Formal Languages, Coinductively Formalized

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Contents

Formal Languages

- A language is a set of strings over some alphabet A.
- Real life examples:
  - Orthographically and grammatically correct English texts (infinite set).
  - Orthographically correct English texts (even bigger set).
  - List of university employees plus their phone extension.
    `AbelAndreas1731,CoquandThierry1030,DybjerPeter1035,...`
- Programming language examples:
  - The set of grammatically correct JAVA programs.
  - The set of decimal numbers.
  - The set of well-formed string literals.
- Languages can describe protocols, e.g. file access.
  - $A = \{o, r, w, c\}$ (open, read, write, close)
  - Read-only access: orc, oc, $orrrc$, $orcorrrcoc$, ...
  - Illegal sequences: c, rr, orr, oco, ooc, ...

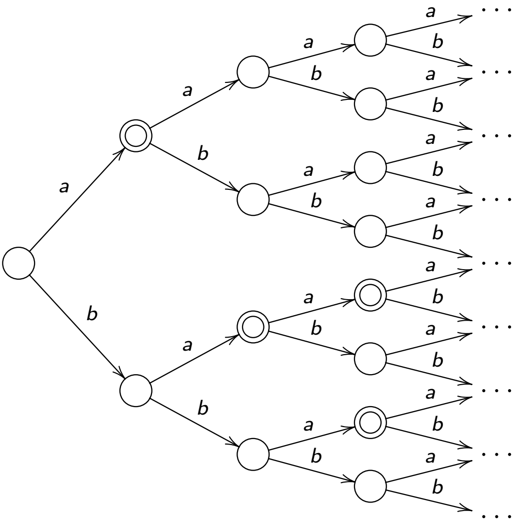
Running Example: Even binary numbers

- Even binary numbers: 0, 10, 100, 110, 1000, 1010, ...
- Excluded: 00, 010 (non-canonical); 1, 11 (odd) ...
- Alphabet $A = \{a, b\}$ where a is zero and b is one.
- So $E = \{a, ba, baa, bba, baaa, baba, \dots\}$.

Tries

- An infinite trie is a node-labeled A-branching tree.
- I.e., each node has one branch for each letter $a \in A$.
- Languages: representable by infinite Bool-labelled tries.
- To check whether word $a_1 \cdots a_n$ is in the language:
  - We start at the root.
  - At step i, we choose branch a_i.
  - At the final node, the label tells us whether the word is in the language or not.
- A trie memoizes a function $f : \mathrm{List}\, A \to \mathsf{Bool}$.

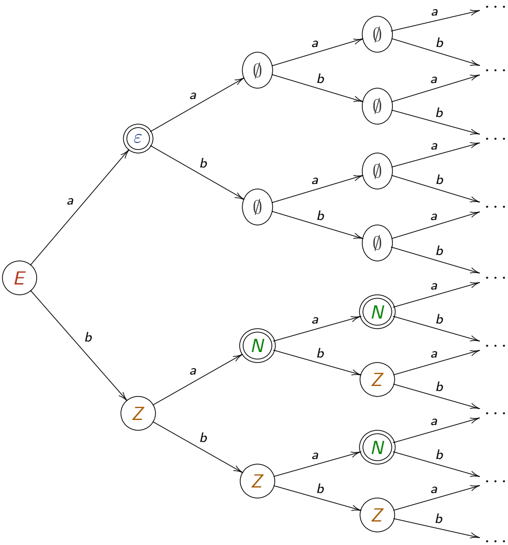
Trie of E



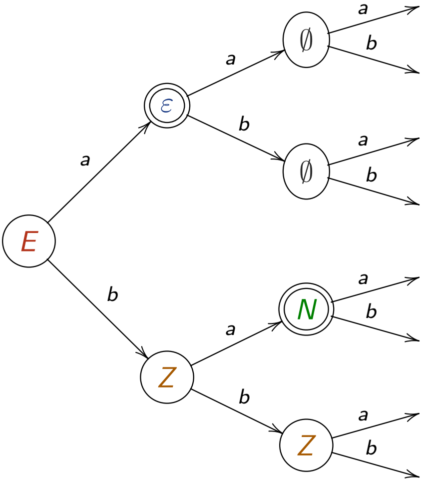
Regular Languages

- A trie is regular if it has only *finitely* many different *subtrees*.
- Each node of the trie corresponds to one of these languages:

