A Model of Type Theory in Cubical Sets

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Univalent Foundations

- Vladimir Voevodsky formulated the *Univalence Axiom* (UA) in Martin-Löf Type Theory as a strong form of the Axiom of Extensionality.
- UA is *classically* justified by the interpretation of types as *Kan simplicial sets*
- However, this justification uses non-constructive steps. Hence this does not provide a way to compute with univalence.

Result

We give a model of dependent type theory (Π, Σ, U, N, ...) in a *constructive metatheory* with:

- $\texttt{refl}\, a : \texttt{Id}_A(a, a)$
- $\texttt{J}(a) : C(a, \texttt{refl}\, a) \to (\Pi\, x : A)(\Pi\, p : \texttt{Id}_A(a, x))\, C(x, p)$
- $\texttt{JEq}(a, e) : \texttt{Id}_{C(a, \texttt{refl}\, a)}(J(a, e, a, \texttt{refl}\, a), e)$
- Univalence Axiom
- Propositional Truncation + Circle + Interval

Implementation: Cubical

(jww C. Cohen, T. Coquand, A. Mörtberg)

- Prototype proof assistant implemented in Haskell
- The Univalence Axiom and functional extensionality are available and compute!
- Try it! `http://github.com/simhu/cubical`

Overview

Cubical Category

We define the cubical category $\mathcal{C}$ as follows.
Fix a countable set of *symbols* (or atoms) $x, y, z, \ldots$ distinct from $0, 1$.

$\mathcal{C}$ is given by:

- objects are finite (decidable) sets of symbols $I, J, K, \ldots$
- a morphism $f\colon I \to J$ is given by a set map

  $$f\colon I \to J \cup \{0, 1\}$$

  such that if $f(x), f(y) \in J$, then $f(x) = f(y)$ implies $x = y$
  (f is injective on its *defined* elements.)
  This represents a substitution: assign values 0 or 1 to variables or rename them.

Cubical Category

- Composition of $f\colon I \to J$ and $g\colon J \to K$ defined by

$$(g \circ f)(x) = \begin{cases} g(fx) & f \text{ defined on } x, \\ fx & \text{otherwise;} \end{cases}$$

We write fg for $g \circ f$.

Cubical Sets

Definition

A *cubical set* X is a functor $X\colon \mathcal{C} \to \mathbf{Set}$.

So a cubical set X is given by sets $X(I)$ for each I, and maps $X(I) \to X(J)$, $a \mapsto af$ for $f\colon I \to J$ with

$$a\mathbf{1} = a \quad \text{and} \quad (af)g = a(fg).$$

Call an element of $X(I)$ and *I-cube*.

Example: Polynomial Ring (P. Aczel)

If k is a ring, then $k[x, y, z, \ldots]$ is a cubical set.

- a $\emptyset$-cube, or point, is an element of k
- a x-cube, or line, is an element of $k[x]$
- a x, y-cube, or square, is an element of $k[x, y]$
- ...
- an I-cube for $I = x_1, \ldots, x_n$ is an element of $k[x_1, \ldots, x_n]$

Cubical Sets

Think of a symbol x as a name for an indeterminate and

- $X(\emptyset)$ as points,
- $X(\{x\})$ as lines in dimension x,
- $X(\{x, y\})$ as squares in the dimensions x, y,
- $X(\{x, y, z\})$ as cubes,
- $\ldots$

Cubical Sets: Faces

For $x \in I$ the morphisms $(x=0), (x=1)\colon I \to I - x$ in $\mathcal{C}$ sending x to 0 and 1 respectively induce the face maps

$$X(x=0), X(x=1)\colon X(I) \to X(I-x)$$

An I-cube θ of X connects its two *faces* $\theta(x=0)$ and $\theta(x=1)$:

$$\theta(x=0) \xrightarrow[x]{\theta} \theta(x=1)$$

Cubical Sets: Degeneracies

$f\colon I \to J$ is a degeneracy morphism if f is defined on all elements in I and $I \subsetneq J$.

If $x \notin I$, consider the inclusion $s_x\colon I \to I, x$. We have $s_x(x=0) = \mathbf{1} = s_x(x=1)$, and so for an I-cube α of X:

$$\alpha \xrightarrow[x]{\alpha s_x} \alpha$$

If $\beta = \alpha s_x$ is such a degenerate I, x-cube, we can think of β to be *independent of the indeterminate* x.

Cubical Sets

Remark

- Kan's original approach (1955) to combinatorial homotopy theory used cubical sets

- Our notion is equivalent to nominal sets with 01-substitions (Pitts, Staton). This is a nominal set equipped with operations $(x = b)$ for $b \in \{0, 1\}$ s.t.
  1. $(u(x = b))\pi = u\pi(\pi(x) = b)$,
  2. $u(x = b) \# x$,
  3. $u \# x$ implies $u(x = b) = u$,
  4. $u(x = b)(y = c) = u(y = c)(x = b)$ if $x \neq y$.

