An Interpretation of Kleene's Slash in Type Theory

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Abstract

Kleene introduced the notion of slash to investigate the disjunction and existence properties under implication for intuitionistic arithmetic. In this paper Kleene's slash is translated to type theory. Besides translations of Kleene's results, the main application of the slash in type theory is that conditions are given for a typable term, containing free variables, to have a normal form beginning with a constructor.

1 Introduction

The disjunction and existence properties, that is, $\vdash A \vee B$ implies $\vdash A$ or $\vdash B$ and $\vdash \exists x A(x)$ implies $\vdash A(t)$ for some term t, respectively, were first proved for intuitionistic arithmetic by Kleene [9] using a modification of recursive realizability. Harrop [8] extended Kleene's result by also considering derivations depending on assumptions. Harrop proved

$$C \vdash A \vee B \text{ implies } C \vdash A \text{ or } C \vdash B \tag{ED}$$

$$C \vdash \exists x A(x) \text{ implies } C \vdash A(t) \text{ for some term } t \tag{EE}$$

where C is a closed formula not containing any strictly positive occurrences of $\vee$ and $\exists$; such a formula is called a Harrop formula.

In [10] Kleene gives much simplified proofs of the disjunction and existence properties as well as (ED) and (EE) for a more extensive class of formulas than the Harrop formulas. I will in this paper translate Kleene's method to type theory and thereby obtain conditions on a typable term to have a canonical value also when it may contain free variables.

Tait [17] introduced a powerful method for proving normalization of typed terms, which was adopted by Martin-Löf [11] to prove normalization of derivations in natural deduction. These ideas are behind Prawitz' [15, 16] suggestion of a general notion of validity for derivations. In Hallnäs [7] there is an extension of Prawitz' notion of validity which is closely related to slash in type theory.

In fact, Tait's computability method, when applied to type theory for proving normalization of closed terms, can be seen as a special case of the translation below of slash.

I will use Martin-Löf's type theory [13, 14], formulated so that $a = b \in A$ is understood as definitional equality; the rules are those of the intensional theory in [14]. In order not to obscure the ideas, I will not treat universes; to define slash for a universe is straightforward, but to prove the main result, theorem 2, involve the same additional complications as when proving normalization, using Tait's method, for Martin-Löf's type theory with universes, see Coquand [2].

Friedman [4] defined slash for higher order logic and I expect no problems in translating that notion of slash to impredicative type theories like Girard's system F [5] and Coquand and Huet's Calculus of Constructions [3].

2 Kleene's slash

The modification of recursive realizability in Kleene [9], used for the proofs of the disjunction and existence properties, consists in adding derivability conditions to the definition of realizability. The crucial observation in Kleene [10] is that the proofs of the disjunction and existence properties in [9] do not rely on any recursion theoretic elements, which, hence, can be left out, thereby giving very simple proofs of these properties.

Let Γ be a list of closed formulas and C a closed formula. The relation $\Gamma \mid C$, "Γ slashes C", is defined by induction on the number of logical constants in C by the clauses

1. If A is an atomic formula, $\Gamma \mid A$ if $\Gamma \vdash A$.
2. $\Gamma \mid A \vee B$, if $\Gamma \mid A$ or $\Gamma \mid B$.
3. $\Gamma \mid A \& B$, if $\Gamma \mid A$ and $\Gamma \mid B$.
4. $\Gamma \mid A \supset B$, if $\Gamma \vdash A \supset B$, and $\Gamma \mid A$ implies $\Gamma \mid B$.
5. $\Gamma \mid \forall x A(x)$, if $\Gamma \vdash \forall x A(x)$ and $\Gamma \mid A(\overline{n})$ for each numeral $\overline{n}$.
6. $\Gamma \mid \exists x A(x)$, if $\Gamma \mid A(\overline{n})$ for some numeral $\overline{n}$.

This definition of slash is a simplification, due to Aczel [1], of Kleene's original definition. The main result in [10] for the slash is the following theorem.

