Gluing for type theory

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Abstract

The relationship between categorical gluing and proofs using the logical relation technique is folklore. In this paper we work out this relationship for Martin-Löf type theory and show that parametricity and canonicity arise as special cases of gluing. The input of gluing is two models of type theory and a pseudomorphism between them and the output is a displayed model over the first model. A pseudomorphism preserves the categorical structure strictly, the empty context and context extension up to isomorphism, and there are no conditions on preservation of type formers. We look at three examples of pseudomorphisms: the identity on the syntax, the interpretation into the set model and the global section functor. Gluing along these result in syntactic parametricity, semantic parametricity and canonicity, respectively.

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1 Introduction

Categorical gluing [9, Section 4.10] is a method to form a new category from two categories and a functor between them. Given a functor F from category $\mathcal{S}$ to $\mathcal{M}$, an object in the glued category is a triple $\Gamma : |\mathcal{S}|$, $\Delta : |\mathcal{M}|$ and a morphism $\mathcal{M}(\Delta, F\,\Gamma)$. Models of logics and type theories can be given as categories with extra structure and gluing can be extended to these models. Gluing was used to prove properties of closed proofs in intuitionistic higher order logic [17] and normalisation for simple type theory [12, 20] and System F [2]. In programming language semantics, similar results are proved more syntactically using the technique of logical relations, see [13] for an introduction and [11, 1] for example proofs using this technique. It is folklore that logical relations correspond to gluing. Logical relations scale to real-world systems [21, 15] while gluing is a more abstract construction which can be applied to systems with well-understood categorical semantics.

In this paper we develop the correspondence between proof-relevant logical predicates and gluing for Martin-Löf type theory. Logical relations were defined for type theory to prove free theorems in syntactic [5] and semantic (Reynolds-style) [4] ways. Proof-relevant logical

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predicates were employed to prove normalisation and canonicity for type theory [3, 8, 16]. We unify these approaches by defining gluing in an abstract way, for any pseudomorphism between two models of type theory. An important characteristic of our approach is using an algebraic syntax of type theory. By this we mean the well-typed syntax of type theory given as a quotient inductive-inductive type (QIIT, [14]). A model of this syntax is just an algebra of the QIIT which turns out to be the same as a category with families (CwF, [10]) with extra structure. A pseudomorphism of models is a map from sorts in one model to sorts in the other model which preserves the categorical structure strictly and the empty context and context extension up to isomorphism. We show that gluing can be performed along any pseudomoprhism and gluing preserves Π, Σ, Bool and an infinite hierarchy of Russell-universes.

Our motivational guideline for this paper is the following.

1. Gluing over identity is syntactic parametricity.
2. Gluing over the interpretation into the set model is semantic parametricity.
3. Gluing over the global section functor is canonicity.
4. Gluing over Yoneda is normalisation.
5. Gluing over Yoneda composed with the set interpretation is definability/completeness.

In this paper we only generalise steps 1–3. The Yoneda embedding (from the syntax to the presheaf model over a wide subcategory of contexts and substitutions) is also a pseudomorphism, so our paper applies to steps 4–5 as well. However, Yoneda has extra structure that we do not employ in this paper. This extra structure is needed to obtain full normalisation or completeness.

Structure of the paper

After summarizing related work and the metatheory, we define our object type theory in Section 2 and as an example we define its set model (Section 3). Then we define the notion of pseudomorphism (Section 4) and gluing for any pseudomorphism (Section 5). Afterwards, in Section 6 we define a non-trivial pseudomorphism: the global section functor which goes from the syntax to the set model and maps types to terms of the type in the empty context. We put together the pieces in Section 7 by obtaining parametricity and canonicity for our object theory using gluing. We conclude and summarize further work in Section 8.

Contribution

The contribution of this paper is showing that gluing can be defined for any pseudomorphism for Martin-Löf type theory. To our knowledge, this is the first general construction from which both parametricity and canonicity arise.

Related work

Sterling and Spitters [20] developed gluing for simple type theory and show how it relates to syntactic proofs of normalisation by logical relations and semantic proofs based on normalisation by evaluation. Altenkirch, Hofmann and Streicher developed gluing for System F and prove normalisation in their unpublished note [2]. Rabe and Sojakova [18] defined a syntactic framework for logical relations which applies to theories formulated in the Edinburgh Logical Framework (LF). Shulman [19] developed gluing for type theory in the context of type-theoretic fibration categories and proves homotopy canonicity for a 1-truncated version of homotopy type theory. Compared to Shulman, we work with a notion of model closer to the syntax of type theory: categories with families. In previous work [3] we proved normalisation for type theory with Π, a base type and a base family. The logical predicate used in that proof is an instance of the abstract gluing technique presented in this paper. Coquand [8] proves canonicity and normalisation for a richer type theory with Bool and a hierarchy of universes. His canonicity proof is an unfolding of the canonicity proof given in this paper.

Gluing along a strict morphism is straightforward and probably there are many examples of this construction in the literature. For example, Clairambault and Dybjer [7, right to left direction of Prop. 3] define gluing for CwFs with extra structure (however not by this name). Using the results of [14], gluing can be defined for any quotient inductive-inductive type (QIIT). Given an algebra morphism F from S to M, working in the internal language of the CwF model of the QIIT-signature defined in [14, Section 7], the glued displayed model is given by $\Sigma\,(\mathsf{K}\,S)\,(\mathsf{Eq}\,(\mathsf{mk}\,(F\circ\mathsf{unk}\,\mathsf{vz}))\,(\mathsf{mk}\,\mathsf{wk}))$. These general constructions however fail for the global section functor which is not strict, but still allows gluing.

Metatheory and notation

Our metatheory is extensional type theory. We have a cumulative hierarchy of universes Set_0, Set_1, ... with Set_ω on top. Sometimes we omit the universe indices. Function space is denoted by $\to$ with constructor λ and application written as juxtaposition. We use implicit arguments extensively, e.g. we would write the type of function composition as $(B\to C)\to(A\to B)\to(A\to C)$ instead of $(A:\mathsf{Set})\to(B:\mathsf{Set})\to(C:\mathsf{Set})\to(B\to C)\to(A\to B)\to(A\to C)$. When a metavariable is not quantified explicitly (such as A, B, C), we assume implicit quantification and implicit application as well. Sometimes we omit explicit arguments for readability, in this case we write undescore _ instead of the argument. Pairs are denoted by $\times$ with constructor $-,-$ and destructors $._1$ and $._2$. Both $\to$ and $\times$ come with η laws. The one-element type is denoted $\mathbb{1}$ with constructor $*$, the two-element type is denoted $\mathbb{2}$, its constructors being $*$ and $**$ and its eliminator case. Equality is denoted $=$ and we use equational reasoning to write equality proofs.

2 Type theory

By *type theory* we mean the (generalised) algebraic structure in Figure 1 with four sorts, 26 operators and 34 equations. The four sorts are those of contexts, types, substitutions and terms. Types are indexed by a universe level which is a metatheoretic natural number. Furthermore, types are indexed by contexts and terms by a context and a type in that context so that we can only mention well-typed terms. Substitutions are indexed by their domain and codomain, both contexts.

We explain the operators and laws for the substitution calculus (first column, operators id to $,\circ$) as follows: Con and Sub form a category (id to idr); there is a contravariant, functorial action of substitutions on types and terms ($-[-]$ to $[\circ]$), there is an empty context $\cdot$ with a unique ($\cdot\eta$) empty substitution ϵ into it ($\cdot$ is the terminal object of the category); extended contexts can be formed using $-\triangleright-$ and there is a natural isomorphism between substitutions into $\Delta\triangleright A$ and a pair of a substitution σ into Δ and a term of type $A[\sigma]$. The substitution calculus is the same as the structure of a predicative category with families (CwF, [10]). In the CwF language, context extension is called comprehension. We denote n-times iteration of the weakening substitution p by p^n where $\mathsf{p}^0=\mathsf{id}$, and we denote De Bruijn indices by natural numbers, i.e. $0:=\mathsf{q}$, $1:=\mathsf{q}[\mathsf{p}]$, ..., $n:=\mathsf{q}[\mathsf{p}^n]$. We define lifting of a substitution

Con	$: \mathsf{Set}$
Ty	$: \mathbb{N} \to \mathsf{Con} \to \mathsf{Set}$
Sub	$: \mathsf{Con} \to \mathsf{Con} \to \mathsf{Set}$
Tm	$: (\Gamma : \mathsf{Con}) \to \mathsf{Ty}\, i\, \Gamma \to \mathsf{Set}$
id	$: \mathsf{Sub}\, \Gamma\, \Gamma$
$- \circ -$	$: \mathsf{Sub}\, \Theta\, \Delta \to \mathsf{Sub}\, \Gamma\, \Theta \to \mathsf{Sub}\, \Gamma\, \Delta$
ass	$: (\sigma \circ \delta) \circ \nu = \sigma \circ (\delta \circ \nu)$
idl	$: \mathsf{id} \circ \sigma = \sigma$
idr	$: \sigma \circ \mathsf{id} = \sigma$
$-[-]$	$: \mathsf{Ty}\, i\, \Delta \to \mathsf{Sub}\, \Gamma\, \Delta \to \mathsf{Ty}\, i\, \Gamma$
$-[-]$	$: \mathsf{Tm}\, \Delta\, A \to (\sigma : \mathsf{Sub}\, \Gamma\, \Delta) \to \mathsf{Tm}\, \Gamma\, (A[\sigma])$
$[\mathsf{id}]$	$: A[\mathsf{id}] = A$
$[\circ]$	$: A[\sigma \circ \delta] = A[\sigma][\delta]$
$[\mathsf{id}]$	$: t[\mathsf{id}] = t$
$[\circ]$	$: t[\sigma \circ \delta] = t[\sigma][\delta]$
$\cdot$	$: \mathsf{Con}$
ϵ	$: \mathsf{Sub}\, \Gamma\, \cdot$
$\cdot\eta$	$: (\sigma : \mathsf{Sub}\, \Gamma\, \cdot) = \epsilon$
$- \rhd -$	$: (\Gamma : \mathsf{Con}) \to \mathsf{Ty}\, i\, \Gamma \to \mathsf{Con}$
$-,-$	$: (\sigma : \mathsf{Sub}\, \Gamma\, \Delta) \to \mathsf{Tm}\, \Gamma\, (A[\sigma]) \to \mathsf{Sub}\, \Gamma\, (\Delta \rhd A)$
p	$: \mathsf{Sub}\, (\Gamma \rhd A)\, \Gamma$
q	$: \mathsf{Tm}\, (\Gamma \rhd A)\, (A[\mathsf{p}])$
$\rhd\beta_1$	$: \mathsf{p} \circ (\sigma, t) = \sigma$
$\rhd\beta_2$	$: \mathsf{q}[\sigma, t] = t$
$\rhd\eta$	$: (\mathsf{p}, \mathsf{q}) = \mathsf{id}$
$,\circ$	$: (\sigma, t) \circ \nu = (\sigma \circ \nu, t[\nu])$
Π	$: (A : \mathsf{Ty}\, i\, \Gamma) \to \mathsf{Ty}\, j\, (\Gamma \rhd A) \to \mathsf{Ty}\, (i \sqcup j)\, \Gamma$
lam	$: \mathsf{Tm}\, (\Gamma \rhd A)\, B \to \mathsf{Tm}\, \Gamma\, (\Pi\, A\, B)$
app	$: \mathsf{Tm}\, \Gamma\, (\Pi\, A\, B) \to \mathsf{Tm}\, (\Gamma \rhd A)\, B$
$\Pi\beta$	$: \mathsf{app}\, (\mathsf{lam}\, t) = t$
$\Pi\eta$	$: \mathsf{lam}\, (\mathsf{app}\, t) = t$
$\Pi[]$	$: (\Pi\, A\, B)[\sigma] = \Pi\, (A[\sigma])\, (B[\sigma^{\uparrow}])$
$\mathsf{lam}[]$	$: (\mathsf{lam}\, t)[\sigma] = \mathsf{lam}\, (t[\sigma^{\uparrow}])$