  Used in the implementation

Model of Type Theory

Type theory is a generalized algebraic theory (Cartmell).

- Given by: Sorts, Operations, and Equations
- Sorts are interpreted by *sets*
- Interpretation of each operation
- Check the required equations

We use the notion of categories with families (Dybjer) to give our model.

Cubical Sets as a Category with Families

Cubical sets form (as any presheaf category) a model of type theory:

- The category of contexts $\Gamma \vdash$ and substitutions $\sigma\colon \Delta \to \Gamma$ is the category of cubical sets.
- Types $\Gamma \vdash A$ are given by

$$\begin{aligned} &A\alpha \text{ a set}, && \text{for } \alpha \in \Gamma(I), I \in \mathcal{C}, \\ &A\alpha \to A\alpha f \text{ a map}, && \text{for } f\colon I \to J \text{ in } \mathcal{C}, \\ &\quad a \mapsto af && \end{aligned}$$

  such that $a\mathbf{1} = a$, $(af)g = a(fg)$.
- Terms $\Gamma \vdash t\colon A$ are given by $t\alpha \in A\alpha$ such that $(t\alpha)f = t(\alpha f)$.

Cubical Sets as a Category with Families

- For $\Gamma \vdash A$ the context extension $\Gamma.A \vdash$ is defined as

$$(\alpha, a) \in (\Gamma.A)(I) \text{ iff } \alpha \in \Gamma(I) \text{ and } a \in A\alpha,$$
$$(\alpha, a)f = (\alpha f, af).$$

  We can define the projections $\mathsf{p} \colon \Gamma.A \to \Gamma$ and $\Gamma.A \vdash \mathsf{q} : A\,\mathsf{p}$ by

$$\mathsf{p}(\alpha, a) = \alpha,$$
$$\mathsf{q}(\alpha, a) = a.$$

This gives a model of Π and Σ but will not get us the identity type we want!

Identity Types

Let $\Gamma \vdash A$, $\Gamma \vdash a : A$, and $\Gamma \vdash b : A$.

We define $\Gamma \vdash \mathtt{Id}_A(a, b)$:
For $\alpha \in \Gamma(I)$ we define $\langle x \rangle \omega \in (\mathtt{Id}_A(a, b))\alpha$ for x fresh if

$$\omega \in A\alpha s_x \text{ s.t. } \omega(x = 0) = a\alpha \text{ and } \omega(x = 1) = b\alpha.$$

Identify $\langle x \rangle \omega = \langle x' \rangle \omega'$ iff $\omega(x = x') = \omega'$.

Identity Types

For $f : I \to J$ define

$$(\langle x \rangle \omega) f =_{\text{def}} \langle y \rangle (\omega (f, x = y)) \quad \in A \alpha f s_y$$

where y is fresh for J, and $(f, x = y) \colon I, x \to J, y$ extends f.

Identity Types

This immediately justifies the introduction rule

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash \texttt{refl}\, a : \texttt{Id}_A(a, a)}$$

by setting $(\texttt{refl}\, a)\alpha = \langle x \rangle\, a\, \alpha s_x$ for $\alpha \in \Gamma(I)$ and $x \notin I$.

Identity Types

For modeling the elimination principle we need: if $\Gamma \vdash A$ and there is a path between α_0 and α_1 in Γ, then the fibers $A\alpha_0$ and $A\alpha_1$ should be equivalent!

In the Kan simplicial set model this is provably *not* constructive (T. Coquand/M. Bezem).

To justify the elimination principle for `Id` we need additional structure on types!

Example: Polynomial Ring (contd.)

In the polynomial ring cubical set $P = k[x, y, z, \ldots]$ we can define a term $\alpha : (\Pi p\ q : P)\ \mathtt{Id}_P(p, q)$ by:

$$\alpha\ p\ q = \langle x \rangle t(x)$$

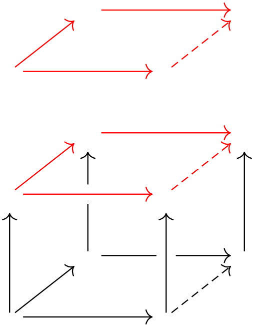
where $t(x) = (1 - x)p + xq$.
E.g., if p and q depend at most on y, z, then

$$(\alpha\ p(y, z)\ q(y, z))(y = 0) = \alpha\ p(0, z)\ q(0, z)$$

This operation is *uniform*!

Kan Structure

A Kan structure on a cubical set is a uniform choice of fillers of open boxes.



