Constructive Kan Fibrations

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HDACT, Ljubljana, June 2012

Univalent Foundations

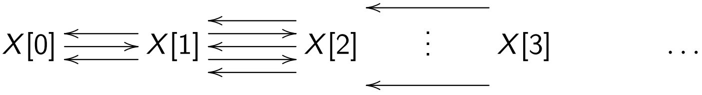
- Vladimir Voevodsky formulated the *Univalence Axiom* (UA) in Martin-Löf Type Theory as a strong form of the Axiom of Extensionality.
- UA is *classically* justified by the interpretation of types as *Kan simplicial sets*
- However, this justification uses non-constructive steps. Hence this does not provide a way to compute with univalence.
- Goal: give a constructive version of this model.

Simplicial Sets

The simplicial category $\mathbf{\Delta}$ is the category of finite non-zero ordinals, i.e., with

- objects $[n] = \{0, \ldots, n\}$ (as totally ordered set), $n \geq 0$, and
- morphisms the order preserving maps $\alpha \colon [n] \to [m]$.

A *simplicial set* $X \in \mathbf{sSet}$ is a presheaf on the category $\mathbf{\Delta}$, i.e., a functor $X \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$.



points lines triangles tetrahedra

Simplicial Sets

- A n-simplex $x \in X_n$ is *degenerate* if there is a surjective $s\colon [n] \twoheadrightarrow [m]$ with $n > m$ and $y \in X_m$ such that

$$x = y\, s.$$

- For example, the degenerate line of a point $p \in X_0$ is

$$p \xrightarrow{p\, s^0} p$$

  where $s^0\colon [1] \to [0]$.
- Degeneracy is in general *not* decidable (e.g., $\Delta_1{}^{\mathbf{N}}$).

Presheaf Models of Type Theory

It is possible to interpret type theory in any presheaf category $\mathrm{Psh}(\mathcal{C}) := \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ (of which **sSet** is a special case):

- The category of contexts $\Gamma \vdash$ and substitutions $\sigma \colon \Delta \to \Gamma$ is $\mathrm{Psh}(\mathcal{C})$; so a context Γ is given by

$$\begin{aligned} &\Gamma_X \text{ a set,} && \text{for } X \in \mathcal{C}, \\ &\Gamma_X \to \Gamma_Y \text{ a map,} && \text{for } f \colon Y \to X \text{ in } \mathcal{C}, \\ &\quad \rho \mapsto \rho f && \end{aligned}$$

such that $\rho 1 = \rho$, $(\rho f)g = \rho(fg)$.

Presheaf Models of Type Theory

- Types $\Gamma \vdash A$ are given by

$$
\begin{array}{ll}
A\rho \text{ a set,} & \text{for } \rho : \Gamma_X, X \in \mathcal{C}, \\
A\rho \to A\rho f \text{ a map,} & \text{for } f : Y \to X \text{ in } \mathcal{C}, \\
\quad a \mapsto af &
\end{array}
$$

  such that $a1 = a$, $(af)g = a(fg)$.
- Terms $\Gamma \vdash t : A$ are given by $t\rho : A\rho$ such that $(t\rho)f = t(\rho f)$.

Presheaf Models of Type Theory

- For a map $\sigma : \Delta \to \Gamma$ and $\Gamma \vdash A$ we define $\Delta \vdash A\sigma$ by

$$(A\sigma)\rho =_{\text{def}} A(\sigma\rho),$$

for $\rho : \Delta_X$.