Σ	$: (A : \mathsf{Ty}\, i\, \Gamma) \to \mathsf{Ty}\, j\, (\Gamma \rhd A) \to \mathsf{Ty}\, (i \sqcup j)\, \Gamma$
$-,-$	$: (u : \mathsf{Tm}\, \Gamma\, A) \to \mathsf{Tm}\, \Gamma\, (B[\mathsf{id}, u]) \to \mathsf{Tm}\, \Gamma\, (\Sigma\, A\, B)$
projl	$: \mathsf{Tm}\, \Gamma\, (\Sigma\, A\, B) \to \mathsf{Tm}\, \Gamma\, A$
projr	$: (t : \mathsf{Tm}\, \Gamma\, (\Sigma\, A\, B)) \to \mathsf{Tm}\, \Gamma\, (B[\mathsf{id}, \mathsf{projl}\, t])$
$\Sigma\beta_1$	$: \mathsf{projl}\, (u, v) = u$
$\Sigma\beta_2$	$: \mathsf{projr}\, (u, v) = v$
$\Sigma\eta$	$: (\mathsf{projl}\, t, \mathsf{projr}\, t) = t$
$\Sigma[]$	$: (\Sigma\, A\, B)[\sigma] = \Sigma\, (A[\sigma])\, (B[\sigma^{\uparrow}])$
$,[]$	$: (u, v)[\sigma] = (u[\sigma], v[\sigma])$
$\top$	$: \mathsf{Ty}\, 0\, \Gamma$
tt	$: \mathsf{Tm}\, \Gamma\, \top$
$\top\eta$	$: (t : \mathsf{Tm}\, \Gamma\, \top) = \mathsf{tt}$
$\top[]$	$: \top[\sigma] = \top$
$\mathsf{tt}[]$	$: \mathsf{tt}[\sigma] = \mathsf{tt}$
U	$: (i : \mathbb{N}) \to \mathsf{Ty}\, (i+1)\, \Gamma$
El	$: \mathsf{Tm}\, \Gamma\, (\mathsf{U}\, i) \to \mathsf{Ty}\, i\, \Gamma$
c	$: \mathsf{Ty}\, i\, \Gamma \to \mathsf{Tm}\, \Gamma\, (\mathsf{U}\, i)$
$\mathsf{U}\beta$	$: \mathsf{El}\, (\mathsf{c}\, A) = A$
$\mathsf{U}\eta$	$: \mathsf{c}\, (\mathsf{El}\, a) = a$
$\mathsf{U}[]$	$: (\mathsf{U}\, i)[\sigma] = (\mathsf{U}\, i)$
$\mathsf{El}[]$	$: (\mathsf{El}\, a)[\sigma] = \mathsf{El}\, (a[\sigma])$
Bool	$: \mathsf{Ty}\, 0\, \Gamma$
true	$: \mathsf{Tm}\, \Gamma\, \mathsf{Bool}$
false	$: \mathsf{Tm}\, \Gamma\, \mathsf{Bool}$
if	$: (C : \mathsf{Ty}\, i\, (\Gamma \rhd \mathsf{Bool})) \to \mathsf{Tm}\, \Gamma\, (P[\mathsf{id}, \mathsf{true}]) \to \mathsf{Tm}\, \Gamma\, (P[\mathsf{id}, \mathsf{false}]) \to (t : \mathsf{Tm}\, \Gamma\, \mathsf{Bool}) \to \mathsf{Tm}\, \Gamma\, (C[\mathsf{id}, t])$
$\mathsf{Bool}\beta_1$	$: \mathsf{if}\, C\, u\, v\, \mathsf{true} = u$
$\mathsf{Bool}\beta_2$	$: \mathsf{if}\, C\, u\, v\, \mathsf{false} = v$
$\mathsf{Bool}[]$	$: \mathsf{Bool}[\sigma] = \mathsf{Bool}$
$\mathsf{true}[]$	$: \mathsf{true}[\sigma] = \mathsf{true}$
$\mathsf{false}[]$	$: \mathsf{false}[\sigma] = \mathsf{false}$
$\mathsf{if}[]$	$: (\mathsf{if}\, C\, u\, v\, t)[\sigma] = \mathsf{if}\, (C[\sigma^{\uparrow}])\, (u[\sigma])\, (v[\sigma])\, (t[\sigma])$

Figure 1 Type theory as a generalised algebraic structure. $\sigma^{\uparrow}$ abbreviates $(\sigma \circ \mathsf{p}, \mathsf{q})$.

$\sigma : \mathsf{Sub}\,\Gamma\,\Delta$ by $\sigma^\uparrow : \mathsf{Sub}\,(\Gamma \triangleright A[\sigma])\,(\Delta \triangleright A) := (\sigma \circ \mathsf{p}, \mathsf{q})$. We observe that it has the property ${}^\uparrow[] : (\sigma^\uparrow)[\delta, t] = (\sigma \circ \delta, t)$.

Π types are given by a natural isomorphism between lam and app. We define the usual application as $t \,\$\, u := (\mathsf{app}\,t)[\mathsf{id}, u]$. $A \Rightarrow B$ abbreviates $\Pi\,A\,(B[\mathsf{p}])$. Σ types are given by the constructor $-,-$ and projections projl and projr and they support an η law. There is a unit type $\top$ with one constructor tt and an η law and there is a hierarchy of Russell-universes, given by natural isomorphisms between $\mathsf{Ty}\,i\,\Gamma$ and $\mathsf{Tm}\,\Gamma\,(\mathsf{U}\,i)$ for every i.[1] As Π, Σ and U are given by natural isomorphisms, we only stated one substitution law, the others can be derived. We illustrate how to do this for app and state the other laws.

$$\begin{aligned}
\mathsf{app}[] &: (\mathsf{app}\,t)[\sigma^\uparrow] \overset{\Pi\beta}{=} \mathsf{app}\,(\mathsf{lam}\,((\mathsf{app}\,t)[\sigma^\uparrow])) \overset{\mathsf{lam}[]}{=} \mathsf{app}\,((\mathsf{lam}\,(\mathsf{app}\,t))[\sigma]) \overset{\Pi\eta}{=} \mathsf{app}\,(t[\sigma]) \\
\$[] &: (t \,\$\, u)[\sigma] = t[\sigma] \,\$\, u[\sigma] \\
\mathsf{projl}[] &: (\mathsf{projl}\,t)[\sigma] = \mathsf{projl}\,(t[\sigma]) \\
\mathsf{projr}[] &: (\mathsf{projr}\,t)[\sigma] = \mathsf{projr}\,(t[\sigma]) \\
\mathsf{c}[] &: (c\,A)[\sigma] = c\,(A[\sigma])
\end{aligned}$$

Finally, we have booleans with a dependent eliminator if into any universe. Sometimes for readability we omit the first argument (C) of if and write _ instead.

As an example we write the polymorphic identity function as $\mathsf{lam}\,(\mathsf{lam}\,\mathsf{q})$. Note that lam and q have several implicit arguments that we did not write down. However when we write a term, these implicit arguments should be clear from the context. In this example by saying that it has type $\mathsf{Tm}\,\cdot\,(\Pi\,(\mathsf{U}\,0)\,(\mathsf{El}\,\mathsf{q} \Rightarrow \mathsf{El}\,\mathsf{q}))$ fixes all its implicit arguments. We don't have raw terms with a type assignment or type inference system, we only work with fully annotated well-typed terms where lots of information is implicit (as usual in mathematics).

We call algebras of this algebraic structure a *model* of type theory. When referring to different models, we put the model in lower index, i.e. Con_M refers to contexts in model M, $\mathsf{id}_M : \mathsf{Sub}_M\,\Gamma_M\,\Gamma_M$ refers to the identitity substitution in this model. For metavariables, we usually use the same lower index as for the two occurrences of Γ_M in the type of id_M.

We assume the existence of the quotient inductive-inductive type (QIIT, [14]) specified by this algebraic structure. This entails the following:

- A (strict) *morphism* H between models M and N consists of four functions between the sorts which preserve all the 26 operators (up to equality). We use lower indices to mark which component we mean, e.g. some of the components are the following.

$$\begin{aligned}
H_{\mathsf{Con}} &: \mathsf{Con}_M \to \mathsf{Con}_N \\
H_{\mathsf{Ty}} &: \mathsf{Ty}_M\,i\,\Gamma \to \mathsf{Ty}_N\,i\,(H\,\Gamma) \\
H_{\mathsf{Sub}} &: \mathsf{Sub}_M\,\Gamma\,\Delta \to \mathsf{Sub}_N\,(H\,\Gamma)\,(H\,\Delta) \\
H_{\mathsf{Tm}} &: \mathsf{Tm}_M\,\Gamma\,A \to \mathsf{Tm}_N\,(H\,\Gamma)\,(H\,A) \\
H_{[]} &: H_{\mathsf{Ty}}\,(A[\sigma]_M) = (H_{\mathsf{Ty}}\,A)[H_{\mathsf{Sub}}\,\sigma]_N \\
H_{\triangleright} &: H_{\mathsf{Con}}\,(\Gamma \triangleright_M A) = H_{\mathsf{Con}}\,\Gamma \triangleright_N H_{\mathsf{Ty}}\,A \\
H_{\Pi} &: H_{\mathsf{Ty}}\,(\Pi_M\,A\,B) = \Pi_N\,(H_{\mathsf{Ty}}\,A)\,(H_{\mathsf{Ty}}\,B) \\
H_{\mathsf{lam}} &: H_{\mathsf{Tm}}\,(\mathsf{lam}_M\,t) = \mathsf{lam}_N\,(H_{\mathsf{Tm}}\,t) \\
H_{\mathsf{app}} &: H_{\mathsf{Tm}}\,(\mathsf{app}_M\,t) = \mathsf{app}_N\,(H_{\mathsf{Tm}}\,t)
\end{aligned}$$

[1] We learned this representation of Russell-universes from Thierry Coquand.

Sometimes we omit lower indices for readability, e.g. above we wrote $H\,\Gamma$ instead of $H_{\mathsf{Con}}\,\Gamma$ and we also did not decorate metavariables with lower indices, all Γs were meant in Con_M, σ in Sub_M etc. We will follow this convention later.

