Normalization by Evaluation for Martin-Löf Type Theory

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Normalization by Evaluation

- Slogan: implement computation in the *object language (syntax)* by mapping it to computation in the *meta language (semantics)*
- For the λ-calculus: interpret object-level abstraction and application by meta-level abstraction and application
- Berger Schwichtenberg (LICS91): objects of the meta-language can be brought "down" to the object-language

Related Work

- Martin-Löf 1975: NbE for Type Theory (weak conversion)
- Martin-Löf 2004: Talk on NbE (philosophical justification)
- Danvy et al: Type-directed partial evaluation
- Altenkirch Hofmann Streicher 1996: NbE for λ-free System F
- Berger Eberl Schwichtenberg 2003: Term rewriting for NbE
- Aehlig Joachimski 2004: Untyped NbE, operationally
- Filinski Rohde 2004: Untyped NbE, denotationally
- Danielsson 2006: strongly typed NbE for LF

Our Approach

- Interpretation of Martin-Löf Type Theory (MLTT) in Scott Domain $[\mathsf{D} \to \mathsf{D}] \subseteq \mathsf{D}$.
- Each type A is mapped to a set of total elements in D
- Total elements can be brought down to syntax in long $\beta\eta$-normal form

The Language

- Set Tm of raw terms:

$$r, s, t, A, B, C ::= x \mid \lambda x t \mid r\, s \mid \Pi x\!:\!A.B \mid Set$$

- Plus natural numbers and primitive recursion like in Gödel's T
- Non-dependent function space $A \to B := \Pi \_\!:\!A.\, B$
- $\beta\eta$-reduction: Congruence closure of

$$\begin{array}{lcll} (\lambda x t)\, s & \longrightarrow & t[s/x] & \\ \lambda x.\, (t\, x) & \longrightarrow & t & \text{if } x \notin \mathsf{FV}(t) \end{array}$$

- Plus computation rules for primitive recursion

Example: Functions with Variable Arity

- A dependent type

$$
\begin{array}{lcl}
T & : & \mathit{Nat} \to \mathit{Set} \\
T\,0 & = & \mathit{Nat} \\
T\,(n+1) & = & \mathit{Nat} \to T\ n \\
 & & \\
T\,3 & = & \mathit{Nat} \to \mathit{Nat} \to \mathit{Nat} \to \mathit{Nat}
\end{array}
$$

- Summation of $n+1$ natural numbers

$$
\begin{array}{lclcl}
\mathit{sum}' & : & \Pi n : \mathit{Nat}.\, \mathit{Nat} \to T\ n & & \\
\mathit{sum}'\,0\,a & = & a & : & T\,0 \\
\mathit{sum}'\,(n+1)\,a\,x & = & \mathit{sum}'\,n\,(a+x) & : & T\ n \\
 & & & & \\
\mathit{sum}'\,3\,0\,6\,7\,8 & = & 21 & &
\end{array}
$$

Inference Rules

- Judgements

$\Gamma \vdash$	Γ is a well-formed context
$\Gamma \vdash A$	A is a well-formed type
$\Gamma \vdash t : A$	t has type A

- Wellformed types

$$\frac{\Gamma \vdash}{\Gamma \vdash \mathit{Set}} \qquad \frac{\Gamma \vdash A : \mathit{Set}}{\Gamma \vdash A} \qquad \frac{\Gamma \vdash A \qquad \Gamma, x\!:\!A \vdash B}{\Gamma \vdash \Pi x\!:\!A.\, B}$$

Inference Rules II

- Lambda-calculus

$$\frac{\Gamma \vdash}{\Gamma \vdash x : \Gamma(x)} \qquad \frac{\Gamma, x{:}A \vdash t : B}{\Gamma \vdash \lambda x t : \Pi x{:}A.\, B}$$

$$\frac{\Gamma \vdash r : \Pi x{:}A.\, B \qquad \Gamma \vdash s : A}{\Gamma \vdash r\, s : B[s/x]}$$