Theorem 1. *Let Γ be a list of closed formulas and A a closed formula. Assume that $\Gamma \mid C$ for each C in Γ. Then $\Gamma \vdash A$ implies $\Gamma \mid A$.*

The proof is by induction on the length of the derivation $\Gamma \vdash A$. From this theorem it easy to obtain (ED) and (EE) for a closed formula C such that $C \mid C$, using the observation that $\Gamma \mid A$ implies $\Gamma \vdash A$; since $C \mid C$ holds if C is a Harrop formula, this implies Harrop's result for (ED) and (EE).

3 Translation of Kleene's slash to type theory

In arithmetic, slash is a relation between a list of closed formulas and a closed formula; this will in type theory be translated to a relation between a context Γ and a judgement $t \in A$ in the context Γ.

The definition of $\Gamma \mid t \in A$ is made by induction on the derivation $\Gamma \vdash A\ set$ by the clauses

1. $\Gamma \mid t \in A{+}B$, if $\Gamma \vdash t \in A{+}B$, $\Gamma \vdash t = inl(a) \in A{+}B$ for some term a such that $\Gamma \mid a \in A$, or $\Gamma \vdash t = inr(b) \in A{+}B$ for some term b such that $\Gamma \mid b \in B$.

2. $\Gamma \mid t \in \Pi(A, B)$, if $\Gamma \vdash t \in \Pi(A, B)$, there exists a term $b(x)$ such that $\Gamma, x \in A \vdash b(x) \in B(x)$, for all terms a, $\Gamma \mid a \in A$ implies $\Gamma \mid b(a) \in B(a)$, and $\Gamma \vdash t = \lambda x.b(x) \in \Pi(A, B)$.

3. $\Gamma \mid t \in \Sigma(A, B)$, if $\Gamma \vdash t \in \Sigma(A, B)$, there exist terms a and b such that $\Gamma \mid a \in A$, $\Gamma \mid b \in B(a)$, and $\Gamma \vdash t = \langle a, b\rangle \in \Sigma(A, B)$.

4. $\Gamma \mid t \in Id(A, a, b)$, if $\Gamma \vdash t \in Id(A, a, b)$, $\Gamma \mid a \in A$, $\Gamma \mid b \in A$, $\Gamma \vdash a = b \in A$, and $\Gamma \vdash t = id(a) \in Id(A, a, b)$.

5. $\Gamma \mid t \in N$, if $\Gamma \vdash t \in N$ and $\Gamma \vdash t = \overline{n} \in N$ for some numeral $\overline{n}$.

6. $\Gamma \mid t \in N_k$, if $\Gamma \vdash t \in N_k$ and $\Gamma \vdash t = n_k \in N_k$ for some n, $0 \leq n < k$, $k = 0, 1, \ldots$.

It follows directly from the definition that $\Gamma \mid t \in A$ implies $\Gamma \vdash t \in A$. Note that, since $\bot$ is defined to be the empty set N_0, the second condition in the definition of $\Gamma \mid t \in N_0$ gives that there is no term t such that $\Gamma \mid t \in \bot$ holds. This is different from formulation of arithmetic used by Kleene where $\bot$ is defined to be $0 = 1$ and, hence, $\Gamma \mid \bot$ for any inconsistent Γ. If we consider the absurdity $\bot_{\text{M}}$ of minimal logic, we can extend the definition of slash by

7. $\Gamma \mid t \in \bot_{\text{M}}$, if $\Gamma \vdash t \in \bot_{\text{M}}$.

The rules for $\bot_{\text{M}}$ differs from the rules for $\bot$ in that there is no elimination rule for $\bot_{\text{M}}$.

Kleene [10] also defined slash for predicate calculus and propositional calculus. Translations of slash to type theory can also be made for these logics, and I will indicate the translation in the case of propositional calculus; this is a real simplification compared with arithmetic since no dependent types are needed.

In order to express propositional calculus in type theory, using the Curry-Howard interpretation, we have to add set variables $X_1, X_2, \ldots$ and build up sets from these variables and N_0, using the disjoint union $+$, the function arrow $\rightarrow$ and cartesian product $\times$ of two sets. To obtain a translation of slash in this case we leave out clauses 4, 5, and 6, except for N_0, and define slash for set variables by

8. $\Gamma \mid t \in X_i$, if $\Gamma \vdash t \in X_i$, $i = 1, 2, \ldots$.

In clause 2, Π is then replaced by $\rightarrow$ and, in clause 3, Σ is replaced by $\times$. All results below also holds for this restricted calculus.