- A *displayed model* Q over a model M is given by four families over the sorts in M, 26 operations and 34 equalities which are all over those of M, e.g.

$$\begin{aligned}
&\mathsf{Con}_Q &&: \mathsf{Con}_M \to \mathsf{Set}\\
&\mathsf{Ty}_Q &&: (i : \mathbb{N}) \to \mathsf{Con}_Q\,\Gamma \to \mathsf{Ty}_M\,i\,\Gamma \to \mathsf{Set}\\
&\mathsf{Sub}_Q &&: \mathsf{Con}_Q\,\Gamma \to \mathsf{Con}_Q\,\Delta \to \mathsf{Sub}_M\,\Gamma\,\Delta \to \mathsf{Set}\\
&\mathsf{Tm}_Q &&: (\Gamma_Q : \mathsf{Con}_Q\,\Gamma) \to \mathsf{Ty}_Q\,j\,\Gamma_Q\,A \to \mathsf{Tm}_M\,\Gamma\,A \to \mathsf{Set}\\
&-[-]_Q &&: \mathsf{Ty}_Q\,i\,\Delta_Q\,A \to \mathsf{Sub}_Q\,\Gamma_Q\,\Delta_Q\,\sigma \to \mathsf{Ty}_Q\,i\,\Gamma_Q\,(A[\sigma]_M)\\
&-\triangleright_Q - &&: (\Gamma_Q : \mathsf{Con}_Q\,\Gamma) \to \mathsf{Ty}_Q\,i\,\Gamma_Q\,A \to \mathsf{Con}_Q\,(\Gamma \triangleright_M A)\\
&\Pi_Q &&: (A_Q : \mathsf{Ty}_Q\,i\,\Gamma_Q\,A) \to \mathsf{Ty}_Q\,j\,(\Gamma_Q \triangleright_Q A_Q)\,B \to \mathsf{Ty}_Q\,(i \sqcup j)\,\Gamma_Q\,(\Pi_M\,A\,B)\\
&\mathsf{lam}_Q &&: \mathsf{Tm}_Q\,(\Gamma_Q \triangleright_Q A_Q)\,B_Q\,t \to \mathsf{Tm}_Q\,\Gamma_Q\,(\Pi_Q\,A_Q\,B_Q)\,(\mathsf{lam}_M\,t)\\
&\mathsf{app}_Q &&: \mathsf{Tm}_Q\,\Gamma_Q\,(\Pi_Q\,A_Q\,B_Q)\,t \to \mathsf{Tm}_Q\,(\Gamma_Q \triangleright_Q A_Q)\,B_Q\,(\mathsf{app}_M\,t)\\
&\Pi\beta_Q &&: \mathsf{app}_Q\,(\mathsf{lam}_Q\,t_Q) = t_Q
\end{aligned}$$

- A *section* I of a displayed model Q over M is like a dependent morphism and contains, among others, the following components.

$$\begin{aligned}
&I_{\mathsf{Con}} &&: (\Gamma : \mathsf{Con}_M) \to \mathsf{Con}_Q\,\Gamma\\
&I_{\mathsf{Ty}} &&: (A : \mathsf{Ty}_M\,j\,\Gamma) \to \mathsf{Ty}_Q\,j\,(I\,\Gamma)\,A\\
&I_{\mathsf{Sub}} &&: (\sigma : \mathsf{Sub}_M\,\Gamma\,\Delta) \to \mathsf{Sub}_Q\,(I\,\Gamma)\,(I\,\Delta)\,\sigma\\
&I_{A[\sigma]} &&: I\,(A[\sigma]_M) = (I\,A)[I\,\sigma]_Q\\
&I_{\triangleright} &&: I\,(\Gamma \triangleright_M A) = I\,\Gamma \triangleright_Q I\,A\\
&I_{\Pi} &&: I\,(\Pi_M\,A\,B) = \Pi_Q\,(I\,A)\,(I\,B)\\
&I_{\mathsf{lam}} &&: I\,(\mathsf{lam}_M\,t) = \mathsf{lam}_Q\,(I\,t)\\
&I_{\mathsf{app}} &&: I\,(\mathsf{app}_M\,t) = \mathsf{app}_Q\,(I\,t)
\end{aligned}$$

- There is a model S called the *syntax* and for every model M, there is a morphism rec^M from S to M called the *recursor*. For every displayed model Q over S there is a section elim^Q of Q called the *eliminator*.

2.1 The identity type

In our construction of gluing we will assume that the target model has identity types. Identity types extend type theory as given in Figure 1 with the following operators and equations.

$$\begin{aligned}
&\mathsf{Id} &&: (A : \mathsf{Ty}\,i\,\Gamma) \to \mathsf{Tm}\,\Gamma\,A \to \mathsf{Tm}\,\Gamma\,A \to \mathsf{Ty}\,i\,\Gamma\\
&\mathsf{refl} &&: (u : \mathsf{Tm}\,\Gamma\,A) \to \mathsf{Tm}\,\Gamma\,(\mathsf{Id}\,A\,u\,u)\\
&\mathsf{J} &&: \big(C : \mathsf{Ty}\,i\,(\Gamma \triangleright A \triangleright \mathsf{Id}\,(A[\mathsf{p}])\,(u[\mathsf{p}])\,0)\big) \to \mathsf{Tm}\,\Gamma\,(C[\mathsf{id}, u, \mathsf{refl}\,u]) \to\\
&&&\quad (e : \mathsf{Tm}\,\Gamma\,(\mathsf{Id}\,A\,u\,v)) \to \mathsf{Tm}\,\Gamma\,(C[\mathsf{id}, v, e[\mathsf{p}]])\\
&\mathsf{Id}\beta &&: \mathsf{J}\,C\,w\,(\mathsf{refl}\,u) = w\\
&\mathsf{Id}[] &&: (\mathsf{Id}\,A\,u\,v)[\sigma] = \mathsf{Id}\,(A[\sigma])\,(u[\sigma])\,(v[\sigma])\\
&\mathsf{refl}[] &&: (\mathsf{refl}\,u)[\sigma] = \mathsf{refl}\,(u[\sigma])
\end{aligned}$$

$$\mathsf{J}[] \quad : (\mathsf{J}\, C\, w\, e)[\sigma] = \mathsf{J}\,(C[\sigma^{\uparrow\uparrow}])\,(w[\sigma])\,(e[\sigma])$$

$\mathsf{Id}\, A\, u\, v$ expresses that u is equal to v, there is one constructor refl expressing reflexivity and there is the eliminator J which says that given a family over identities and a witness of that family for refl we get that there is an element of that family for every identity proof.

3 The Set model

As an example of a simple model, we define the set model (standard model, metacircular model). In this model, contexts are sets, types are families over their contexts, substitutions are functions and terms are dependent functions. Context extension is metatheoretic Σ, otherwise everything is modelled by its metatheoretic counterparts, e.g. Π types are dependent functions, lam is λ, app is metatheoretic application. We list a few components for illustration.

$$
\begin{aligned}
&\mathsf{Con} &&:= \mathsf{Set}_\omega\\
&\mathsf{Ty}\, i\, \Gamma &&:= \Gamma \to \mathsf{Set}_i\\
&\mathsf{Sub}\, \Gamma\, \Delta &&:= \Gamma \to \Delta\\
&\mathsf{Tm}\, \Gamma\, A &&:= (\gamma : \Gamma) \to A\, \gamma\\
&A[\sigma] &&:= \lambda\gamma.A\,(\sigma\,\gamma)\\
&\cdot &&:= \mathbb{1}\\
&\epsilon &&:= \lambda\_.*\\
&\Gamma \rhd A &&:= (\gamma : \Gamma) \times A\,\gamma\\
&(\sigma, t) &&:= (\sigma, t)\\
&\mathsf{p} &&:= \mathsf{projl}\\
&\mathsf{q} &&:= \mathsf{projr}\\
&\Pi\, A\, B &&:= \lambda\gamma.(\alpha : A\,\gamma) \to B\,(\gamma, \alpha)\\
&\mathsf{lam}\, t &&:= \lambda\gamma.\lambda\alpha.t\,(\gamma, \alpha)\\
&\mathsf{app}\, t &&:= \lambda\gamma.t\,\gamma_{.1}\,\gamma_{.2}\\
&\Pi\beta &&: \mathsf{app}\,(\mathsf{lam}\, t) = \lambda\gamma'.(\lambda\gamma.\lambda\alpha.t\,(\gamma,\alpha))\,\gamma'_{.1}\,\gamma'_{.2} \overset{\to\beta}{=} \lambda\gamma'.t\,(\gamma'_{.1}, \gamma'_{.2}) \overset{\times\eta}{=} \lambda\gamma'.t\,\gamma' \overset{\to\eta}{=} t\\
&\mathsf{U}\, i &&:= \lambda\_.\mathsf{Set}_i\\
&\mathsf{El}\, a &&:= a\\
&\mathsf{c}\, a &&:= a\\
&\mathsf{Bool} &&:= \mathbb{2}\\
&\mathsf{true} &&:= *\\
&\mathsf{false} &&:= **\\
&\mathsf{if}\, C\, t\, u\, v &&:= \mathsf{case}\, t\, u\, v\\
&\mathsf{Id}\, A\, u\, v &&:= (u = v)
\end{aligned}
$$

The β law for Π uses the metatheoretic β and η laws for the functions and η for pairs.

Using the recursor we can define an interpreter for our syntax which maps syntactic terms to metatheoretic objects.

$$
\begin{aligned}
&[\![-]\!] : \mathsf{Con}_\mathsf{S} &&\to \mathsf{Set}_\omega &&:= \mathsf{rec}^{\mathsf{Set}}_{\mathsf{Con}}\\
&[\![-]\!] : \mathsf{Ty}_\mathsf{S}\, i\, \Gamma &&\to [\![\Gamma]\!] \to \mathsf{Set}_i &&:= \mathsf{rec}^{\mathsf{Set}}_{\mathsf{Ty}}
\end{aligned}
$$

$$\llbracket - \rrbracket : \mathsf{Sub_S}\,\Gamma\,\Delta \to \llbracket\Gamma\rrbracket \to \llbracket\Delta\rrbracket \qquad := \mathsf{rec}^{\mathsf{Set}}_{\mathsf{Sub}}$$
$$\llbracket - \rrbracket : \mathsf{Tm_S}\,\Gamma\,A \ \to (\gamma : \llbracket\Gamma\rrbracket) \to \llbracket A\rrbracket\,\gamma := \mathsf{rec}^{\mathsf{Set}}_{\mathsf{Tm}}$$

For example, the interpretation of the polymorphic identity function is mapped to

$$\llbracket \mathsf{lam}\,(\mathsf{lam}\,\mathsf{q})\rrbracket : (\gamma : \mathbb{1}) \to (A : \mathsf{Set}_0) \to A \to A = \lambda\gamma.\lambda A.\lambda a.a.$$

4 Pseudomorphism

In this section we define morphisms of models of type theory which are strict on the category structure and weak on $\cdot$ and $- \triangleright -$. We call such a morphism a *pseudomorphism*.