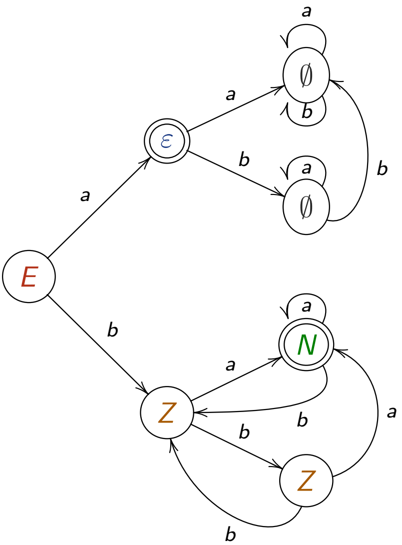
E	even binary numbers
Z	strings ending in a
N	strings not ending in b
ε	the empty string
$\emptyset$	nothing (empty language)


E
a
b
ε
Z
∅
N
Z
a
b
...

Cutting duplications at depth 3



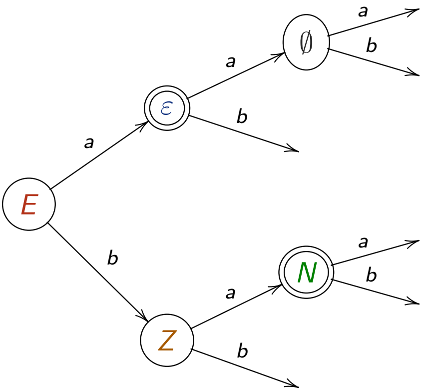
Bending branches . . .



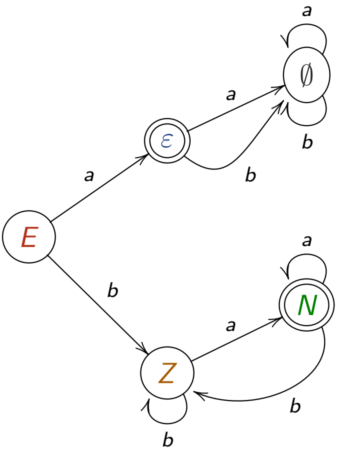
Final Automata

- We have arrived at a familiar object: a final automaton.
- Depending on what we cut, we get different automata for E.
- If we cut *all* duplicate subtrees, we get the *minimal* automaton.

Removing duplicate subtrees II. . .



Bending branches II . . .



Extensional Equality of Automata

- All automata for E unfold to the same trie.
- This gives a extensional notion of automata *equality*:
  1. Recognizing the same language.
  2. I.e., unfold to the same trie.

Automata, Formally

- An automaton consists of
  1. A set of states S.
  2. A function $\nu : S \to \mathsf{Bool}$ singling out the accepting states.
  3. A transition function $\delta : S \to A \to S$.

$s \in S$	νs	$\delta\, s\, a$	$\delta\, s\, b$
E	✗	ε	Z
ε	✓	$\emptyset$	$\emptyset$
$\emptyset$	✗	$\emptyset$	$\emptyset$
Z	✗	N	Z
N	✓	N	Z

- Language automaton
  1. State = language ℓ accepted when starting from that state.
  2. $\nu\ell$: Language ℓ is nullable (accepts the empty word)?
  3. $\delta \ell a = \{w \mid aw \in \ell\}$: Brzozowski derivative.