In fact, we could define slash already for a calculus with only set variables and the function arrow, corresponding to simply typed λ-calculus. But to obtain anything substantially new from the results below, compared with an ordinary normalization proof, we need a set with more than one constructor, like the disjoint union of two sets, interpreting disjunction.

From the definition of slash, we get the following corollary, in which we assume that the set A is neither $\perp_{\text{M}}$ nor a set variable; this restriction has to be made simply because there are no canonical element in these sets.

Corollary 1. *If $\Gamma \mid t \in A$ then there exists a canonical value can of the set A such that $\Gamma \vdash t = can \in A$.*

A canonical value of a set is a term beginning with a constructor of the set. Using the fact that every typable term, which in general may be open, has a normal form, we may also require that a canonical value should be on normal form. In the formulation of Martin-Löf's type theory we are considering, $a = b \in A$ means definitional equality; so $\Gamma \vdash t = can \in A$ implies that t can be computed to the value can. When we in the sequel talk about canonical elements of a set we will tacitly assume that this set is not $\perp_{\text{M}}$ or a set variable.

The main result for the interpretation of slash in type theory is the following theorem.

Theorem 2. *Let Δ be a context $x_1 \in D_1, \ldots, x_m \in D_m(x_1, \ldots, x_{m-1})$ and $d_1, \ldots, d_m$ terms such that $\Gamma \mid d_i \in D_i(d_1, \ldots, d_{i-1})$, $0 < i \leq m$. Then $\Delta \vdash a(x_1, \ldots, x_m) \in A(x_1, \ldots, x_m)$ implies $\Gamma \mid a(d_1, \ldots, d_m) \in A(d_1, \ldots, d_m)$.*

In the proof we need the following lemma.

Lemma 1. *Let $\Gamma \mid a \in A$, $\Gamma \vdash b \in A$ and $\Gamma \vdash B$ set. Then* (i) *$\Gamma \vdash a = b \in A$ implies $\Gamma \mid b \in A$ and* (ii) *$\Gamma \vdash A = B$ implies $\Gamma \mid a \in B$.*

Proof. (i) follows easily from the definition of slash. Since we do not include universes, $\Gamma \vdash A = B$ holds only if A and B have the same outermost set constructor and equal parts; hence we can use induction on the length of A to prove (ii); the proof is then straightforward.

Proof of theorem 2. By induction on the length of the derivation $\Delta \vdash a(x_1, \ldots, x_m) \in A(x_1, \ldots, x_m)$. As an illustration, we treat the rules for disjoint union. For the introduction rule

$$\frac{\Delta \vdash a(x_1, \ldots, x_m) \in A(x_1, \ldots, x_m)}{\Delta \vdash inl(a(x_1, \ldots, x_m)) \in A(x_1, \ldots, x_m) + B(x_1, \ldots, x_m)}$$

we have, by induction hypothesis, $\Gamma \mid a(d_1,\ldots,d_m) \in A(d_1,\ldots,d_m)$, which gives, by the definition of slash for disjoint unions, $\Gamma \mid \mathit{inl}(a(d_1,\ldots,d_m)) \in A(d_1,\ldots,d_m) + B(d_1,\ldots,d_m)$. The other introduction rule is treated in the same way. For the elimination rule

$$\frac{\begin{array}{l}\Delta \vdash c(x_1,\ldots,x_m) \in A(x_1,\ldots,x_m) + B(x_1,\ldots,x_m) \\ \Delta,\ x \in A(x_1,\ldots,x_m) \vdash d(x_1,\ldots,x_m,x) \in C(x_1,\ldots,x_m,\mathit{inl}(x)) \\ \Delta,\ y \in B(x_1,\ldots,x_m) \vdash e(x_1,\ldots,x_m,y) \in C(x_1,\ldots,x_m,\mathit{inr}(y))\end{array}}{\begin{array}{l}\Delta \vdash \mathit{when}(c(x_1,\ldots,x_m),d(x_1,\ldots,x_m),e(x_1,\ldots,x_m)) \in \\ \qquad\qquad C(x_1,\ldots,x_m,c(x_1,\ldots,x_m))\end{array}}$$