The components of a pseudomorphism F from S to M are the following.

$$\begin{aligned}
&F_{\mathsf{Con}} &&: \mathsf{Con}_S \to \mathsf{Con}\\
&F_{\mathsf{Ty}} &&: \mathsf{Ty}_S\,j\,\Gamma \to \mathsf{Ty}\,j\,(F\,\Gamma)\\
&F_{\mathsf{Sub}} &&: \mathsf{Sub}_S\,\Gamma\,\Delta \to \mathsf{Sub}\,(F\,\Gamma)\,(F\,\Delta)\\
&F_{\mathsf{Tm}} &&: \mathsf{Tm}_S\,\Gamma\,A \to \mathsf{Tm}\,(F\,\Gamma)\,(F\,A)\\
&F_{\mathsf{id}} &&: F\,\mathsf{id}_S = \mathsf{id}\\
&F_{\circ} &&: F\,(\sigma \circ_S \delta) = F\,\sigma \circ F\,\delta\\
&F_{[]} &&: F\,(A[\sigma]_S) = (F\,A)[F\,\sigma]\\
&F_{[]} &&: F\,(t[\sigma]_S) = (F\,t)[F\,\sigma]\\
&F_{\cdot} &&: F\,\cdot_S \cong \cdot\\
&F_{\epsilon} &&: F\,\epsilon_S = F_{\cdot.2} \circ \epsilon\\
&F_{\triangleright} &&: F\,(\Gamma \triangleright_S A) \cong F\,\Gamma \triangleright F\,A\\
&F_{\triangleright.1\circ} &&: F_{\triangleright.1} \circ F\,(\sigma^{\uparrow_S}) = (F\,\sigma)^{\uparrow} \circ F_{\triangleright.1}\\
&F_{,} &&: F\,(\sigma,_S\,t) = F_{\triangleright.2} \circ (F\,\sigma, F\,t)\\
&F_{\mathsf{p}} &&: F\,\mathsf{p}_S = \mathsf{p} \circ F_{\triangleright.1}\\
&F_{\mathsf{q}} &&: F\,\mathsf{q}_S = \mathsf{q}[F_{\triangleright.1}]
\end{aligned}$$

For readability, we omit the lower indices $_S$ from the metavariable names and all the $_M$ lower indicies. That is, when we write Con or id we mean Con_M and id_M. We overload the different parts of F, i.e. write F for F_{Con}, F_{Ty}, F_{Sub} etc.

Categorically, this pseudomorphism is a functor on categories of contexts with natural transformations on types and terms such that given $A : \mathsf{Ty}_S\,j\,\Delta$ with $\sigma : \mathsf{Sub}_S\,\Gamma\,\Delta$ and $t : \mathsf{Tm}_S\,\Gamma\,(A[\sigma]_S)$ with the pair (σ, t) having the universal property of the context extension of Δ with A in S, then $(F_{\mathsf{Sub}}\sigma, F_{\mathsf{Tm}}t)$ has the universal property of the context extension of $F_{\mathsf{Con}}\Delta$ with $F_{\mathsf{Ty}}A$ in M.

Just as a strict morphism (described in Section 2), a pseudomorphism maps contexts in S to contexts in M, types in S to types in M, etc. Identity, composition and action on substitution are preserved strictly (F_{id}, $F_{\circ}$, $F_{[]}$ and $F_{[]}$). The empty context and context extension are preserved up to definitional isomorphism. Definitional isomorphism between two contexts $\Gamma, \Delta : \mathsf{Con}$ is defined as follows.

$$(f : \Gamma \cong \Delta) := (f_{.1} : \mathsf{Sub}\,\Gamma\,\Delta) \times (f_{.2} : \mathsf{Sub}\,\Delta\,\Gamma) \times (f_{.12} : f_{.1} \circ f_{.2} = \mathsf{id}) \times (f_{.21} : f_{.2} \circ f_{.1} = \mathsf{id})$$

$F_{\triangleright.1\circ}$ denotes a naturality condition that $F_{\triangleright}$ has to satisfy. The empty substitution ϵ and the comprehension operators $-,-$, p, q are preserved strictly, but this is up to the weakness of $\triangleright$.

We derive the following naturality condition.

$$F_{\triangleright.2\circ} : F_{\triangleright.2} \circ_M (F\,\sigma)^{\uparrow_M} \overset{F_{\triangleright.12}}{=} F_{\triangleright.2} \circ (F\,\sigma)^{\uparrow_M} \circ F_{\triangleright.1} \circ F_{\triangleright.2} \overset{F_{\triangleright.1}\circ}{=}$$
$$F_{\triangleright.2} \circ F_{\triangleright.1} \circ F\,(\sigma^{\uparrow_S}) \circ F_{\triangleright.2} \overset{F_{\triangleright.21}}{=} F\,(\sigma^{\uparrow_S}) \circ_M F_{\triangleright.2}$$

We also note that $F\,(\sigma^{\uparrow_S}) = F_{\triangleright.2} \circ_M (F\,\sigma)^{\uparrow_M} \circ_M F_{\triangleright.1}$.

Note that every strict morphism F is automatically pseudo, with $F_{.1} = F_{.2} = \mathsf{id}_M$ and $F_{\triangleright.1} = F_{\triangleright.2} = \mathsf{id}_M$.

5 Gluing

In this section, given a pseudomorphism F from model S to model M, we define a displayed model P^F (P for short) over S. We call this model *gluing* along F and its components are given in Figure 2. We omit the $_S$ indices of metavariables and all the $_M$ indices for readability.

In the introduction we remarked that in categorical gluing an object in the glued model consists of a triple $\Gamma : |\mathcal{S}|$, $\Delta : |\mathcal{M}|$ and a morphism $\mathcal{M}(\Delta, F\,\Gamma)$. We could follow this line and define contexts in the glued model as such triples. This would be called the fibrational or display map approach. Instead our definition is more type theoretic, it uses indexed families, doubly (for the correspondence between fibrations and families see e.g. [6, p. 221]). Firstly, the glued model is given as a displayed model, that is, for each $\Gamma : \mathsf{Con}_S$ we have a set $\mathsf{Con}_\mathsf{P}\,\Gamma$. Secondly, instead of setting $\mathsf{Con}_\mathsf{P}\,\Gamma$ to $(\Delta : \mathsf{Con}_M) \times \mathsf{Sub}_M\,\Delta\,(F\,\Gamma)$, we use the built-in notion of indexed families in M, that is: types. Hence a context over Γ is an M-type in context $F\,\Gamma$. We remark that the glueing construction also works with the former choice of contexts.

Types in type theory can be thought of as proof-relevant predicates over their context and this is the intuition we adopt for describing the glued model. This is in line with the logical predicate view of gluing. We start with $\mathsf{Con}_\mathsf{P}\,\Gamma$: a predicate at Γ is indexed over the F-image of Γ. A predicate at a type A is indexed over the image of Γ for which the predicate holds and the image of A. For a substitution σ, we state the fundamental lemma: if the predicate holds at Γ, the predicate holds at Δ for the F-image of the substitution. In short, images of substitutions respect the predicate. For terms, we similarly state that the image of a term respects the predicate.

We continue by explaining what the logical predicate says at different contexts and types. The predicate at the empty context $\cdot_\mathsf{P}$ is always true. At extended contexts the predicate is given pointwise by a Σ type. $\Gamma_\mathsf{P} \triangleright A_\mathsf{P}$ is in context $F\,(\Gamma \triangleright A)$, but Γ_P only needs the component $F\,\Gamma$ which we obtain using the isomorphism $F_{\triangleright.1}$ from $F\,(\Gamma \triangleright_S A)$ to $F\,\Gamma \triangleright F\,A$ followed by first projection. A_P is first indexed over $F\,\Gamma$ which is given by $\mathsf{p} \circ F_{\triangleright.1} \circ \mathsf{p}$, then over Γ_P which is the first component of the Σ type referenced by q, then over $F\,A$ which is provided by the $F_{\triangleright.1}$ part of the isomorphism.

The predicate at a Π type holds for a function of type $F\,(\Pi\,A\,B)$ if whenever it holds for an input, it holds for the output. Let's look at how we express that the predicate holds at B for the output! We are in context

$$\Theta := F\,\Gamma \triangleright \underbrace{\Gamma_\mathsf{P}}_{3} \triangleright \underbrace{F\,(\Pi_S\,A\,B)}_{2} \triangleright \underbrace{F\,A[\mathsf{p}^2]}_{1} \triangleright \underbrace{A_\mathsf{P}[\mathsf{p}^2, \mathsf{q}]}_{0}$$

where we wrote the De Bruijn indices referring to each component underneath. B_P is a predicate indexed over $F\,(\Gamma \triangleright_S A)$, $\Gamma_\mathsf{P} \triangleright_\mathsf{P} A_\mathsf{P}$ and $F\,B[\mathsf{p}]$. The first index is given by $F_{\triangleright.2}$ which puts together the $F\,\Gamma$ (forgetting the last four elements in Θ by p^4) and the $F\,A$ components (last but one element in Θ, i.e. 1). The second index is given by De Bruijn