- Π in Set, conversion:

$$\frac{\Gamma \vdash A : Set \qquad \Gamma, x{:}A \vdash B : Set}{\Gamma \vdash \Pi x{:}A.\, B : Set} \qquad \frac{\Gamma \vdash t : A \qquad \Gamma \vdash A'}{\Gamma \vdash t : A'}\; A =_{\beta\eta} A'$$

- Plus rules for natural numbers

Interpretation Domain

- Scott domain

$$\mathsf{D} \quad = \quad \mathsf{Lam}\,[\mathsf{D} \to \mathsf{D}] + \mathsf{Pi}\,\mathsf{D}\,[\mathsf{D} \to \mathsf{D}] + \mathsf{Set} + \mathsf{Ne}\,\mathit{Tm}$$

- Application

$$\begin{array}{lcll} \_\cdot\_ & : & [\mathsf{D} \to [\mathsf{D} \to \mathsf{D}]] & \\ (\mathsf{Lam}\,f)\cdot d & = & f(d) & \\ e \cdot d & = & \bot & \text{if } e \text{ is not a } \mathsf{Lam} \end{array}$$

- "Bad guy" in D is, e. g., $\mathsf{Lam}\,f$ where

$$\begin{array}{lcl} f(\mathsf{Ne}\,x) & = & \mathsf{Ne}\,y \\ f(d) & = & \bot \end{array}$$

Evaluation

- Let $\rho \in \mathit{Var} \to \mathsf{D}$ an environment.

$$
\begin{array}{rcll}
[\![x]\!]_\rho & = & \rho(x) & \\
[\![\lambda x t]\!]_\rho & = & \mathsf{Lam}\, f & \text{where } f(d) = [\![t]\!]_{\rho[x \mapsto d]} \\
[\![r\, s]\!]_\rho & = & [\![r]\!]_\rho \cdot [\![s]\!]_\rho & \\
[\![\Pi x : A.\, B]\!]_\rho & = & \mathsf{Pi}\, [\![A]\!]_\rho\, g & \text{where } g(d) = [\![B]\!]_{\rho[x \mapsto d]} \\
[\![\mathit{Set}]\!]_\rho & = & \mathsf{Set} &
\end{array}
$$

- $(\mathsf{D}, \_ \cdot \_, [\![\_]\!]_\_)$ is a weakly extensional λ-model

$$t =_\beta t' \ \text{ implies } \ [\![t]\!]_\rho = [\![t']\!]_\rho$$

Reification and Reflection

- Bringing semantic objects down to $\beta\eta$-long forms

$$\downarrow\,:\,[\mathsf{D}\times\mathsf{D}\to \mathit{Tm}]$$

$$\begin{array}{lcll}
\downarrow^{\mathsf{Set}}\mathsf{Pi}\,a\,g & = & \Pi x\!:\!\downarrow^{\mathsf{Set}} a.\ \downarrow^{\mathsf{Set}} g(d) & \text{where } d=\uparrow^{a} x \text{ and } x \text{ ``fresh''}\\
\downarrow^{\mathsf{Set}} a & = & \downarrow^{\mathsf{Set}} a & \\
\downarrow^{\mathsf{Pi}\,a\,g} e & = & \lambda x.\ \downarrow^{g(d)}(e\cdot d) & \text{where } d=\uparrow^{a} x \text{ and } x \text{ ``fresh''}\\
\downarrow^{c}(\mathsf{Ne}\,t) & = & t & \text{if } c \text{ not a } \mathsf{Pi}
\end{array}$$

- Embedding neutral terms $x\,t_1\ \ldots\ t_n$ into D

$$\uparrow\,:\,[\mathsf{D}\times \mathit{Tm}\to\mathsf{D}]$$

$$\begin{array}{lcll}
\uparrow^{\mathsf{Pi}\,a\,g} t & = & \mathsf{Lam}\,f & \text{where } f(d)=\uparrow^{g(d)}(t\,(\downarrow^{a} d))\\
\uparrow^{c} t & = & \mathsf{Ne}\,t & \text{if } c \text{ not a } \mathsf{Pi}
\end{array}$$