Differential equations

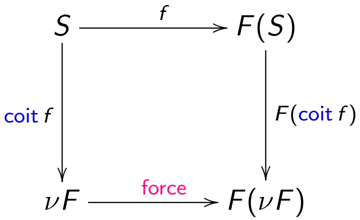
- Language E and friends can be specified by *differential equations*:
- ν gives the *initial value*.

$$
\begin{array}{lcl}
\nu\,\emptyset & = & \mathsf{false} \\
\delta\,\emptyset\,x & = & \emptyset \\
\\
\nu\,\varepsilon & = & \mathsf{true} \\
\delta\,\varepsilon\,x & = & \emptyset \\
\\
\nu\,E & = & \mathsf{false} \\
\delta\,E\,a & = & \varepsilon \\
\delta\,E\,b & = & Z
\end{array}
\qquad
\begin{array}{lcl}
\nu\,N & = & \mathsf{true} \\
\delta\,N\,a & = & N \\
\delta\,N\,b & = & Z \\
\\
\nu\,Z & = & \mathsf{false} \\
\delta\,Z\,a & = & N \\
\delta\,Z\,b & = & Z
\end{array}
$$

- For these simple forms, solutions exist always.
  What is the general story?

Final Coalgebras

- (Weakly) final coalgebra.



- Coiteration = finality witness.

$$\text{force} \circ \text{coit}\, f = F\,(\text{coit}\, f) \circ f$$

- Copattern matching *defines* coit by corecursion:

$$\text{force}\,(\text{coit}\, f\ s) = F\,(\text{coit}\, f)\,(f\ s)$$

Automata as Coalgebra

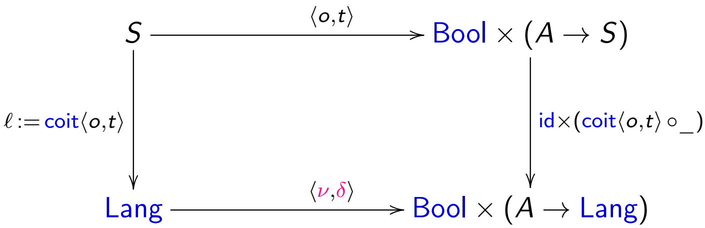
- Arbib & Manes (1986), Rutten (1998), Traytel (2016).
- Automaton structure over set of states S:

$$\begin{array}{lcll} o & : & S \to \mathsf{Bool} & \text{“output”: acceptance} \\ t & : & S \to (A \to S) & \text{transition} \end{array}$$

- Automaton is coalgebra with $F(S) = \mathsf{Bool} \times (A \to S)$.

$$\langle o, t \rangle \quad : \quad S \longrightarrow \mathsf{Bool} \times (A \to S)$$

Formal Languages as Final Coalgebra



$$\begin{aligned} \nu \circ \ell &= o \\ \nu\ (\ell\, s) &= o\, s \end{aligned}$$

"nullable"

$$\begin{aligned} \delta \circ \ell &= (\ell \circ \_) \circ t \\ \delta\ (\ell\, s) &= \ell \circ (t\, s) \\ \delta\ (\ell\, s)\ a &= \ell\,(t\, s\, a) \end{aligned}$$

(Brzozowski) derivative

Languages – Rule-Based

- Coinductive tries Lang defined via observations/projections ν and δ:
- Lang is the greatest type consistent with these rules:

$$\frac{l : \mathsf{Lang}}{\nu\, l : \mathsf{Bool}} \qquad \frac{l : \mathsf{Lang} \qquad a : A}{\delta\, l\, a : \mathsf{Lang}}$$

- Empty language $\emptyset : \mathsf{Lang}$.
- Language of the empty word $\varepsilon : \mathsf{Lang}$ defined by copattern matching:

$$\begin{array}{lcll} \nu\, \varepsilon & = & \mathsf{true} & : \mathsf{Bool} \\ \delta\, \varepsilon\, a & = & \emptyset & : \mathsf{Lang} \end{array}$$