$$
\begin{aligned}
\mathsf{Con}_\mathsf{P}\,\Gamma &:= \mathsf{Ty}\,\omega\,(F\,\Gamma)\\
\mathsf{Ty}_\mathsf{P}\,i\,\Gamma_\mathsf{P}\,A &:= \mathsf{Ty}\,i\,(F\,\Gamma \triangleright \Gamma_\mathsf{P} \triangleright F\,A[\mathsf{p}])\\
\mathsf{Sub}_\mathsf{P}\,\Gamma_\mathsf{P}\,\Delta_\mathsf{P}\,\sigma &:= \mathsf{Tm}\,(F\,\Gamma\triangleright\Gamma_\mathsf{P})\,(\Delta_\mathsf{P}[F\,\sigma\circ\mathsf{p}])\\
\mathsf{Tm}_\mathsf{P}\,\Gamma_\mathsf{P}\,A_\mathsf{P}\,t &:= \mathsf{Tm}\,(F\,\Gamma\triangleright\Gamma_\mathsf{P})\,(A_\mathsf{P}[\mathsf{id},F\,t[\mathsf{p}]])\\
\mathsf{id}_\mathsf{P} &:= \mathsf{q}\\
\sigma_\mathsf{P}\circ_\mathsf{P}\delta_\mathsf{P} &:= \sigma_\mathsf{P}[F\,\delta\circ\mathsf{p},\delta_\mathsf{P}]\\
A_\mathsf{P}[\sigma_\mathsf{P}]_\mathsf{P} &:= A_\mathsf{P}[F\,\sigma\circ\mathsf{p}^2,\sigma_\mathsf{P}[\mathsf{p}],\mathsf{q}]\\
t_\mathsf{P}[\sigma_\mathsf{P}]_\mathsf{P} &:= t_\mathsf{P}[F\,\sigma\circ\mathsf{p},\sigma_\mathsf{P}]\\
\cdot_\mathsf{P} &:= \top\\
\epsilon_\mathsf{P} &:= \mathsf{tt}\\
\Gamma_\mathsf{P}\triangleright_\mathsf{P}A_\mathsf{P} &:= \Sigma\,\big(\Gamma_\mathsf{P}[\mathsf{p}\circ F_{\triangleright.1}]\big)\,\big(A_\mathsf{P}[\mathsf{p}\circ F_{\triangleright.1}\circ\mathsf{p},0,\mathsf{q}[F_{\triangleright.1}\circ\mathsf{p}]]\big)\\
\sigma_\mathsf{P},_\mathsf{P}t_\mathsf{P} &:= (\sigma_\mathsf{P},t_\mathsf{P})\\
\mathsf{p}_\mathsf{P} &:= \mathsf{projl}\,\mathsf{q}\\
\mathsf{q}_\mathsf{P} &:= \mathsf{projr}\,\mathsf{q}\\
\Pi_\mathsf{P}\,A_\mathsf{P}\,B_\mathsf{P} &:= \Pi\,\big(F\,A[\mathsf{p}^2]\big)\\
&\qquad\Big(\Pi\,\big(A_\mathsf{P}[\mathsf{p}^2,\mathsf{q}]\big)\\
&\qquad\quad\big(B_\mathsf{P}\big[F_{\triangleright.2}\circ(\mathsf{p}^4,1),(3,0),F\,(\mathsf{app}_S\,\mathsf{q})[F_{\triangleright.2}\circ(F_{\triangleright.2}\circ(\mathsf{p}^4,2),1)]\big]\big)\Big)\\
\mathsf{lam}\,t_\mathsf{P} &:= \mathsf{lam}\,\Big(\mathsf{lam}\,\big(t_\mathsf{P}\big[F_{\triangleright.2}\circ(\mathsf{p}^3,1),(2,0)\big]\big)\Big)\\
\mathsf{app}\,t_\mathsf{P} &:= \big(\mathsf{app}\,(\mathsf{app}\,t_\mathsf{P})\big)\big[\mathsf{p}\circ F_{\triangleright.1}\circ\mathsf{p},\mathsf{projl}\,0,0[F_{\triangleright.1}\circ\mathsf{p}],\mathsf{projr}\,0\big]\\
\Sigma_\mathsf{P}\,A_\mathsf{P}\,B_\mathsf{P} &:= \Sigma\,\big(A_\mathsf{P}\big[\mathsf{p},F\,(\mathsf{projl}_S\,\mathsf{q})[F_{\triangleright.2}\circ(\mathsf{p}^2,\mathsf{q})]\big]\big)\\
&\qquad\big(B_\mathsf{P}\big[F_{\triangleright.2}\circ(\mathsf{p}^3,F\,(\mathsf{projl}_S\,\mathsf{q})[F_{\triangleright.2}\circ(\mathsf{p}^3,1)]),(2,0),\\
&\qquad\quad F\,(\mathsf{projr}_S\,\mathsf{q})[F_{\triangleright.2}\circ(\mathsf{p}^3,1)]\big]\big)\\
(u_\mathsf{P},_\mathsf{P}v_\mathsf{P}) &:= (u_\mathsf{P},v_\mathsf{P})\\
\mathsf{projl}_\mathsf{P}\,t_\mathsf{P} &:= \mathsf{projl}\,t_\mathsf{P}\\
\mathsf{projr}_\mathsf{P}\,t_\mathsf{P} &:= \mathsf{projr}\,t_\mathsf{P}\\
\top_\mathsf{P} &:= \top\\
\mathsf{tt}_\mathsf{P} &:= \mathsf{tt}\\
\mathsf{U}_\mathsf{P}\,i &:= F\,(\mathsf{El}_S\,\mathsf{q})[F_{\triangleright.2}\circ(\mathsf{p}^2,\mathsf{q})]\Rightarrow\mathsf{U}\,i\\
\mathsf{El}_\mathsf{P}\,a_\mathsf{P} &:= \mathsf{El}\,(\mathsf{app}\,a_\mathsf{P})\\
\mathsf{c}_\mathsf{P}\,A_\mathsf{P} &:= \mathsf{lam}\,(\mathsf{c}\,A_\mathsf{P})\\
\mathsf{Bool}_\mathsf{P} &:= \Sigma\,\mathsf{Bool}\,\big(\mathsf{Id}\,(F\,\mathsf{Bool}_S[\mathsf{p}^3])\,(\mathsf{if}\,(F\,\mathsf{Bool}_S[\mathsf{p}^4])\,0\,(F\,\mathsf{true}_S[\mathsf{p}^3])\,(F\,\mathsf{false}_S[\mathsf{p}^3]))\,1\big)\\
\mathsf{true}_\mathsf{P} &:= (\mathsf{true},\mathsf{refl}\,(F\,\mathsf{true}_S[\mathsf{p}]))\\
\mathsf{false}_\mathsf{P} &:= (\mathsf{false},\mathsf{refl}\,(F\,\mathsf{false}_S[\mathsf{p}]))\\
\mathsf{if}_\mathsf{P}\,C_\mathsf{P}\,t_\mathsf{P}\,u_\mathsf{P}\,v_\mathsf{P} &:= \mathsf{J}\,\_\,(\mathsf{if}\,\_\,(\mathsf{projl}\,t_\mathsf{P})\,u_\mathsf{P}\,v_\mathsf{P})\,(\mathsf{projr}\,t_\mathsf{P})
\end{aligned}
$$

Figure 2 The displayed model P^F obtained by gluing along F. We write P instead of P^F, we omit the $_S$ indices of metavariables and all the $_M$ indices for readability. The full version of if_P (with the _s filled in) is given in Appendix A.

indices 3 and 0. The last index is the result of applying the function given by De Bruijn index 2. We cannot just use app since 2 does not have a function type, it has an image of a function type. However we can still apply it by observing that

$$\mathsf{app}\,\mathsf{q} : \mathsf{Tm}_S\,(\Gamma \triangleright_S \Pi_S\,A\,B \triangleright_S A[\mathsf{p}_S]_S)\,(B[{\mathsf{p}_S}^2,_S \mathsf{q}_S]_S)$$

and applying F to this results in

$$F\,(\mathsf{app}\,\mathsf{q}) : \mathsf{Tm}\,\big(F\,(\Gamma \triangleright_S \Pi_S\,A\,B \triangleright_S A[\mathsf{p}_S]_S)\big)\,(F\,B[F_{\triangleright.2} \circ (\mathsf{p} \circ F_{\triangleright.1} \circ \mathsf{p} \circ F_{\triangleright.1}, \mathsf{q}[\mathsf{F}_{\triangleright.1}])]).$$

Substituting this term by $F_{\triangleright.2} \circ (F_{\triangleright.2} \circ (\mathsf{p}^4, 2), 1)$ and using the fact that $F_\triangleright$ is an isomorphism results in exactly what we need.

The predicate at a Σ type holds if it holds pointwise. Here we use $F\,(\mathsf{projl}_S\,\mathsf{q})$ and $F\,(\mathsf{projr}_S\,\mathsf{q})$ to obtain $F\,A$ and $F\,B$ from $F\,(\Sigma_S\,A\,B)$. The predicate at $\top$ is trivial. The predicate at the universe is predicate space expressed as functions into $\mathsf{U}\,i$. The domain of this function space is again obtained by applying F to $\mathsf{El}_S\,\mathsf{q}$. The predicate at Bool for a b in $F\,\mathsf{Bool}$ says that b is either $F\,\mathsf{true}_S$ or $F\,\mathsf{false}_S$. We express this by encoding the sum type as a Σ over Bool.

The substition and term part of the gluing model is fairly straightforward. The most interesting component is if_P where we use J to eliminate the right projection of t_P (which is the equality in the second component of Bool_P), then we case split on the first projection by if_M and return u_P and v_P in the true and false cases, respectively. We omitted some arguments of J and if for readability, the full version is given in Appendix A. There we also verify that all equalities of the displayed model of type theory hold.

6 Global section functor

In this section we show that the global section functor is a pseudomorphism. In the next section we will use this property to derive canonicity for type theory.

The *global section functor* G is a pseudomorphism from S to Set defined as follows (S is the syntax, the set model Set is defined in Section 3).

$$\begin{aligned}
\mathsf{G}_{\mathsf{Con}}\,\Gamma &: \underbrace{\mathsf{Con}_{\mathsf{Set}}}_{=\mathsf{Set}_\omega} && := \mathsf{Sub}_\mathsf{S}\,\cdot\,\Gamma \\
\mathsf{G}_{\mathsf{Ty}}\,A &: \underbrace{\mathsf{Ty}_{\mathsf{Set}}\,j\,(\mathsf{G}\,\Gamma)}_{=\mathsf{G}\,\Gamma\to\mathsf{Set}_j} && := \lambda\rho\,.\,\mathsf{Tm}_\mathsf{S}\,\cdot\,(A[\rho]_\mathsf{S}) \\
\mathsf{G}_{\mathsf{Sub}}\,\sigma &: \underbrace{\mathsf{Sub}_{\mathsf{Set}}\,(\mathsf{G}\,\Gamma)\,(\mathsf{G}\,\Delta)}_{=\mathsf{G}\,\Gamma\to\mathsf{G}\,\Delta} && := \lambda\rho\,.\,\sigma \circ_\mathsf{S} \rho \\
\mathsf{G}_{\mathsf{Tm}}\,t &: \underbrace{\mathsf{Tm}_{\mathsf{Set}}\,(\mathsf{G}\,\Gamma)\,(\mathsf{G}\,A)}_{=(\rho:\mathsf{G}\,\Gamma)\to\mathsf{G}\,A\,\rho} && := \lambda\rho\,.\,t[\rho]_\mathsf{S}
\end{aligned}$$

A context is mapped to the set of closed syntactic substitutions into that context. A type is mapped to a function which takes a closed substitution and returns a set. This is the set of closed terms of that type substituted by the input substitution. Substitutions and terms are mapped to postcomposition.

