Cubical Interpretations of Type Theory

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Intensional Type Theory

Martin-Löf type theory with intensional identity types lacks *principles of extensionality* such as:

- function extensionality

$$(\Pi(x : A)\ f\ x =_B g\ x) \to f =_{\Pi(x:A)B} g$$

- isomorphic types are equal; gives

$$A \cong B \to P(A) \to P(B)$$

Both principles make type theory more modular for both programming and proofs!

Univalent Foundations

Voevodsky formulated the *Univalence Axiom* in 2009

- refinement of the principle that isomorphic types are equal
- UA implies function extensionality
- A new, surprising connection of type theory with homotopy theory! *“Proofs of equalities are paths!”*
- **classical** model using Kan simplicial sets; does not explain UA computationally

This Thesis

I. A model of dependent type theory in cubical sets, formulated in a *constructive metatheory*

II. Cubical Type Theory inspired by a refinement of this model where the Univalence Axiom is provable

Part I.

Cubical Sets: Intuition

- introduced by Kan (1955)
- A cubical set X is specified by points, lines, squares, cubes, ...
- Intuition: n-cubes should represent maps

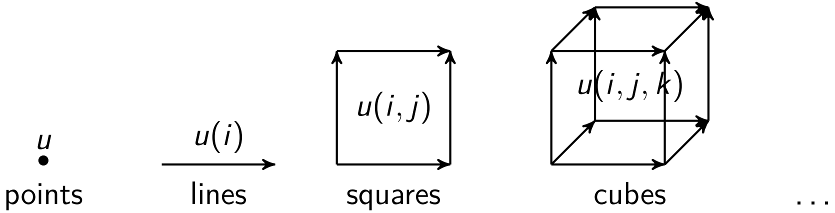
$$u\colon \mathbb{I}^n \to X, \quad \text{where } \mathbb{I} = [0,1]$$

- Here: take $\{i_1, \ldots, i_n\}$ instead of n

$$u(i_1, \ldots, i_n) \in X \quad (i_1 \in \mathbb{I}, \ldots, i_n \in \mathbb{I})$$

"values depending on names $i_1, \ldots, i_n$"

Cubical Sets: Intuition



Basic operations are substitutions on names:

- taking a face: $\begin{cases} (u(i,j))(j/0) = u(i,0) \\ (u(i,j))(j/1) = u(i,1) \end{cases}$
- considering $u(i,j)$ as degenerate cube $v(i,j,k) = u(i,j)$ constant in direction k
- renaming a name $(u(i,j))(j/k) = u(i,k)$ (k fresh)

Cubical Sets

Fix countably infinite set of *names/atoms/directions* $i, j, k, \ldots$ distinct from $0, 1$.

A **cubical set** is a presheaf $X\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ where $\mathcal{C}$ is the category of cubes given by:

- objects are finite sets of names $I = \{i_1, \ldots, i_n\}$, $n \geqslant 0$
- morphisms $f\colon J \to I$ given by maps $I \to J \cup \{0, 1\}$ injective on the preimage of J

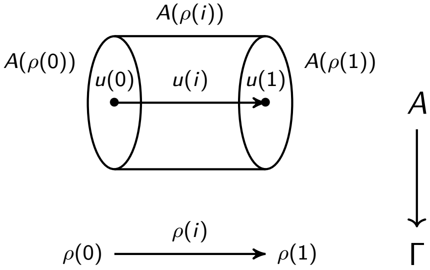
X given by:

- sets $X(I)$ (called *I-cubes*), $I = \{i_1, \ldots, i_n\}$
- maps $X(I) \to X(J), u \mapsto uf$ for $f\colon J \to I$ with $u\mathbf{1} = u$ and $(uf)g = u(fg)$

Presheaf Models of Type Theory

Cubical sets form a model of type theory (as does any presheaf category):

- contexts $\Gamma \vdash$ are cubical sets
- types $\Gamma \vdash A$: sets of “heterogeneous” cubes $A\rho$ over $\rho \in \Gamma(I)$



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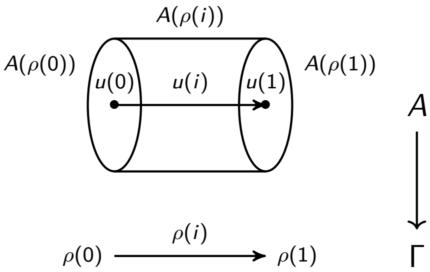


But equality not interesting. . .