Open Boxes

Let $x \notin J$ and $a \in \{0,1\}$. Define

$$\mathcal{O}^a(x;J) = \{(x,a)\} \cup \{(y,c) \mid y \in J \text{ and } c \in \{0,1\}\}$$

For $I = x, J, K$ (disjoint) an open box in a cubical set X is given by a family $\vec{u}$ of elements $u_{yc} \in X(I-y)$ for $(y,c) \in \mathcal{O}^a(x;J)$ such that

$$u_{yc}(z=d) = u_{zd}(y=c)$$

Note: K can be non-empty!

Kan Structure

A Kan structure on a cubical set X is given by operations $X{\uparrow}$ (and $X{\downarrow}$) for each $I = x, J, K$, such that

$$X{\uparrow}\vec{u} \in X(I) \qquad \text{for } \vec{u} \text{ open box of shape } \mathcal{O}^0(x; J) \text{ in } X$$

such that for $(y, c) \in \mathcal{O}^0(x; J)$

$$(X{\uparrow}\vec{u})(y = c) = u_{yc} \in X(I - y)$$

and for $f : I \to K$ defined on x, J

$$(X{\uparrow}\vec{u})f = X{\uparrow}(\vec{u}\, f)$$

where $\vec{u}\, f$ is the $\mathcal{O}^0(fx; fJ)$ open box given by $u_{(fy)c} = u_{yc}(f - y) \in X(K - fy)$ with $(f - y) \colon I - y \to K - fy$.

Kan Structure

(Similarly we require operations for $X{\downarrow}$.)

We set

$$X^+\vec{u} = (X{\uparrow}\vec{u})(x = 1)$$
$$X^-\vec{u} = (X{\downarrow}\vec{u})(x = 0)$$

Kan Structure on a Type

A *Kan structure* on a type $\Gamma \vdash A$ is given by operations for all $\alpha \in \Gamma(I)$

$$A\alpha{\uparrow}\vec{u} \;\in\; A\alpha \qquad \text{for open boxes } \vec{u}$$

where $u_{yc} \in A\alpha(y = c), (y, c) \in \mathcal{O}^0(x; J)$ such that $(A\alpha{\uparrow}\vec{u})(y = c) = u_{yc}$ and for $f : I \to K$ defined on x, J

$$(A\alpha{\uparrow}\vec{u})f = (A\alpha f){\uparrow}(\vec{u}\; f).$$

(Similarly we require operations $A\alpha{\downarrow}\vec{u}$.)

Model of Type Theory

By restricting types $\Gamma \vdash A$ to those with a Kan structure, we get a model of type theory.

Theorem

Having a Kan structure is closed under Π-, Σ- *and* $\mathtt{Id}$-*types.*

Identity Type (cont.)

Theorem

If $\Gamma.A \vdash P$ *has a Kan structure, then there is a term* `subst` *s.t.*

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash a : A \qquad \Gamma \vdash b : A \qquad \Gamma \vdash p : \mathtt{Id}_A(a, b) \qquad \Gamma \vdash u : P[a]}{\Gamma \vdash \mathtt{subst}(p, u) : P[b]}$$

Proof.

Let $\alpha \in \Gamma(I)$; then $p\alpha = \langle x \rangle \omega$ and ω connects $a\alpha$ and $b\alpha$ in dimension x with $x \notin I$. So we get an I, x-cube in $\Gamma.A$:

$$[a]\alpha \xrightarrow{(\alpha s_x, \omega)} [b]\alpha$$

We define $\mathtt{subst}(p, u)\alpha = P(\alpha s_x, \omega)^+(u\alpha)$.

Identity Type (cont.)

Note that we have a line:

$$u\alpha \xrightarrow{P(\alpha s_x, \omega)\uparrow(u\alpha)} \mathtt{subst}(p, u)\alpha$$

In particular, if $p = \mathtt{refl}\, a$, then $\omega = a\alpha s_x$ and this gives a term of

$$\Gamma \vdash \mathtt{Id}_{P[a]}(u, \mathtt{subst}(\mathtt{refl}\, a, u)).$$

One can also show that the singleton type $(\Sigma x : A)\ \mathtt{Id}_A(a, x)$ is contractible.

Universe

Notation: $\mathbb{I}^J = \mathrm{Hom}_{\mathcal{C}}(J, -)\colon \mathcal{C} \to \mathbf{Set}$ for the representable cset

Definition

As a cubical set the universe U is given by J-cubes being types $\mathbb{I}^J \vdash A$ *with Kan structure* such that all the A_f's are small sets ($f\colon J \to K$).

- $\mathrm{U}(\emptyset)$ are small Kan cubical sets
- A line in U between A and B can be seen as "heterogeneous" notion of lines, squares, cubes, ... $a \to b$ where $a \in A(I)$ and $b \in B(I)$.