Corecursion

- Empty language $\emptyset : \mathsf{Lang}$ defined by corecursion:

$$\begin{array}{lcl} \nu\,\emptyset & = & \mathsf{false} \\ \delta\,\emptyset\,a & = & \emptyset \end{array}$$

- Language union $k \cup l$ is pointwise disjunction:

$$\begin{array}{lcl} \nu\,(k \cup l) & = & \nu\,k \vee \nu\,l \\ \delta\,(k \cup l)\,a & = & \delta\,k\,a \cup \delta\,l\,a \end{array}$$

- Language composition $k \cdot l$ à la Brzozowski:

$$\begin{array}{lcl} \nu\,(k \cdot l) & = & \nu\,k \wedge \nu\,l \\ \delta\,(k \cdot l)\,a & = & \begin{cases} (\delta\,k\,a \cdot l) \cup \delta\,l\,a & \text{if } \nu\,k \\ (\delta\,k\,a \cdot l) & \text{otherwise} \end{cases} \end{array}$$

- Not accepted because $\cup$ is not a constructor.

Construction of greatest fixed-points

- Iteration to greatest fixed-point.

$$\top \supseteq F(\top) \supseteq F^2(\top) \supseteq \cdots \supseteq F^\omega(\top) = \bigcap_{n<\omega} F^n(\top)$$

- Naming $\nu^i F = F^i(\top)$.

$$\begin{array}{lcl} \nu^0 \; F & = & \top \\ \nu^{n+1} \; F & = & F\,(\nu^n F) \\ \nu^\omega \; F & = & \bigcap_{n<\omega} \nu^n F \end{array}$$

- Deflationary iteration.

$$\nu^i \; F \;\; = \;\; \bigcap_{j<i} F\,(\nu^j F)$$

Sized coinductive types

- Add to syntax of type theory

Size	type of ordinals
i	ordinal variables
$\nu^i F$	sized coinductive type
Size$<i$	type of ordinals below i

- Bounded quantification $\forall j{<}i.\, A = (j : \mathsf{Size}{<}\, i) \to A$.
- Well-founded recursion on ordinals, roughly:

$$\frac{f : \forall\, i.\, (\forall\, j{<}i.\, \nu^j F) \to \nu^i F}{\mathsf{fix}\, f : \forall\, i.\, \nu^i F}$$

Sized coinductive type of languages

- $\mathsf{Lang}\,i \cong \mathsf{Bool} \times (\forall j{<}i.\ A \to \mathsf{Lang}\,j)$

$$\frac{l : \mathsf{Lang}\,i}{\nu\,l : \mathsf{Bool}} \qquad \frac{l : \mathsf{Lang}\,i \qquad j < i \qquad a : A}{\delta\,l\,\{j\}\,a : \mathsf{Lang}\,j}$$

- $\emptyset : \forall i.\,\mathsf{Lang}\,i$ by copatterns and induction on i:

$$\begin{array}{lcll} \nu\,(\emptyset\,\{i\}) & = & \mathsf{false} & : \mathsf{Bool} \\ \delta\,(\emptyset\,\{i\})\,\{j\}\,a & = & \emptyset\,\{j\} & : \mathsf{Lang}\,j \end{array}$$

- Note $j < i$.
- On right hand side, $\emptyset : \forall j{<}i.\,\mathsf{Lang}\,j$ (coinductive hypothesis).