we have, by induction hypothesis,

$$\Gamma \mid c(d_1,\ldots,d_m) \in A(d_1,\ldots,d_m) + B(d_1,\ldots,d_m)\,, \tag{1}$$

$$\Gamma \mid d(d_1,\ldots,d_m,a) \in C(d_1,\ldots,d_m,\mathit{inl}(a)) \quad \text{for all } a \text{ such that } \Gamma \mid a \in A(d_1,\ldots,d_m)\,, \tag{2}$$

$$\Gamma \mid e(d_1,\ldots,d_m,b) \in C(d_1,\ldots,d_m,\mathit{inl}(b)) \quad \text{for all } b \text{ such that } \Gamma \mid b \in B(d_1,\ldots,d_m)\,. \tag{3}$$

From (1) we get, by the definition of slash for disjoint unions, that one of

$$\Gamma \vdash c(d_1,\ldots,d_m) = \mathit{inl}(a) \in A(d_1,\ldots,d_m) + B(d_1,\ldots,d_m) \quad \text{for some } a \text{ such that } \Gamma \mid a \in A(d_1,\ldots,d_m)\,, \tag{4}$$

$$\Gamma \vdash c(d_1,\ldots,d_m) = \mathit{inr}(b) \in A(d_1,\ldots,d_m) + B(d_1,\ldots,d_m) \quad \text{for some } b \text{ such that } \Gamma \mid b \in B(d_1,\ldots,d_m)\,. \tag{5}$$

holds. Let us assume that it is (4) that holds, the case of (5) is handled in the same way. By the definition of *when*, we have

$$\Gamma \vdash \mathit{when}(\mathit{inl}(a),d(d_1,\ldots,d_m),e(d_1,\ldots,d_m)) = d(d_1,\ldots,d_m,a) \in C(d_1,\ldots,d_m,\mathit{inl}(a)) \tag{6}$$

and from (2) and (4) we get

$$\Gamma \mid d(d_1,\ldots,d_m,a) \in C(d_1,\ldots,d_m,\mathit{inl}(a))\,. \tag{7}$$

The lemma, (4), (6), and (7) finally give

$$\Gamma \mid \mathit{when}(c(d_1,\ldots,d_m),d(d_1,\ldots,d_m),e(d_1,\ldots,d_m)) \in C(d_1,\ldots,d_m,c(d_1,\ldots,d_m))\,.$$

As a special case of theorem 2 we get the translated version of theorem 1:

Corollary 2. *Let* Γ *be a context* $x_1 \in C_1,\ \ldots,\ x_n \in C_n(x_1,\ldots,x_{n-1})$ *and* $c_1,\ldots,\ c_n$ *terms such that* $\Gamma \mid c_i \in C_i(c_1,\ldots,c_{i-1})$, $0 < i \leq n$. *Then* $\Gamma \vdash a(x_1,\ldots,x_n) \in A(x_1,\ldots,x_n)$ *implies* $\Gamma \mid a(c_1,\ldots,c_n) \in A(c_1,\ldots,c_n)$.

Corollaries 1 and 2 give that if $t \in A$ can be derived in the the empty context, then t can be computed to a canonical value of A. This result can also be obtained by a normalization proof, using Tait's computability method;

actually, if Γ is empty then the definition of $\Gamma \mid t \in A$ is the same as the definition of Tait's computability predicate $Comp_A(t)$, provided we only consider normalization of closed terms as in Martin-Löf [12].

Because of the strong rules for Σ in Martin-Löf's type theory, the existence property is trivially satisfied in any context Γ since, by Σ-elimination, $\Gamma \vdash t \in \Sigma(A, B)$ implies $\Gamma \vdash \mathit{fst}(t) \in A$ and $\Gamma \vdash \mathit{snd}(t) \in B(\mathit{fst}(t))$. Note, however, that the term $\mathit{fst}(t)$ cannot in general be computed to a canonical value. The disjunction property cannot be proved under such general conditions. Corresponding to Kleene's result for disjunction in arithmetic we have the following consequence of corollary 2.