Notice that this functor is weak on the empty context: $\mathsf{Sub}_\mathsf{S}\,\cdot\,\cdot$ is isomorphic to $\mathbb{1}$ (this is the empty context in the Set model), but not equal. Similarly, $\mathsf{Sub}_S\,\cdot\,(\Gamma \triangleright A)$ is isomorphic to $(\rho : \mathsf{Sub}_S\,\cdot\,\Gamma) \times \mathsf{Tm}_\mathsf{S}\,\cdot\,(A[\rho])$ by the isomorphism given by comprehension, but not equal. We list the components needed to show that G is a pseudomorphism in Figure 3.

$$
\begin{array}{lll}
\mathsf{G}_{\mathsf{id}} & : & \mathsf{G}\,\mathsf{id}_{\mathsf{S}} = \lambda\rho.\mathsf{id}\circ_{\mathsf{S}}\rho \overset{\mathsf{idl}_{\mathsf{S}}}{=} \lambda\rho.\rho = \mathsf{id}_{\mathsf{Set}} \\
\mathsf{G}_{\circ} & : & \mathsf{G}\,(\sigma\circ_{\mathsf{S}}\delta) = \lambda\rho.(\sigma\circ_{\mathsf{S}}\delta)\circ_{\mathsf{S}}\rho \overset{\mathsf{ass}_{\mathsf{S}}}{=} \lambda\rho.\sigma\circ_{\mathsf{S}}(\delta\circ_{\mathsf{S}}\rho) = \mathsf{G}\,\sigma\circ_{\mathsf{Set}}\mathsf{G}\,\delta \\
\mathsf{G}_{[]} & : & \mathsf{G}\,(A[\delta]_{\mathsf{S}}) = \lambda\rho.\mathsf{Tm}_{\mathsf{S}}\cdot(A[\delta][\rho]) \overset{[\circ]_{\mathsf{S}}}{=} \lambda\rho.\mathsf{Tm}_{\mathsf{S}}\cdot(A[\delta\circ\rho]) = (\mathsf{G}\,A)[\mathsf{G}\,\sigma]_{\mathsf{Set}} \\
\mathsf{G}_{[]} & : & \mathsf{G}\,(t[\delta]_{\mathsf{S}}) = \lambda\rho.t[\delta][\rho] \overset{[\circ]_{\mathsf{S}}}{=} \lambda\rho.t[\delta\circ\rho] = (\mathsf{G}\,t)[\mathsf{G}\,\sigma]_{\mathsf{Set}} \\
\mathsf{G}_{\cdot} & : & \mathsf{G}\cdot_{\mathsf{S}} \cong \cdot_{\mathsf{Set}} := (\lambda\rho.*, \lambda\_.\epsilon, \mathrm{trivial}, \mathrm{trivial}) \\
\mathsf{G}_{\epsilon} & : & \mathsf{G}\,\epsilon_{\mathsf{S}} = \lambda\rho.\epsilon\circ\rho \overset{\cdot\eta_{\mathsf{S}}}{=} \lambda\rho.\epsilon = \mathsf{G}_{\cdot.2}\circ_{\mathsf{Set}}\epsilon_{\mathsf{Set}} \\
\mathsf{G}_{\rhd} & : & \mathsf{G}\,(\Gamma\rhd_{\mathsf{S}}A) \cong \mathsf{G}\,\Gamma\rhd_{\mathsf{Set}}\mathsf{G}\,A := (\lambda\rho.(\mathsf{p}\circ\rho, \mathsf{q}[\rho]), \lambda(\rho,u).(\rho,_{\mathsf{S}}u),) \\
\mathsf{G}_{\rhd.2} & : & \mathsf{Sub}_{\mathsf{Set}}\,(\mathsf{G}\,\Gamma\rhd_{\mathsf{Set}}\mathsf{G}\,A)\,(\mathsf{G}\,(\Gamma\rhd_{\mathsf{S}}A)) := \lambda(\rho,u).(\rho,_{\mathsf{S}}u) \\
\mathsf{G}_{\rhd.12} & : & \mathsf{G}_{\rhd.1}\circ_{\mathsf{Set}}\mathsf{G}_{\rhd.2} = \lambda(\rho,u).(\mathsf{p}\circ(\rho,_{\mathsf{S}}u),_{\mathsf{S}}\mathsf{q}[\rho,_{\mathsf{S}},u]) \overset{\rhd\beta_1,\rhd\beta_2}{=} \lambda(\rho,u).(\rho,u) = \mathsf{id}_{\mathsf{Set}} \\
\mathsf{G}_{\rhd.21} & : & \mathsf{G}_{\rhd.2}\circ_{\mathsf{Set}}\mathsf{G}_{\rhd.1} = \lambda\rho.(\mathsf{p}\circ\rho,_{\mathsf{S}}\mathsf{q}[\rho]) \overset{,\circ_{\mathsf{S}}}{=} \lambda\rho.(\mathsf{p},\mathsf{q})\circ\rho \overset{\rhd\eta}{=} \lambda\rho.\mathsf{id}\circ\rho \overset{\mathsf{idl}_{\mathsf{S}}}{=} \lambda\rho.\rho = \mathsf{id}_{\mathsf{Set}} \\
\mathsf{G}_{\rhd.1\circ} & : & \mathsf{G}_{\rhd.1}\circ_{\mathsf{Set}}\mathsf{G}\,(\sigma^{\uparrow_{\mathsf{S}}}) = \lambda\rho.(\sigma\circ\mathsf{p}\circ\rho, \mathsf{q}[\rho]) \overset{,\circ_{\mathsf{S}}}{=} \lambda\rho.(\sigma\circ\mathsf{p},\mathsf{q})\circ\rho \overset{\mathsf{idl}_{\mathsf{S}},\rhd\eta_{\mathsf{S}}}{=} \\
 & & \rho.(\sigma\circ\mathsf{p},\mathsf{q})\circ(\mathsf{p}\circ\rho,\mathsf{q}[\rho]) = (\mathsf{G}\,\sigma)^{\uparrow_{\mathsf{Set}}}\circ_{\mathsf{Set}}\mathsf{G}_{\rhd.1} \\
\mathsf{G}_{,} & : & \mathsf{G}\,(\sigma,_{\mathsf{S}}t) = \lambda\rho.(\sigma,t)\circ\rho \overset{,\circ_{\mathsf{S}}}{=} \rho.(\sigma\circ\rho, t[\rho]) = \mathsf{G}_{\rhd.2}\circ_{\mathsf{Set}}(\mathsf{G}\,\sigma,_{\mathsf{Set}}\mathsf{G}\,t) \\
\mathsf{G}_{\mathsf{p}} & : & \mathsf{G}\,\mathsf{p}_{\mathsf{S}} = \lambda\rho.\mathsf{p}\circ\rho = \lambda\rho.(\mathsf{p}\circ\rho,\mathsf{q}[\rho]).1 = \mathsf{p}_{\mathsf{Set}}\circ_{\mathsf{Set}}\mathsf{G}_{\rhd.1} \\
\mathsf{G}_{\mathsf{q}} & : & \mathsf{G}\,\mathsf{q}_{\mathsf{S}} = \lambda\rho.\mathsf{q}[\rho] = \lambda\rho.(\mathsf{p}\circ\rho,\mathsf{q}[\rho]).2 = \mathsf{q}_{\mathsf{Set}}[\mathsf{G}_{\rhd.1}]_{\mathsf{Set}}
\end{array}
$$

Figure 3 Proof that the global section functor is a pseudomorphism.

7 Reaping the fruits

Let I be the identity morphism from S to S which is obviously a strict morphism, hence pseudo.[2] Gluing along I produces a function whose input is a term t in context Γ and whose output is a term in context Γ extended by $\mathsf{elim}^{\mathsf{P}^{\mathsf{I}}}_{\mathsf{Tm}}\,\Gamma$ which expresses that the predicate holds at Γ. The type of the output term says that the predicate holds at A for t. This is the fundamental lemma or parametricity theorem.

$$\mathsf{elim}^{\mathsf{P}^{\mathsf{I}}}_{\mathsf{Tm}} : (t : \mathsf{Tm}_{\mathsf{S}}\,\Gamma\,A) \to \mathsf{Tm}_{\mathsf{S}}\,(\Gamma\rhd\mathsf{elim}^{\mathsf{P}^{\mathsf{I}}}_{\mathsf{Tm}}\,\Gamma)\,\big((\mathsf{elim}^{\mathsf{P}^{\mathsf{I}}}_{\mathsf{Tm}}\,A)[\mathsf{id}, t[\mathsf{p}]]\big)$$

Let us look at the "hello world" example of parametricity, the case where $\Gamma = \cdot$ and $A = \Pi\,(\mathsf{U}\,i)\,(\mathsf{El}\,\mathsf{q}\Rightarrow\mathsf{El}\,\mathsf{q})$. Now using the fact that $\mathsf{elim}^{\mathsf{P}^{\mathsf{I}}}$ is a section, the type of $\mathsf{elim}^{\mathsf{P}^{\mathsf{I}}}_{\mathsf{Tm}}\,t$ computes to

$$\mathsf{Tm}_{\mathsf{S}}\,(\cdot\rhd\top)\,\Big(\Pi\,(\mathsf{U}\,i)\,\Big(\Pi\,(\mathsf{El}\,\mathsf{q}\Rightarrow\mathsf{U}\,i)\,\Big(\Pi\,(\mathsf{El}\,1)\,\big(\mathsf{El}\,(1\,\$\,0)\Rightarrow\mathsf{El}\,(1\,\$(t\,\$\,2\,\$\,0))\big)\Big)\Big)\Big),$$

where the type is the object theoretic syntax for

$$(A : \mathsf{Set}_i)(C : A \to \mathsf{Set}_i)(a : A) \to C\,a \to C\,(t\,A\,a).$$

[2] Note that the target of the pseudomorphism needs to have identity types, so technically I is the embedding of the syntax without identity types into the syntax with identity types. Alternatively, we can extend gluing for identity types.

Given a fixed type $A : \mathsf{Ty_S}\, i \cdot$ and an element $u : \mathsf{Tm_S} \cdot A$ we have

$$(\mathsf{elim}^{\mathsf{P^I}}_{\mathsf{Tm}}\, t)[\epsilon, \mathsf{tt}] \,\$\, \mathsf{c}\, A \,\$\, \mathsf{lam}\, (\mathsf{c}\, (\mathsf{Id}\, (A[\epsilon])\, 0\, u[\epsilon]))\, \$\, u \,\$\, \mathsf{refl}\, u : \mathsf{Tm} \cdot \big(\mathsf{Id}\, A\, (t \,\$\, c\, A \,\$\, u)\, u\big),$$

that is, we get that for any A and u, $t \,\$\, \mathsf{c}\, A \,\$\, u$ is equal to u.

Gluing along $\mathsf{rec}^{\mathsf{Set}}$ (the interpretation into the set model, see end of Section 3) produces Reynolds-style parametricity. It says that if there is an interpretation of the context Γ for which the predicate holds at Γ, the predicate holds at A for the interpretation of t.

$$\mathsf{elim}^{\mathsf{P}^{\mathsf{rec}^{\mathsf{Set}}}}_{\mathsf{Tm}} : (t : \mathsf{Tm_S}\, \Gamma\, A) \to (\gamma : [\![\Gamma]\!]) \times (\bar{\gamma} : \mathsf{elim}^{\mathsf{P}^{\mathsf{rec}^{\mathsf{Set}}}}_{\mathsf{Con}}\, \Gamma\, \gamma) \to \mathsf{elim}^{\mathsf{P}^{\mathsf{rec}^{\mathsf{Set}}}}_{\mathsf{Ty}}\, A\, (\gamma, \bar{\gamma}, [\![t]\!]\, \gamma)$$

Gluing along the global section functor G produces the following.

$$\mathsf{elim}^{\mathsf{P^G}}_{\mathsf{Tm}} : (t : \mathsf{Tm_S}\, \Gamma\, A) \to (\rho : \mathsf{Sub_S} \cdot \Gamma) \times (\bar{\rho} : \mathsf{elim}^{\mathsf{P^G}}_{\mathsf{Con}}\, \Gamma\, \rho) \to \mathsf{elim}^{\mathsf{P^G}}_{\mathsf{Ty}}\, A\, (\rho, \bar{\rho}, t[\rho])$$

If t is a boolean in the empty context, the type of $\mathsf{elim}^{\mathsf{P^G}}_{\mathsf{Tm}}\, t\, (\mathsf{id}, *)$ is $\mathsf{elim}^{\mathsf{P^G}}_{\mathsf{Ty}}\, \mathsf{Bool}\, (\rho, *, t)$ which is equal to $(b : \mathbb{2}) \times (\mathsf{case}\, b\, \mathsf{true_S}\, \mathsf{false_S} = t)$, i.e. canonicity.