Type-based guardedness checking

- Union preserves size/guardeness:

$$\frac{k : \mathsf{Lang}\, i \qquad l : \mathsf{Lang}\, i}{k \cup l : \mathsf{Lang}\, i}$$

$$\begin{array}{lcl} \nu\,(k \cup l) & = & \nu\, k \vee \nu\, l \\ \delta\,(k \cup l)\,\{j\}\, a & = & \delta\, k\,\{j\}\, a \cup \delta\, l\,\{j\}\, a \end{array}$$

- Composition is accepted and also guardedness-preserving:

$$\frac{k : \mathsf{Lang}\, i \qquad l : \mathsf{Lang}\, i}{k \cdot l : \mathsf{Lang}\, i}$$

$$\begin{array}{lcl} \nu\,(k \cdot l) & = & \nu\, k \wedge \nu\, l \\ \delta\,(k \cdot l)\,\{j\}\, a & = & \begin{cases} (\delta\, k\,\{j\}\, a \cdot l) \cup \delta\, l\,\{j\}\, a & \text{if } \nu\, k \\ (\delta\, k\,\{j\}\, a \cdot l) & \text{otherwise} \end{cases} \end{array}$$

(Not Necessarily Finite) Automata

- Recapitulate automata à la Rutten (1998):

S	:	Set	state set (could be infinite)
ν	:	$S \to$ Bool	accepting state?
δ	:	$S \times A \to S$	transition function

- Automaton is record/object.

```
record DA (S : Set) : Set where
  field ν : (s : S) → Bool
        δ : (s : S) (a : A) → S

  νs : ∀{i} (ss : List i S) → Bool
  νs ss = List.any ν ss

  δs : ∀{i} (ss : List i S) (a : A) → List i S
  δs ss a = List.map (λ s → δ s a) ss
```

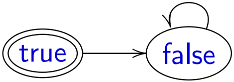
Constructing Automata

- Automaton for the empty language $\emptyset$:

$$
\begin{aligned}
&\emptyset A : DA\ \top \\
&\nu\ \ \emptyset A\ \ s\ \ \ \ \ = \text{false} \\
&\delta\ \ \emptyset A\ \ s\ \ a = s
\end{aligned}
$$

- Automaton for the empty word ε:

$$
\begin{aligned}
&\varepsilon A : DA\ \text{Bool} \\
&\nu\ \ \varepsilon A\ \ b\ \ \ \ \ = b \\
&\delta\ \ \varepsilon A\ \ b\ \ a = \text{false}
\end{aligned}
$$

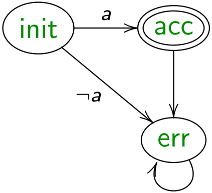


Constructing Automata

Accepting a specific character a.

```
data 3States : Set where
  init acc err : 3States

charA : (a : A) → DA 3States
ν (charA a) init   = false
ν (charA a) acc    = true
ν (charA a) err    = false
δ (charA a) init x =
  if ⌊ a ≟ x ⌋ then acc else err
δ (charA a) acc x  = err
δ (charA a) err x  = err
```



Unioning Automata

Union automaton.

- Transition in lock-step in $S_1 \times S_2$.
- Accept if s_1 or s_2 is accepting.

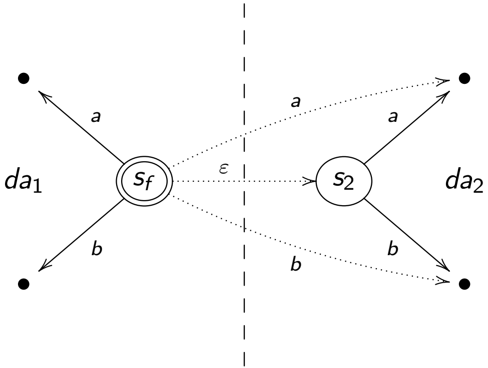
$$
\begin{array}{l}
\_\oplus\_ : \forall\{S_1\ S_2\}\ (da_1 : \mathsf{DA}\ S_1)\ (da_2 : \mathsf{DA}\ S_2) \to \mathsf{DA}\ (S_1 \times S_2) \\
\nu\ \ (da_1 \oplus da_2)\ \ (s_1\ ,\ s_2)\ \ \ \ = \nu\ da_1\ s_1\ \ \vee\ \ \nu\ da_2\ s_2 \\
\delta\ \ (da_1 \oplus da_2)\ \ (s_1\ ,\ s_2)\ \ a\ = \delta\ da_1\ s_1\ a\ \ ,\ \ \delta\ da_2\ s_2\ a
\end{array}
$$