. . . We Want: Proofs of Equalities are Paths!

A cubical set A has a path type:

$$x : A, y : A \vdash \mathsf{Path}_A\, x\, y$$

For $u, v \in A(I)$ the elements of $\mathsf{Path}_A\, u\, v$ are of the form

$$\langle i \rangle w$$

where

- $w \in A(I, i)$ and i fresh
- $w(i/0) = u$ and $w(i/1) = v$
- i is *bound*, so $\langle i \rangle w = \langle j \rangle w'$ iff $w(i/j) = w'$

Equality as Path?

The path type is reflexive $x : A \vdash \mathsf{refl}\, x : \mathsf{Path}_A\, x\, x$ interpreted by the constant path $\mathsf{refl}\, u = \langle i \rangle u$.

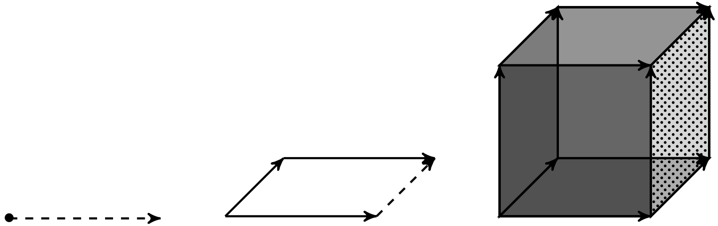
To justify the usual elimination principle for identity types we need in particular Leibniz's *indiscernibility of identicals*: given $p : \mathsf{Path}_A\, u\, v$ and a type $x : A \vdash B(x)$ we want a map:

$$\mathsf{transp}\, p : B(u) \to B(v)$$

We need to require additional structure on types!!

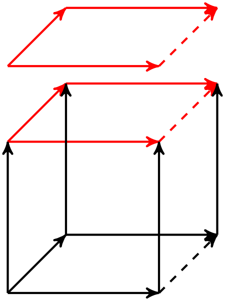
Kan's Extension Property

Kan (1955) formulated an extension property on a cubical set: "any open box can be filled"



Kan Structure

- refines Kan's extension property
- structure, not a property
- *uniform* choice of fillers of open boxes
- allow more general open boxes



Results

Theorem (Bezem/Coquand/SH 2013)

There is a model of type theory based on cubical sets with Kan structure supporting Π, Σ, *data types like* N *(naturals), and identity types (interpreted as path types).*

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Remark

- The usual definitional equalities for the identity type hold only as *propositional* equalities. This can be fixed (Swan).
- Function extensionality is valid in the model.
- The model is formulated in constructive metatheory and we can read of operational semantics. Type checker implemented in Haskell.[1]

[1] `github.com/simhu/cubical`

(jww Cohen, Coquand, Mörtberg 2013)

Universes

A universe U can be interpreted by setting $\mathrm{U}(I)$ to be all small types $I \vdash A$ (with Kan structure).
Points in U are small cubical sets with Kan structure.

Theorem (SH)

U *has a Kan structure.*

This universe also satisfies the Univalence Axiom (not treated in this thesis).

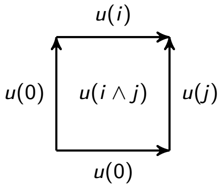
Part II.