Presheaf Models of Type Theory

- For $\Gamma \vdash A$ the context extension $\Gamma.A \vdash$ is defined as

$$(\rho, a) : (\Gamma.A)_X \text{ iff } \rho : \Gamma_X \text{ and } a : A\rho,$$
$$(\rho, a)f =_{\text{def}} (\rho f, af).$$

We can define the projections $\mathsf{p} \colon \Gamma.A \to \Gamma$ and $\Gamma.A \vdash \mathsf{q} : A\,\mathsf{p}$ by

$$\mathsf{p}(\rho, a) =_{\text{def}} \rho,$$
$$\mathsf{q}(\rho, a) =_{\text{def}} a.$$

Presheaf Models of Type Theory

It is also possible to interpret Π and Σ:

$$\frac{\Gamma \vdash A \quad \Gamma.A \vdash B}{\Gamma \vdash \Pi A B} \qquad \frac{\Gamma \vdash A \quad \Gamma.A \vdash B}{\Gamma \vdash \Sigma A B}$$

Types as Simplicial Sets

In the simplicial set model the interpretation of the equality type is the path space, i.e., an equality proof of a_0 and a_1 is a *path* connecting a_0 with a_1.

We fix the standard 1-simplex Δ_1 $(= \mathrm{Hom}_{\boldsymbol{\Delta}}(\cdot, [1]))$ serving as an interval

$$\mathbb{I} := \Delta_1.$$

This has two (global) elements $\vdash 0, 1 : \mathbb{I}$.

Path Space

For a simplicial set A the exponent $A^{\mathbb{I}}$ has a concrete description:

$$\begin{aligned} A^{\mathbb{I}}[0] &= A[1], \text{ i.e., lines in } A, \\ A^{\mathbb{I}}[1] &= \text{squares in } A, \\ A^{\mathbb{I}}[2] &= \text{prisms in } A, \\ &\dots \end{aligned}$$

Path Spaces

For $\Gamma \vdash A$ and $\Gamma \vdash a, b : A$ the path space

$$\Gamma \vdash \mathsf{Path}_A\ a\ b$$

is defined as

$$(\mathsf{Path}_A\ a\ b)\rho := \{\alpha \in A^{\mathbb{I}}\,\rho \mid \alpha(0) = a\rho \text{ and } \alpha(1) = b\rho\}$$

for $\rho \in \Gamma[n]$.

Path Spaces

We want Path_A to satisfy of the axioms of the identity type.

Reflexivity: for $a : A$ the constant map $\mathsf{ref}_a : \mathbb{I} \to A$, $\mathsf{ref}_a = \lambda i.a$ gives an element of $\mathsf{Path}_A\ a\ a$.

Path Spaces: Extensionality

This path space verifies the axiom of extensionality

$$\frac{\Gamma.A \vdash p : \mathsf{Path}_B\ u\ v}{\Gamma \vdash \mathsf{ext}\ p : \mathsf{Path}_{\Pi A B}\ (\lambda u)\ (\lambda v)}$$

(ext is basically the dependent version of $A \to (\mathbb{I} \to B)$ implies $\mathbb{I} \to (A \to B)$.)

Path Space

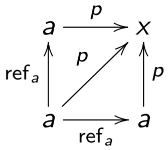
In **sSet** we also have that the “singleton” type of $\Gamma \vdash a : A$ is contractible, i.e.,

$$\mathrm{iscontr}\Big(\sum_{x:A} \mathrm{Path}_A\ a\ x\Big)$$

i.e.,

$$\prod_{(x,p):S} \mathrm{Path}_S\ (a, \mathrm{ref}_a)\ (x, p)$$

with $S := \Sigma x : A\,.\,\mathrm{Path}_A\ a\ x$.
In **sSet** we have the square:



Univalence?

Given two simplicial sets A and B, and a map $\sigma \colon A \to B$ we can associate a dependent type $\mathbb{I} \vdash E$ with $E0 = A$ and $E1 = B$.

This will serve as path connecting A and B.

For $\rho \in \mathbb{I}[n]$, i.e., monotone $\rho \colon [n] \to [1]$ we have to define the set $E\rho$.

There are $n + 2$ such maps $0 < \rho_1 < \cdots < \rho_n < 1$.