Corollary 3. *Let C be a set for which there exists a term $c(x)$ such that $x \in C \mid c(x) \in C$. Then $x \in C \vdash t(x) \in A{+}B$ implies $x \in C \vdash a(x) \in A$ for some term $a(x)$ or $x \in C \vdash b(x) \in B$ for some term $b(x)$.*

In arithmetic, disjunction can be expressed by using the existential quantifier; in Martin-Löf's type theory $+$ can in a similar way be expressed by Σ. But it is easy to see that the existence property obtained by the strong Σ-elimination does not imply the disjunction property.

4 Harrop sets

The interest of corollary 2 depends on for what sets we have $x \in C \mid c(x) \in C$ and on the complexity of the term $c(x)$. In arithmetic the main examples of formulas C such that $C \mid C$ are the Harrop formulas. A Harrop formula is defined to be a formula not containing any strictly positive occurrences of $\vee$ and $\exists$. Alternatively, the Harrop formulas can be inductively defined by (i) atomic formulas are Harrop formulas, (ii) if A and B are Harrop formulas then $A \& B$ is a Harrop formula, (iii) if B is a Harrop formula and A an arbitrary formula then $A \supset B$ is a Harrop formula, and (iv) if A is a Harrop formula then $\forall x A$ is a Harrop formula.

In type theory, we define the notion corresponding to Harrop formulas in an arbitrary context. The Harrop sets H_Γ in the context Γ are inductively defined by

(i) N_1, the one element set, is in H_Γ,

(ii) if $\Gamma \mid a \in A$, $\Gamma \mid b \in A$, and $\Gamma \vdash a = b \in A$ then $Id(A, a, b)$ is in H_Γ,

(iii) if A is in H_Γ and $B(x)$ in $H_{\Gamma,\, x \in A}$ then $\Sigma(A, B)$ is in H_Γ,

(iv) if A is a set in Γ and $B(x)$ in $H_{\Gamma,\, x \in A}$ then $\Pi(A, B)$ is in H_Γ.

Note that the sets in (i)-(iv) are those with exactly one constructor. By clause (iii), $A \times B$ is a Harrop set if A and B are Harrop sets and, by clause (iv), $A \to B$ is Harrop set if B is a Harrop set. Because of clause (iii) it is possible for a Harrop set to contain a strictly positive occurrence of a set of the form $\Sigma(A, B)$, but that requires A to be a Harrop set; so there cannot be any strictly positive occurrences of sets of the form $\Sigma(N, B)$, corresponding to an existentially quantified proposition on the natural numbers. If we consider minimal logic and set variables, we can extend the Harrop sets by

(v) $\perp_{\mathrm{M}}$ is in H_Γ,

(vi) X_i is in H_Γ, $i = 1, 2, \ldots$.

We cannot, however, include $\perp$ in the definition of Harrop sets since we would then not have the following theorem, corresponding to that, in arithmetic, $C \mid C$ when C is a Harrop formula.

Theorem 3. *If C is a Harrop set in the context Γ then there exists a term $c(x)$ such that*

$$\Gamma,\ x \in C \mid c(x) \in C$$

The term $c(x)$ can be recursively constructed by

(i) $\Gamma,\ x \in N_1 \mid 0_1 \in N_1$,

(ii) *if* $\Gamma,\ x \in A \mid a \in A$, $\Gamma, y \in A \mid b \in A$, *and* $\Gamma \vdash a = b \in A$ *then* $\Gamma,\ z \in Id(A, a, b) \mid id(a) \in Id(A, a, b)$,

(iii) *if* $\Gamma,\ x \in A \mid a(x) \in A$ *and* $\Gamma,\ x \in A,\ y \in B(x) \mid b(x, y) \in B(x)$ *then* $\Gamma,\ z \in \Sigma(A, B) \mid \langle a(\mathit{fst}(z)), b(\mathit{fst}(z), \mathit{snd}(z))\rangle \in \Sigma(A, B)$,

(iv) *if* $\Gamma,\ x \in A,\ y \in B(x) \mid b(x, y) \in B(x)$ *then* $\Gamma,\ z \in \Pi(A, B) \mid \lambda x.b(x, \mathit{apply}(z, x)) \in \Pi(A, B)$,

(v) $\Gamma,\ x \in \perp_M \mid x \in \perp_M$,

(vi) $\Gamma,\ x \in X_i \mid x \in X_i$, $i = 1, 2, \ldots$.