8 Conclusions and further work

In this paper we defined gluing for pseudomorphisms of models of type theory thus generalising parametricity and canonicity. We did not try to derive the most general notion of gluing, e.g. we require that the target model supports $\top$, Σ, Id types in addition to what we have in the domain model. It would have been possible to give a less indexed variant of gluing where $\top$ and Σ are not needed, but Id types (or Bool-indexed inductive families) are required to support gluing for Bool. A less indexed variant however would be more tedious to work with due to all the naturality conditions that one would need to keep track of.

In the future we would like to generalise our construction to richer type theories having an identity type, inductive and coinductive types. We believe that this is possible without any extra conditions.

Normalisation by evaluation (NBE) for type theory is also defined using a proof-relevant logical predicate [3]. This logical predicate is given by gluing along the Yoneda embedding from the syntax to the presheaf model over the category of contexts and renamings. This is a pseudomorphism, so we obtain a glued model using our method. However, the universe in this model is not what we want. As a second step after gluing, NBE requires the definition of quote and unquote (sometimes called reify and reflect) functions from terms for which the predicate holds to normal forms and from neutral terms to witnesses of the predicate, respectively. We need to include these as part of the universe in the glued model to make the construction work. The predicate for Bool also needs to be adjusted.

We would also like to investigate examples of non-strict pseudo morphisms apart from global section and Yoneda for which the construction in this paper could be useful. For example, to derive canonicity proofs for type theories justified by models other than the set model.

References

1 Andreas Abel, Joakim Öhman, and Andrea Vezzosi. Decidability of conversion for type theory in type theory. *PACMPL*, 2:23:1–23:29, 2017.

2 Thorsten Altenkirch, Martin Hofmann, and Thomas Streicher. Reduction-free normalisation for system *F*. Unpublished draft, 1997.

3 Thorsten Altenkirch and Ambrus Kaposi. Normalisation by Evaluation for Type Theory, in Type Theory. *Logical Methods in Computer Science*, Volume 13, Issue 4, October 2017. URL: `https://lmcs.episciences.org/4005`, `doi:10.23638/LMCS-13(4:1)2017`.

4 Robert Atkey, Neil Ghani, and Patricia Johann. A relationally parametric model of dependent type theory. In Suresh Jagannathan and Peter Sewell, editors, *The 41st Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, POPL '14, San Diego, CA, USA, January 20-21, 2014*, pages 503–516. ACM, 2014. `doi:10.1145/2535838.2535852`.

5 Jean-Philippe Bernardy, Patrik Jansson, and Ross Paterson. Proofs for free — parametricity for dependent types. *Journal of Functional Programming*, 22(02):107–152, 2012. `doi:10.1017/S0956796812000056`.

6 John Cartmell. Generalised algebraic theories and contextual categories. *Annals of Pure and Applied Logic*, 32:209–243, 1986.

7 Pierre Clairambault and Peter Dybjer. The biequivalence of locally cartesian closed categories and martin-löf type theories. *Mathematical Structures in Computer Science*, 24(6), 2014.

8 Thierry Coquand. Canonicity and normalisation for dependent type theory. *CoRR*, abs/1810.09367, 2018.

9 Roy L. Crole. *Categories for types*. Cambridge mathematical textbooks. Cambridge University Press, Cambridge, New York, 1993. URL: `http://opac.inria.fr/record=b1088776`.

10 Peter Dybjer. Internal type theory. In *Lecture Notes in Computer Science*, pages 120–134. Springer, 1996.

11 Marcelo Fiore and Alex Simpson. Lambda definability with sums via grothendieck logical relations. In Jean-Yves Girard, editor, *Typed Lambda Calculi and Applications*, pages 147–161, Berlin, Heidelberg, 1999. Springer Berlin Heidelberg.

12 Marcelo P. Fiore. Semantic analysis of normalisation by evaluation for typed lambda calculus. In *Proceedings of the 4th international ACM SIGPLAN conference on Principles and practice of declarative programming, October 6-8, 2002, Pittsburgh, PA, USA (Affiliated with PLI 2002)*, pages 26–37. ACM, 2002. URL: `https://doi.org/10.1145/571157.571161`, `doi:10.1145/571157.571161`.

13 Claudio Hermida, Uday S. Reddy, and Edmund P. Robinson. Logical relations and parametricity — a Reynolds programme for category theory and programming languages. *Electronic Notes in Theoretical Computer Science*, 303(0):149 – 180, 2014. Proceedings of the Workshop on Algebra, Coalgebra and Topology (WACT 2013). URL: `http://www.sciencedirect.com/science/article/pii/S1571066114000346`, `doi:http://dx.doi.org/10.1016/j.entcs.2014.02.008`.

14 Ambrus Kaposi, András Kovács, and Thorsten Altenkirch. Constructing quotient inductive-inductive types. *Proc. ACM Program. Lang.*, 3(POPL):2:1–2:24, January 2019. `doi:10.1145/3290315`.

15 Chung kil Hur and Derek Dreyer. A kripke logical relation between ml and assembly. *Conference Record of the Annual ACM Symposium on Principles of Programming Languages*, pages 133–146, 01 2010. `doi:10.1145/1926385.1926402`.

16 András Kovács. Formalisation of canonicity for type theory in agda. ht-tps://github.com/AndrasKovacs/glue, November 2018.

17 J. Lambek and P. J. Scott. *Introduction to higher order categorical logic*. Cambridge University Press, New York, NY, USA, 1986.

18 Florian Rabe and Kristina Sojakova. Logical relations for a logical framework. *ACM Trans. Comput. Logic*, 14(4):32:1–32:34, November 2013. URL: `http://doi.acm.org/10.1145/2536740.2536741`, `doi:10.1145/2536740.2536741`.

19 Michael Shulman. Univalence for inverse diagrams and homotopy canonicity. *Mathematical Structures in Computer Science*, 25:1203–1277, 6 2015. arXiv:1203.3253. URL: `http://journals.cambridge.org/article_S0960129514000565`, `doi:10.1017/S0960129514000565`.

20 Jonathan Sterling and Bas Spitters. Normalization by gluing for free λ-theories. *CoRR*, abs/1809.08646, 2018. URL: `http://arxiv.org/abs/1809.08646`, `arXiv:1809.08646`.

21 Andrew W. Appel and David Mcallester. An indexed model of recursive types for foundational proof-carrying code. *ACM Trans. Program. Lang. Syst.*, 23:657–683, 09 2001. `doi:10.1145/504709.504712`.

A Full version of if_P and equalities in gluing

if_P is part of the glued displayed model P, see Section 5, Figure 2. Its definition is the following including the omitted _ arguments.

$$
\begin{aligned}
&\mathsf{if}_\mathsf{P}\, C_\mathsf{P}\, t_\mathsf{P}\, u_\mathsf{P}\, v_\mathsf{P} :=\\
&\quad \mathsf{J}\,\Big(C_p\big[F_{\triangleright.2}\circ(\mathsf{p}^3,1),(2,(\mathsf{projl}\,t_\mathsf{P}[\mathsf{p}^2],0)),F\,\big(\mathsf{if}_S\,(C[\mathsf{p}^2,\mathsf{q}])\,\mathsf{q}\,(u[\mathsf{p}])\,(v[\mathsf{p}])\big)[F_{\triangleright.2}\circ(\mathsf{p}^3,1)]\big]\Big)\\
&\qquad \Big(\mathsf{if}\,\big(C_\mathsf{P}\big[F_{\triangleright.2}\circ(\mathsf{p}^2,w),(1,(0,\mathsf{refl}\,w)),F\,\big(\mathsf{if}\,(C[\mathsf{p}^2,\mathsf{q}])\,\mathsf{q}\,(u[\mathsf{p}])\,(v[\mathsf{p}])\big)[F_{\triangleright.2}\circ(\mathsf{p}^2,w)]\big]\big)\\
&\qquad\quad (\mathsf{projl}\,t_\mathsf{P})\,u_\mathsf{P}\,v_\mathsf{P}\,\Big)\\
&\qquad (\mathsf{projr}\,t_\mathsf{P})
\end{aligned}
$$

where w abbreviates $\mathsf{if}\,(F\,\mathsf{Bool}_S[\mathsf{p}^2])\,0\,(F\,\mathsf{true}_S[\mathsf{p}^2])\,(F\,\mathsf{false}[\mathsf{p}^2])$.

Here we check that the P satisfies all the equalities of displayed models. We note that $\sigma_\mathsf{P}{}^{\uparrow_\mathsf{P}} = \big(\sigma_\mathsf{P}[\mathsf{p}\circ F_{\triangleright.1}\circ\mathsf{p},\mathsf{projl}\,\mathsf{q}],\mathsf{projr}\,\mathsf{q}\big)$.