- $E0 = A[n]$ and $E1 = B[n]$;
- $E\rho_k = \{(a, b) \mid a : A[n-k], b : B[n], \text{ and } bi = \sigma(a)\}$ with $i \colon [n-k] \to [n]$ the canonical injection.

Path Space

What is missing in order to satisfy the axioms of the identity type?

Substitutivity, or Leibniz's *indiscernibility of identicals*:

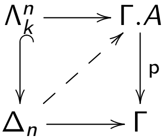
$$\Gamma \vdash \mathsf{transp} : \prod_{a,b:A} \big(\mathsf{Path}_A\ a\ b \to B(a) \to B(b)\big)$$

for $\Gamma \vdash A$ and $\Gamma.A \vdash B$.

There is no reason this should hold in general! We have to require it!

Classical Justification of Transport

To justify the elimination rule for equality one has to restrict types $\Gamma \vdash A$ such that the projection $\mathsf{p}\colon \Gamma.A \to \Gamma$ is a Kan fibration, i.e.,



(If $\Gamma = 1$, then A is called Kan complex.)

Constructively this will be expressed by a filling operator.

Classical Justification of Transport

Classically, this lifting property can then be extended to the wider class of so called *anodyne maps* than just the horn inclusions $\Lambda^n_k \hookrightarrow \Delta_n$.

For example, for any simplicial set X the canonical maps

$$\Lambda^n_k \times X \hookrightarrow \Delta_n \times X \text{ and}$$
$$X \to X^{\mathbb{I}}$$

are anodyne.

What can we say constructively?

Definition

A simplicial set X has *decidable degeneracy* if given $x \in X[n]$ we can decide whether x is degenerate or not, and if it is find $y \in X[n-1]$ and $\eta \colon [n] \twoheadrightarrow [n-1]$ such that

$$x = y\eta$$

In this case we also say X is *decidable*.

Theorem

If X has decidable degeneracy, then $\Lambda^n_k \times X \hookrightarrow \Delta_n \times X$ is anodyne constructively.

Closure of Kan complexes under Π-Types

Closure under exponents: B Kan complex $\Rightarrow B^A$ Kan complex

- direct, combinatorial argument (see the book by May)
- using that $\Lambda^n_k \times A \hookrightarrow \Delta_n \times A$ is anodyne (see the book by Gabriel and Zisman)

What can we do in a constructive meta-theory?

Two possible remedies:

1. modifying the notion of a “Kan fibration” by analyzing what is needed to get the transport property;
2. use simplicial sets where we can decide degeneracy.

First Approach: Results

The first approach provides a model of type theory with Π, Σ, and Path_A justifying extensionality and containing counter-examples to UIP.

Transport Maps

$\Gamma \vdash A$ has *transport maps* if we have two sections φ^+ and φ^-

$$\vdash \varphi^+ : \prod_{\alpha : \Gamma^{\mathbb{I}}} \big(A\alpha(0) \to A\alpha(1)\big)$$
$$\vdash \varphi^- : \prod_{\alpha : \Gamma^{\mathbb{I}}} \big(A\alpha(1) \to A\alpha(0)\big)$$

such that $\varphi^{\pm}\,\alpha\,a = a$ for α constant (where $\alpha : (\Gamma^{\mathbb{I}})[n]$, $i : \mathbb{I}[n], a : A\alpha(i)$).

Properties

Lemma

Assume that $\Gamma \vdash A$ *has transport maps. Then there is a term* transp *justifying the rule*

$$\frac{\Gamma.A \vdash B \quad \Gamma \vdash a, b : A \quad \Gamma \vdash p : \mathrm{Path}_A\ a\ b \quad \Gamma \vdash c : B[a]}{\Gamma \vdash \mathrm{transp}\, p\, c : B[b]}.$$

Moreover, we have $\mathrm{transp}\,\mathrm{ref}_a\, c = c$.