Variation of Cubical Sets

One can extend the allowed operations in cubical sets:

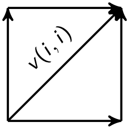
Connections

new "degeneracies":
given a line $u(i)$ we get a square



Diagonals

allows to identify names:
a square $v(i, j)$ has diagonal $v(i, i)$



Refined Model

(j.w.w. Cohen, Coquand, Mörtberg)

- Kan structure simplified: only require the “lid” not the filler of open boxes; (but notion of open box more general)
- glueing operation that justifies univalence and Kan structure for U
- some higher inductive types like spheres and propositional truncation

Cubical Type Theory

(j.w.w. Cohen, Coquand, Mörtberg)

Type theory inspired by this refined model where we directly can argue about n-dimensional cubes.

Intuition

Judgments may depend on names i ranging over a formal interval $\mathbb{I}$:

$$i : \mathbb{I} \vdash t(i) : A(i)$$

is a line connecting

$$t(0) : A(0) \quad \text{to} \quad t(1) : A(1)$$

$$t(0) \xrightarrow{\quad t(i) \quad} t(1)$$

Cubical Type Theory

(j.w.w. Cohen, Coquand, Mörtberg)

- Extends type theory with:
  - names, name abstraction, application
  - path types
  - compositions (Kan structure)
  - glueing
  - some higher inductive types
- Univalence and function extensionality are provable!
- Implementation: `cubicaltt`[2]
  - Examples: univalence, function extensionality, categories, universal algebra, S^1, torus, ...

[2] `github.com/mortberg/cubicaltt`

Meta-Mathematical Properties

Theorem (Cohen/Coquand/SH/Mörtberg)

Cubical Type Theory is consistent.

Conjecture

Cubical Type Theory has decidable type checking.

Canonicity Theorem (SH)

If I is a context of the form $i_1 : \mathbb{I}, \ldots, i_m : \mathbb{I}$ ($m \geqslant 0$) and $I \vdash u : \mathsf{N}$, then $I \vdash u = \mathsf{S}^n\, 0 : \mathsf{N}$ for a unique $n \in \mathbb{N}$.

Summary

- two models of dependent type theory based on cubical sets
- Cubical Type Theory (CTT): type theory where we can argue about n-cubes; univalence and function extensionality provable
- meta-mathematical properties of CTT: canonicity; first step towards normalization and decidability of type checking

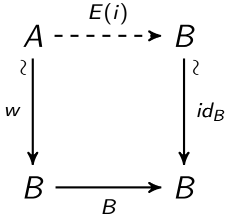
Summary

- two models of dependent type theory based on cubical sets
- Cubical Type Theory (CTT): type theory where we can argue about n-cubes; univalence and function extensionality provable
- meta-mathematical properties of CTT: canonicity; first step towards normalization and decidability of type checking

Thank you!

Example of Glueing

The glueing operation allows to glue types to parts of another type along an equivalence.



$$w : \text{Equiv}\, A\, B$$
$$\text{id}_B : \text{Equiv}\, B\, B$$

$$E(i) = \text{Glue}\, [(i = 0) \mapsto (A, w), \\ (i = 1) \mapsto (B, \text{id}_B)]\; B$$

Ingredients of the Canonicity Proof

- define typed deterministic reduction $I \vdash u \succ v : A$
- adapt computability predicate method (Tait, Martin-Löf)
  Inductive-recursive definition:

$$\begin{cases} I \Vdash A \\ I \Vdash A = B \end{cases} \qquad \begin{cases} I \Vdash u : A & \text{given } I \Vdash A \\ I \Vdash u = v : A & \text{given } I \Vdash A \end{cases}$$

- Expansion Lemma: if $I \vdash u : A$ neutral, $I \Vdash A$,
  $J \vdash uf \succ v_f : Af$ and $J \Vdash v_f : Af$ for $f : J \to I$ such that
  $J \Vdash v_f = v_1 f : Af$, then
  $$I \Vdash u : A \text{ and } I \Vdash u = v_1 : A.$$
  (Similarities to work by Angiuli/Harper/Wilson.)