NbE Algorithm

- Environment $\uparrow\Gamma$ satisfies $(\uparrow\Gamma)(x) = \uparrow^{a} x$ where $a = [\![\Gamma(x)]\!]_{\uparrow\Gamma}$.
- A term $\Gamma \vdash t : A$ is normalized by

$$\mathsf{nbe}_{\Gamma}^{A} t = \downarrow^{[\![A]\!]_{\uparrow\Gamma}} [\![t]\!]_{\uparrow\Gamma}$$

Soundness and Completeness

- Goal: show that $\mathsf{nbe}_\Gamma^A t$ produces a canonical form for $\Gamma \vdash t : A$.
- Consequence: NbE decides $\beta\eta$-equality.
- Completeness: If $\Gamma \vdash t, t' : A$ then

$$t =_{\beta\eta} t' \text{ implies } \mathsf{nbe}_\Gamma^A t \equiv \mathsf{nbe}_\Gamma^A t'.$$

- Soundness: If $\Gamma \vdash t : A$ then

$$t =_{\beta\eta} \mathsf{nbe}_\Gamma^A t$$

Completeness

- Partial equivalence relation (PER) $c = c' \in \mathcal{T}ype$ identifies codes for semantical types $c, c' \in \mathsf{D}$.
- For $c = c \in \mathcal{T}ype$, define PER $e = e' \in [c]$.
- Definition mutually: use Dybjer's induction-recursion.

$$\frac{a = a' \in \mathcal{T}ype \qquad g(d) = g(d') \in \mathcal{T}ype \text{ for all } d = d' \in [a]}{\mathsf{Pi}\, a\, g = \mathsf{Pi}\, a'\, g' \in \mathcal{T}ype}$$

$$[\mathsf{Pi}\, a\, g] = \{(e, e') \mid e \cdot d = e' \cdot d' \in [g(d)] \text{ for all } d = d' \in [a]\}$$

- PERs single out total elements.
- PERs are extensional (handle η).

Completeness Proof (Sketch)

- Let $t, t' : A$.

$$t =_{\beta\eta} t'$$

$$\Downarrow$$

$$[\![t]\!] = [\![t']\!] \in [[\![A]\!]]$$

$$\Downarrow$$

$$\downarrow^{[\![A]\!]}[\![t]\!] \equiv \downarrow^{[\![A]\!]}[\![t']\!]$$

- Generalizes to open terms.

Soundness via Term Model

- For $\sigma \in \mathit{Var} \to \mathit{Tm}$ let $(\!| t |\!)_\sigma$ denote parallel substitution.
- $\mathsf{T} = \mathit{Tm}/{=_{\beta\eta}}$.
- $(\mathsf{T}, \_\, \_, (\!| \_ |\!)_\_)$ is an extensional λ-model.
- Logical Relation $\mathsf{R}^a \subseteq \mathsf{T} \times [a]$ for $a \in \mathcal{T}\mathit{ype}$.

$$
\begin{array}{lcll}
r\ \mathsf{R}^{\mathsf{Pi}\,a\,g}\ e & \iff & r\ s\ \mathsf{R}^{g(d)}\ e \cdot d & \text{for all } s\ \mathsf{R}^a\ d \\
r\ \mathsf{R}^c\ e & \iff & r =_{\beta\eta} \downarrow^c e &
\end{array}
$$

- Relating models: If $t : A$ then $(\!| t |\!)\ \mathsf{R}^{[\![A]\!]}\ [\![t]\!]$.
- Consequence $t =_{\beta\eta} \downarrow^{[\![A]\!]} (\!| t |\!)$.

Conclusion

- NbE is sound and complete for MLTT.
- Draft paper available (email me).
- Freshness formally treated with de Bruijn indices.
- Kripke logical relation needed instead.

Future Work

- Reification not necessary to decide $A =_{\beta\eta} A'$ in conversion rule.
- Instead of $Tm \subseteq \mathsf{D}$, use a D with atoms v_k.
- Neutral $v_k\, d_1\, \ldots\, d_n$ in D.
- Justification seems harder: Kripke-PER model?
- Other direction: work with judgmental equality $\Gamma \vdash t = t' : A$.