Power automaton: being in several states at the same time.

$$
\begin{array}{l}
\mathsf{powA} : \forall\{S\}\ (da : \mathsf{DA}\ S) \to \mathsf{DA}\ (\mathsf{List}\ \infty\ S) \\
\nu\ \ (\mathsf{powA}\ da)\ \ ss\ \ \ \ = \nu\mathsf{s}\ da\ ss \\
\delta\ \ (\mathsf{powA}\ da)\ \ ss\ \ a\ = \delta\mathsf{s}\ da\ ss\ a
\end{array}
$$

Automaton for Language Composition

Compose two automata, picking initial state s_2 of da_2.



Automaton for Language Composition

A composed state is one $s_1 : S_1$ and possibly several $ss_2 \subset S_2$.

$$\text{composeA} : \forall \{S_1\ S_2\}$$
$$(da_1 : \text{DA}\ S_1)\ (s_2 : S_2)\ (da_2 : \text{DA}\ S_2) \to \text{DA}\ (S_1 \times \text{List}\ \infty\ S_2)$$

We accept if in a final state in S_2
or the final state S_1 if the initial state in S_2 is accepting.

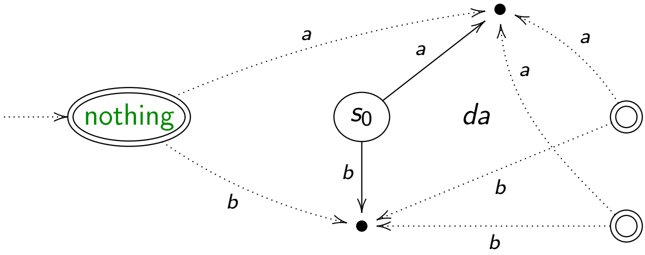
$$\nu\ \ (\text{composeA}\ da_1\ s_2\ da_2)\ \ (s_1\ ,\ ss_2)\ \ =$$
$$(\nu\ da_1\ s_1 \wedge \nu\ da_2\ s_2) \vee \nu s\ da_2\ ss_2$$

If in final state in S_1 we also transition from initial state in S_2.

$$\delta\ \ (\text{composeA}\ da_1\ s_2\ da_2)\ \ (s_1\ ,\ ss_2)\ \ a\ \ =$$
$$\delta\ da_1\ s_1\ a\ ,\ \delta s\ da_2\ (\text{if}\ \nu\ da_1\ s_1\ \text{then}\ s_2 :: ss_2\ \text{else}\ ss_2)\ a$$

Automaton for Language Iteration

- Kleene star of automaton with initial state s_0.
- New initial (and final state) nothing.



- Additionally transit from final states to successors of s_0.

Automata: Taking Stock

- We now can translate regular expressions to deterministic automata.
- Model implementations of automata very direct.
- All constructions preserve finiteness.
- TODO: connect to efficient implementation.
- All constructions have been formally verified in Agda.

Bisimilarity

- Equality of infinite tries is defined coinductively.
- $\_\cong\_$ is the greatest relation consistent with

$$\frac{l \cong k}{\nu\, l \equiv \nu\, k}\ \cong\nu \qquad \frac{l \cong k \qquad a : A}{\delta\, l\, a \cong \delta\, k\, a}\ \cong\delta$$

- Equivalence relation via provable ≅refl, ≅sym, and ≅trans.

$$\begin{array}{lcll} \cong\text{trans} & : & (p : l \cong k) \to (q : k \cong m) \to l \cong m & \\ \cong\nu\,(\cong\text{trans}\, p\, q) & = & \equiv\text{trans}\,(\cong\nu\, p)\,(\cong\nu\, q) & : \ \nu\, l \equiv \nu\, k \\ \cong\delta\,(\cong\text{trans}\, p\, q)\, a & = & \cong\text{trans}\,(\cong\delta\, p\, a)\,(\cong\delta\, q\, a) & : \ \delta\, l\, a \cong \delta\, m\, a \end{array}$$