Kan Structure on U

Theorem

U *has a Kan structure.*

Proof sketch.

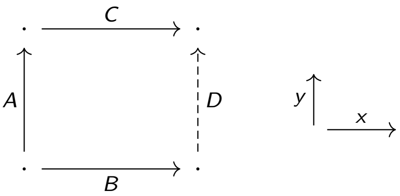
Two steps:

1. U has compositions $\mathrm{U}^{+}\vec{A}$
2. U has fillers $\mathrm{U}{\uparrow}\vec{A}$

Compositions in U

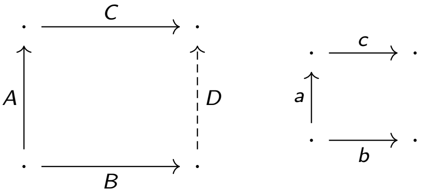
Main idea: composition of relations.

Consider a composition with $J = y$: given $A \in \mathrm{U}(I - x)$ and $B, C \in \mathrm{U}(I - y)$ such that $A(y = 0) = B(x = 0)$ and $A(y = 1) = C(x = 0)$; we want to define $D = \mathrm{U}^+(A, B, C) \in \mathrm{U}(I - x)$.



The main case is to define D_f for $f = \mathbf{1}\colon I - x \to I - x$.

Elements of $D_{\mathbf{1}}$ are triples (a, b, c) where a, b, c are elements of $A_{\mathbf{1}}, B_{\mathbf{1}}, C_{\mathbf{1}}$ respectively such that they are compatible:
$a(y = 0) = b(x = 0)$ and $a(y = 1) = c(x = 0)$



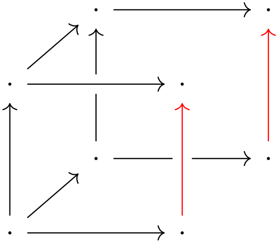
We have to give a Kan structure on D!
Given an open box $\vec{u}$ of shape $\mathcal{O}^0(x'; J')$ in $D_{\mathbf{1}}$ we have to define $D_{\mathbf{1}}\uparrow\vec{u}$. The steps are:

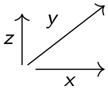
1. W.l.o.g. $J \subseteq x', J'$;
2. Case $x' \notin J$;
3. Case $x' \in J$ (here: $x' = y$).

All of these cases have a concrete combinatorial solution.

Filling in U

We want to give $E = \mathrm{U}\!\uparrow\!(A, B, C) \in \mathrm{U}(I)$. The main case is to give E_f for $f = \mathbf{1} \colon I \to I$. Elements are given by $\langle z\rangle(a, b, c)$ (z fresh) with a, b, c are in $A_{s_z}, B_{s_z}, C_{s_z}$, respectively such that





where the red lines are degenerate. Elements are identified modulo renaming of z.

For the Kan structure on E one has to consider six cases. To fill $\vec{u}$ of shape $\mathcal{O}^a(x'; J')$ in $E_\mathbf{1}$ one has to consider:

1. W.l.o.g. $J \subseteq x', J'$;
2. Case $x' = x$ and $a = 0$;
3. Case $x' = x$ and $a = 1$;
4. Case $x \notin J'$;
5. Case $x' \notin J$;
6. Case $x' \in J$.

This gives an effective proof not relying on minimal fibrations.

Further Work

- Formal correctness proof of the implementation
- Definition of a cubical syntax (Altenkirch/Kaposi, Brunerie, Polonsky)
- Connection to internal parametricity (Bernardy/Moulin)
- Can we get a model with a variation of cubical sets? (E.g., cubical sets with a "diagonal".)
- Resizing rules

Thank you!

Standard Cubes

For a finite set J of names denote the standard J-cube by

$$\mathbb{I}^J = \mathrm{Hom}_{\mathcal{C}}(J, -) \colon \mathcal{C} \to \mathbf{Set}$$

Not well-behaved under product

$$\mathbb{I}^J \times \mathbb{I}^K \ncong \mathbb{I}^{J \cup K} \qquad \text{(for } J, K \text{ disjoint)}$$

But there is a separated product $*$ with

$$\mathbb{I}^J * \mathbb{I}^K \cong \mathbb{I}^{J \cup K}$$

Separated Product

For cubical sets X and Y define

$$(X * Y)(I) = \{(u, v) \in X(I) \times Y(I) \mid u \# v\}$$

where $u \# v$ iff

$$\exists J, K \subseteq I \text{ disjoint } \exists u' \in X(J), v' \in X(K)$$
$$u = u' s_J \text{ and } v = v' s_K$$

with $s_J \colon J \hookrightarrow I$ and $s_K \colon K \hookrightarrow I$.

($*$ also has a right adjoint $\multimap$)