A Harrop set has at most one constructor, which intuitively explains why, in cases (i) - (iv), it is possible to construct a term $c(x)$ beginning with a constructor of the set C by only using the assumption $x \in C$.

In the proof of theorem 3 we need the following lemma.

Lemma 2. *Let Γ be a context $x_1 \in C_1, \ldots, x_n \in C_n(x_1, \ldots, x_{n-1})$, let $C(x_1, \ldots, x_n)$ be a Harrop set in Γ, and let the term $c(x_1, \ldots, x_{n-1}, x)$ be constructed according to* (i) - (vi) *in theorem 3. Let Δ be a context for which there exist terms $a_1, \ldots, a_n$ and a such that $\Delta \vdash a_i \in C_i(a_1, \ldots, a_{i-1})$, $0 < i \leq n$, and $\Delta \vdash a \in C(a_1, \ldots, a_n)$. Then $\Gamma,\ x \in C(x_1, \ldots, x_n) \mid c(x_1, \ldots, x_n, x) \in C(x_1, \ldots, x_n)$ implies $\Delta \mid c(a_1, \ldots, a_n, a) \in C(a_1, \ldots, a_n)$.*

Proof. The proof of the lemma is by induction on the definition of C in H_Γ. We illustrate the proof by the case when C is of the form $\Pi(A, B)$. By the recursive definition of c we know that c is $\lambda x.b(x, \mathit{apply}(z, x))$ for some term b such that $\Gamma,\ x \in A,\ y \in B(x) \mid b(x, y) \in B(x)$ (1). Assume that

$\Delta \vdash d \in \Pi(A,B)$ and $\Delta \mid a \in A$. We then obtain $\Delta \vdash a \in A$ (2) and, hence, $\Delta \vdash apply(d,a) \in B(a)$ (3). From (1), (2), (3), and the induction hypothesis, we get $\Delta \mid b(a, apply(d,a)) \in B(a)$. Hence, by the definition of slash, $\Delta \mid \lambda x.b(x, apply(d,x)) \in \Pi(A,B)$.

Proof of theorem 3. The proof is by induction on the definition of C in H_Γ. We again look at the case when C is of the form $\Pi(A,B)$. By induction hypothesis we have $\Gamma,\ x \in A,\ y \in B(x) \mid b(x,y) \in B(x)$ (1). Assume $\Gamma,\ z \in \Pi(A,B) \mid a \in A$, which gives $\Gamma,\ z \in \Pi(A,B) \vdash a \in A$ (2) and $\Gamma,\ z \in \Pi(A,B) \vdash apply(z,a) \in B(a)$ (3). The lemma applied on (1), (2), and (3) gives $\Gamma,\ z \in \Pi(A,B) \mid b(a, apply(z,a)) \in B(a)$. Hence, by the definition of slash, $\Gamma,\ z \in \Pi(A,B) \mid \lambda x.b(x, apply(z,x)) \in \Pi(A,B)$.

Goad [6] considers derivations, as well as proof terms, depending on Harrop formulas and shows that such a derivation can be normalized so that it ends with an introduction rule. For type theory we have a corresponding result as a corollary to theorem 3 and corollary 1.

Corollary 4. *Let C be a Harrop set and let $c(x)$ be constructed according to* (i)-(vi) *in theorem 3. If $x \in C \vdash a(x) \in A$ then there exists a canonical value $can(x)$ of the set A such that $x \in C \vdash a(c(x)) = can(x) \in A$.*

If we use the negation of minimal logic, that is, $\neg_{\mathrm{M}} C$ is defined to be $C \to \bot_{\mathrm{M}}$, then, by theorem 3, $x \in \neg_{\mathrm{M}} C \mid \lambda y.apply(x,y) \in \neg_{\mathrm{M}} C$; hence we have the following corollary.

Corollary 5. *Let C be an arbitrary set. If $x \in \neg_M C \vdash a(x) \in A$ then there exists a canonical value $can(x)$ of the set A such that $x \in \neg_M C \vdash a(\lambda y.apply(x,y)) = can(x) \in A$.*

So, a term containing a variable $x \in \neg_{\mathrm{M}} C$ can be computed to a canonical value if the variable x is simply η-expanded.

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