- Congruence for language constructions.

$$\frac{k \cong k' \qquad l \cong l'}{(k \cup k') \cong (l \cup l')}\ \cong\cup$$

Proving bisimilarity

- Composition distributes over union.

$$\text{dist} \; : \; \forall\, k\, l\, m.\;\; k \cdot (l \cup m) \cong (k \cdot l) \cup (k \cdot m)$$

- Proof. Observation $\delta \_ \; a$, case k nullable, l not nullable.

$$\begin{array}{lll}
\delta\,(k \cdot (l \cup m))\, a & & \\
\quad = \;\; \boxed{\delta\, k\, a \cdot (l \cup m)} \cup \delta\,(l \cup m)\, a & & \text{by definition} \\
\quad \cong \;\; \boxed{(\delta\, k\, a \cdot l \cup \delta\, k\, a \cdot m)} \cup (\delta\, l\, a \cup \delta\, m\, a) & & \text{by coind. hyp. (wish)} \\
\quad \cong \;\; (\delta\, k\, a \cdot l \cup \delta\, l\, a) \cup (\delta\, k\, a \cdot m \cup \delta\, m\, a) & & \text{by union laws} \\
\quad = \;\; \delta\,((k \cdot l) \cup (k \cdot m))\, a & & \text{by definition}
\end{array}$$

- Formal proof attempt.

$$\cong\!\delta\; \text{dist}\; a \;=\; \cong\!\text{trans}\; (\cong\!\cup\; \boxed{\text{dist}}\; \ldots)\; \ldots$$

- Not coiterative / guarded by constructors!

Guardedness-preserving bisimilarity proofs

- Sized bisimilarity $\cong$ is greatest family of relations consistent with

$$\frac{l \cong^i k}{\nu\, l \equiv \nu\, k}\ \cong\nu \qquad \frac{l \cong^i k \qquad j < i \qquad a : A}{\delta\, l\, a \cong^j \delta\, k\, a}\ \cong\delta$$

- Equivalence and congruence rules are guardedness preserving.

$$\begin{array}{lcll} \cong\text{trans} & : & (p : l \cong^i k) \to (q : k \cong^i m) \to l \cong^i m & \\ \cong\nu\,(\cong\text{trans}\, p\, q) & = & \equiv\text{trans}\,(\cong\nu\, p)\,(\cong\nu\, q) & : \ \nu\, l \equiv \nu\, k \\ \cong\delta\,(\cong\text{trans}\, p\, q)\, j\, a & = & \cong\text{trans}\,(\cong\delta\, p\, j\, a)\,(\cong\delta\, q\, j\, a) & : \ \delta\, l\, a \cong^j \delta\, m\, a \end{array}$$

- Coinductive proof of dist accepted.

$$\cong\delta\ \text{dist}\ j\ a \ = \ \cong\text{trans}\ j\ (\cong\cup\ \boxed{(\text{dist}\ \ j)}\ (\cong\text{refl}\ j))\ \ldots$$

Conclusions

- Tracking guardedness in types allows
  - natural modular corecursive definition
  - natural bisimilarity proof using equation chains
- Implemented in Agda (ongoing)
- Abel et al (POPL 13): Copatterns
- Abel/Pientka (ICFP 13): Well-founded recursion with copatterns
- Abel (CMCS 16): Equational Reasoning about Formal Languages in Coalgebraic Style

Related work

- Hagino (1987): Coalgebraic types
- Cockett et al.: Charity
- Dmitriy Traytel (PhD TU Munich, 2015): Languages coinductively in Isabelle
- Kozen, Silva (2016): Practical coinduction
- Hughes, Pareto, Sabry (POPL 1996)
- Papers on sized types (1998–2015): e.g. Sacchini (LICS 2013)