambda} \mathcal{A}(\alpha) = \mathcal{A}(\lambda)$

# Closure Properties of Semi-Continuity

| *usc* | condition | *lsc* | condition |
|---|---|---|---|
| $\mathcal{A}$ *usc* | $\mathcal{A}$ monotone | $\mathcal{A}$ *lsc* | $\mathcal{A}$ antitone |
| $\mathcal{A} + \mathcal{B}$ *usc* | $\mathcal{A}$, $\mathcal{B}$ *usc* | $\mathcal{A} + \mathcal{B}$ *lsc* | $\mathcal{A}$, $\mathcal{B}$ *lsc* |
| $\mathcal{A} \times \mathcal{B}$ *usc* | $\mathcal{A}$, $\mathcal{B}$ *usc* | $\mathcal{A} \times \mathcal{B}$ *lsc* | $\mathcal{A}$, $\mathcal{B}$ *lsc* |
| $\mathcal{A} \to \mathcal{B}$ *usc* | $\mathcal{A}$ *lsc*, $\mathcal{B}$ *usc* | — | |
| $\nu\mathcal{F}$ *usc* | $\mathcal{F}$ *usc* | $\mu\mathcal{F}$ *lsc* | $\mathcal{F}$ *lsc* |

# Why Upper Semi-Continuity is Vital

Let $\mathsf{pred} : \forall\imath.\, \mathsf{Nat}^{\imath+1} \to \mathsf{Nat}^{\imath}$ such that $\mathsf{pred}\,0$ raises an exception. Define

$$f : \forall\imath.\, \overbrace{(\mathsf{Nat}^{\infty} \to \mathsf{Nat}^{\imath}) \to X}^{A(\imath)}$$

$$f(g : \mathsf{Nat}^{\infty} \to \mathsf{Nat}^{\imath+1}) = f\ ((\mathsf{pred} \circ g \circ \mathsf{succ}) : \mathsf{Nat}^{\infty} \to \mathsf{Nat}^{\imath})$$

Now $f(\mathsf{id})$ loops.

The definition passes the bottom check and the descent criterion, but $A(\imath)$ is neither *usc* nor admissible.

# Related Work

| *Expressivity* | Xi | Par | Ama | Gim | Fra | **A** | Bar | Bla | Buch |
|---|---|---|---|---|---|---|---|---|---|
| term. measures | + | − | − | − | − | − | − | + | o |
| dep. types | o | − | − | + | − | − | + | + | − |
| polymorphism | + | o | − | + | − | **+** | + | + | − |
| infinite branch. | − | − | − | + | + | **+** | + | − | + |
| semi-cont. | − | $\omega$ | − | − | − | **+** | − | − | − |
| productivity | − | + | + | + | + | + | + | − | + |
| *Features* | | | | | | | | | |
| symbolic exec. | − | − | + | + | + | + | + | + | + |
| soundness | V | D | SN | − | SN | SN | o | SN | D |
| ordinals | $< \omega$ | $\leq \omega$ | On | − | $\Omega$ | $\Omega_\omega$ | − | $< \omega$ | $\leq \omega$ |
| equi-rec. | − | + | − | − | − | + | − | − | − |
| size inference | − | + | − | − | − | − | + | − | − |

# Conclusions

- Termination checking can be integrated into type checking
- Especially powerful in combination with polymorphism
- Type-Based Termination is a modern technology, still under active development

# Future Work

- Extend to dependent types
- Investigate semi-continuity for dependent types
- Find intuitive explanations for non-admissibility of types
- Integrate into a theorem prover

# Acknowledgement

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