Lemma

If $\Gamma \vdash A$ *and* $\Gamma.A \vdash B$ *have transport maps, so do* $\Gamma \vdash \Pi A B$ *and* $\Gamma \vdash \Sigma A B$.

What about closure under Path Space?

If we try to prove that $\Gamma \vdash \mathrm{Path}_A\ a\ b$ has transport maps assuming $\Gamma \vdash A$ does, we are left to fill shapes in A like

$$\begin{array}{ccc} a\alpha(1) & & b\alpha(1) \\ \uparrow & & \uparrow \\ a\alpha(0) & \longrightarrow & b\alpha(0) \end{array}$$

to a square.

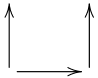
We need more general filling conditions for A!

n-Transport Properties

For $n \geq 1$ and $k = 0, 1$ we define the simplicial set D^n_k as

$$D^n_k[m] := \{(i_1, \ldots, i_n) \in \mathbb{I}^n[m] \mid i_n = k \text{ or } \exists l < n \; i_l \in \{0_{[m]}, 1_{[m]}\}\}.$$

For example, D^2_0 corresponds to



n-Transport Properties

$\Gamma \vdash A$ has the *n-transport property* if we have a sections for $k = 0, 1$

$$\vdash \psi_k : \prod_{\alpha : \Gamma^{\mathbb{I}^n}} \Big(\prod_{i : D^n_k} A\alpha(i) \to \prod_{j : \mathbb{I}^n} A\alpha(j)\Big)$$

such that $\psi_k \, \alpha \, i \, a \, j = a \, j$ for $j : D^n_k$, and $\psi_k \, \alpha \, i \, a$ is constant whenever $\alpha \in \Gamma^{\mathbb{I}^n}$ and $a(i) : A\alpha(i)$ $(i : D^n_k)$ are independent of the last coordinate.

Properties

Lemma

1. *If* $\Gamma \vdash A$ *has the* $(n+1)$*-transport property, then* $\Gamma \vdash \mathsf{Path}_A\ a\ b$ *has the* n*-transport property for all* $\Gamma \vdash a, b : A$.
2. *The* n*-transport property is closed under* Σ.
3. *If* $\Gamma \vdash A$ *has the* 1*-transport property and* $\Gamma.A \vdash B$ *has the* n*-transport property, then* $\Gamma \vdash \Pi A B$ *has the* n*-transport property.*

Transport Fibrations

$\Gamma \vdash A$ is a *transport fibration* if $\Gamma \vdash A$ has the n-transport property for all $n \geq 1$.

Theorem

The transport fibrations form a model of type theory with Π, Σ *and* Path_A, *justifying functional extensionality.*

- In **sSet** the nerve $\mathcal{N}\,G$ of a group G gives us non-trivial examples of a type satisfying this.
- Can be generalized for other presheaf categories and choices of 0, 1 and $\mathbb{I}$.
- Problem: it is rather hard to check this condition.

Second Approach (Work in Progress)

Instead of modifying the notion of Kan fibration, it seems possible to work with simplicial sets where we can decide *degeneracy*, and use the usual notion of Kan fibrations read constructively.

Idea: To $\Gamma \in \mathbf{sSet}$ associate $\Gamma^+ \in \mathbf{sSet}$ where we can decide degeneracy.

One can define

$$\Gamma^+[n] = \coprod_{[n] \twoheadrightarrow [m]} \Gamma[m].$$

Second Approach (Work in Progress)

We always have a morphism $\Gamma^+ \to \Gamma$, $(\eta, a) \mapsto a\eta$. In general, it doesn't seem possible define constructively a map $\Gamma \to \Gamma^+$. This is possible if Γ is decidable.

More generally, any map $\Delta \to \Gamma$ for decidable Δ, induces a map $\Delta \to \Gamma^+$.

The assignment $\Gamma \mapsto \Gamma^+$ is functorial.

If Γ is Kan, so is Γ^+.

This generalizes to types.

Thank you!