$$
\begin{aligned}
&\mathsf{idl}_\mathsf{P} &&: \mathsf{id}_\mathsf{P}\circ_\mathsf{P}\sigma_\mathsf{P} = 0[F\,\sigma\circ\mathsf{p},\sigma_\mathsf{P}] = \sigma_\mathsf{P}\\
&\mathsf{idr}_\mathsf{P} &&: \sigma_\mathsf{P}\circ_\mathsf{P}\mathsf{id}_\mathsf{P} = \sigma_\mathsf{P}[F\,\mathsf{id}\circ\mathsf{p},0] = \sigma_\mathsf{P}[\mathsf{p},\mathsf{q}] = \sigma_\mathsf{P}[\mathsf{id}] = \sigma_\mathsf{P}\\
&\mathsf{ass}_\mathsf{P} &&: (\sigma_\mathsf{P}\circ_\mathsf{P}\delta_\mathsf{P})\circ_\mathsf{P}\nu_\mathsf{P} = \sigma_\mathsf{P}[F\,\delta\circ\mathsf{p},\delta_\mathsf{P}][F\,\nu\circ\mathsf{p},\nu_\mathsf{P}] =\\
& &&\quad \sigma_\mathsf{P}[F\,(\delta\circ_S\nu)\circ\mathsf{p},\delta_\mathsf{P}[F\,\nu\circ\mathsf{p},\nu_\mathsf{P}]] = \sigma_\mathsf{P}\circ_\mathsf{P}(\delta_\mathsf{P}\circ_\mathsf{P}\nu_\mathsf{P})\\
&[\mathsf{id}]_\mathsf{P} &&: A_\mathsf{P}[\mathsf{id}_\mathsf{P}]_\mathsf{P} = A_\mathsf{P}[F\,\mathsf{id}\circ\mathsf{p}^2,\mathsf{q}[\mathsf{p}],\mathsf{q}] = A_\mathsf{P}[(\mathsf{p},\mathsf{q})\circ\mathsf{p},\mathsf{q}] = A_\mathsf{P}[\mathsf{id}\circ\mathsf{p},\mathsf{q}] = A_\mathsf{P}[\mathsf{id}] = A_\mathsf{P}\\
&[\circ]_\mathsf{P} &&: A_\mathsf{P}[\sigma_\mathsf{P}\circ_\mathsf{P}\delta_\mathsf{P}]_\mathsf{P} = A_\mathsf{P}[F\,(\sigma\circ_S\delta)\circ\mathsf{p}^2,\sigma_\mathsf{P}[F\,\delta\circ\mathsf{p},\delta_\mathsf{P}][\mathsf{p}],\mathsf{q}] =\\
& &&\quad A_\mathsf{P}[F\,\sigma\circ\mathsf{p}^2,\sigma_\mathsf{P}[\mathsf{p}],\mathsf{q}][F\,\delta\circ\mathsf{p}^2,\delta_\mathsf{P}[\mathsf{p}],\mathsf{q}] = A_\mathsf{P}[\sigma_\mathsf{P}]_\mathsf{P}[\delta_\mathsf{P}]_\mathsf{P}\\
&[\mathsf{id}]_\mathsf{P} &&: t_\mathsf{P}[\mathsf{id}_\mathsf{P}]_\mathsf{P} = t_\mathsf{P}[F\,\mathsf{id}\circ\mathsf{p},\mathsf{q}] = t_\mathsf{P}[\mathsf{p},\mathsf{q}] = t_\mathsf{P}[\mathsf{id}] = t_\mathsf{P}\\
&[\circ]_\mathsf{P} &&: t_\mathsf{P}[\sigma_\mathsf{P}\circ_\mathsf{P}\delta_\mathsf{P}]_\mathsf{P} = t_\mathsf{P}[F\,(\sigma\circ\delta)\circ\mathsf{p},\sigma_\mathsf{P}[F\,\delta\circ\mathsf{p},\delta_\mathsf{P}]] =\\
& &&\quad t_\mathsf{P}[F\,\sigma\circ\mathsf{p},\sigma_\mathsf{P}][F\,\delta\circ\mathsf{p},\delta_\mathsf{P}] = t_\mathsf{P}[\sigma_\mathsf{P}]_\mathsf{P}[\delta_\mathsf{P}]_\mathsf{P}\\
&\epsilon\eta_\mathsf{P} &&: (\delta_\mathsf{P} : \mathsf{Sub}_\mathsf{P}\,\Gamma_\mathsf{P}\cdot_\mathsf{P}) = (\delta_\mathsf{P} : \mathsf{Tm}\,(F\,\Gamma\triangleright\Gamma_\mathsf{P})\,\top) \overset{\cdot\eta}{=} \mathsf{tt} = \epsilon_\mathsf{P}\\
&\triangleright\beta_{1\mathsf{P}} &&: \mathsf{p}_\mathsf{P}\circ_\mathsf{P}(\sigma_\mathsf{P},_\mathsf{P}t_\mathsf{P}) = (\mathsf{projl}\,\mathsf{q})[F\,(\sigma,_S t)\circ\mathsf{p},(\sigma_\mathsf{P},t_\mathsf{P})] =\\
& &&\quad \mathsf{projl}\,(\mathsf{q}[F\,(\sigma,_S t)\circ\mathsf{p},(\sigma_\mathsf{P},t_\mathsf{P})]) = \mathsf{projl}\,(\sigma_\mathsf{P},t_\mathsf{P}) = \sigma_\mathsf{P}\\
&\triangleright\beta_{2\mathsf{P}} &&: \mathsf{q}_\mathsf{P}[\sigma_\mathsf{P},_\mathsf{P}t_\mathsf{P}]_\mathsf{P} = (\mathsf{projr}\,\mathsf{q})[F\,(\sigma,_S t)\circ\mathsf{p},(\sigma_\mathsf{P},t_\mathsf{P})] =\\
& &&\quad \mathsf{projr}\,(\mathsf{q}[F\,(\sigma,_S t)\circ\mathsf{p},(\sigma_\mathsf{P},t_\mathsf{P})]) = \mathsf{projr}\,(\sigma_\mathsf{P},t_\mathsf{P}) = t_\mathsf{P}\\
&\triangleright\eta_\mathsf{P} &&: (\mathsf{p}_\mathsf{P},_\mathsf{P}\mathsf{q}_\mathsf{P}) = (\mathsf{projl}\,\mathsf{q},\mathsf{projr}\,\mathsf{q}) \overset{\Sigma\eta}{=} \mathsf{q} = \mathsf{id}_\mathsf{P}\\
&,\circ_\mathsf{P} &&: (\sigma_\mathsf{P},_\mathsf{P}t_\mathsf{P})\circ_\mathsf{P}\delta_\mathsf{P} = (\sigma_\mathsf{P},t_\mathsf{P})[F\,\delta\circ\mathsf{p},\delta_\mathsf{P}] \overset{,[]}{=} (\sigma_\mathsf{P}[F\,\delta\circ\mathsf{p},\delta_\mathsf{P}],t_\mathsf{P}[F\,\delta\circ\mathsf{p},\delta_\mathsf{P}]) =\\
& &&\quad (\sigma_\mathsf{P}\circ_\mathsf{P}\delta_\mathsf{P},_\mathsf{P}t_\mathsf{P}[\delta_\mathsf{P}]_\mathsf{P})\\
&\Pi\beta_\mathsf{P} &&: \mathsf{app}_\mathsf{P}\,(\mathsf{lam}_\mathsf{P}\,t_\mathsf{P}) =\\
& &&\quad \big(\mathsf{app}\,(\mathsf{app}\,(\mathsf{lam}\,(\mathsf{lam}\,(t_\mathsf{P}[F_{\triangleright.2}\circ(\mathsf{p}^3,1),(2,0)])))))\\
& &&\qquad \big[\mathsf{p}\circ F_{\triangleright.1}\circ\mathsf{p},\mathsf{projl}\,0,0[F_{\triangleright.1}\circ\mathsf{p}],\mathsf{projr}\,0\big] \overset{\Pi\beta}{=}\\
& &&\quad t_\mathsf{P}[F_{\triangleright.2}\circ(\mathsf{p}^3,1),(2,0)][\mathsf{p}\circ F_{\triangleright.1}\circ\mathsf{p},\mathsf{projl}\,0,0[F_{\triangleright.1}\circ\mathsf{p}],\mathsf{projr}\,0\big] =
\end{aligned}
$$

$$t_{\mathsf{P}}[\mathsf{p}, (\mathsf{projl}\,0, \mathsf{projr}\,0)] = t_{\mathsf{P}}[\mathsf{id}] = t_{\mathsf{P}}$$

$$\begin{aligned}
\Pi\eta_{\mathsf{P}} \quad &: \mathsf{lam}_{\mathsf{P}}\,(\mathsf{app}_{\mathsf{P}}\,t_{\mathsf{P}}) = \\
&\mathsf{lam}\,\Big(\mathsf{lam}\,\big((\mathsf{app}\,(\mathsf{app}\,t_{\mathsf{P}}))\big[\mathsf{p}\circ F_{\triangleright.1}\circ\mathsf{p}, \mathsf{projl}\,0, 0[F_{\triangleright.1}\circ\mathsf{p}], \mathsf{projr}\,0\big] \\
&\qquad\qquad\qquad\qquad\big[F_{\triangleright.2}\circ(\mathsf{p}^3, 1), (2, 0)\big]\big)\Big) = \\
&\mathsf{lam}\,\big(\mathsf{lam}\,((\mathsf{app}\,(\mathsf{app}\,t_{\mathsf{P}}))[\mathsf{p}^3, 2, 1, 0])\big) = \mathsf{lam}\,(\mathsf{lam}\,((\mathsf{app}\,(\mathsf{app}\,t_{\mathsf{P}}))[\mathsf{id}])) \overset{\Pi\eta_S}{=} t_{\mathsf{P}}
\end{aligned}$$

$$\begin{aligned}
\Pi[]_{\mathsf{P}} \quad &: (\Pi_{\mathsf{P}}\,A_{\mathsf{P}}\,B_{\mathsf{P}})[\sigma_{\mathsf{P}}]_{\mathsf{P}} = \\
&\Pi\,\big(F\,A[F\,\sigma\circ\mathsf{p}^2]\big) \\
&\quad\Big(\Pi\,\big(A_{\mathsf{P}}\big[F\,\sigma\circ\mathsf{p}^2, \sigma_{\mathsf{P}}[\mathsf{p}], \mathsf{q}\big][\mathsf{p}^2, \mathsf{q}]\big) \\
&\qquad\big(B_{\mathsf{P}}\big[F_{\triangleright.2}\circ(F\,\sigma\circ\mathsf{p}^4, 1), (\sigma_{\mathsf{P}}[\mathsf{p}^3], 0), \\
&\qquad\qquad F\,(\mathsf{app}\,\mathsf{q})[F_{\triangleright.2}\circ(F_{\triangleright.2}\circ(F\,\sigma\circ\mathsf{p}^4, 2), 1)]\big]\big)\Big) = \\
&\Pi\,\big(F\,(A[\sigma])[\mathsf{p}^2]\big) \\
&\quad\Big(\Pi\,\big(A_{\mathsf{P}}[\sigma_{\mathsf{P}}]_{\mathsf{P}}[\mathsf{p}^2, \mathsf{q}]\big) \\
&\qquad\big(B_{\mathsf{P}}\big[F_{\triangleright.2}\circ(F\,\sigma)^{\uparrow}\circ F_{\triangleright.1}\circ\mathsf{p}^2, \big(\sigma_{\mathsf{P}}[\mathsf{p}\circ F_{\triangleright.1}\circ\mathsf{p}^2, \mathsf{projl}\,1], \mathsf{projr}\,1\big), \mathsf{q}\big] \\
&\qquad\quad\big[F_{\triangleright.2}\circ(\mathsf{p}^4, 1), (3, 0), F\,(\mathsf{app}\,\mathsf{q})[F_{\triangleright.2}\circ(F_{\triangleright.2}\circ(\mathsf{p}^4, 2), 1)]\big]\big)\Big) = \\
&\Pi_{\mathsf{P}}\,(A_{\mathsf{P}}[\sigma_{\mathsf{P}}]_{\mathsf{P}})\,(B_{\mathsf{P}}[\sigma_{\mathsf{P}}{}^{\uparrow_{\mathsf{P}}}]_{\mathsf{P}})
\end{aligned}$$

$$\begin{aligned}
\mathsf{lam}[]_{\mathsf{P}} \quad &: (\mathsf{lam}_{\mathsf{P}}\,t_{\mathsf{P}})[\sigma_{\mathsf{P}}]_{\mathsf{P}} = \mathsf{lam}\,(\mathsf{lam}\,(t_{\mathsf{P}}[F_{\triangleright.2}\circ(F\,\sigma\circ\mathsf{p}^3, 1), (\sigma_{\mathsf{P}}[\mathsf{p}^2], 0)])) = \\
&\mathsf{lam}_{\mathsf{P}}\,(t_{\mathsf{P}}[\sigma_{\mathsf{P}}{}^{\uparrow_{\mathsf{P}}}]_{\mathsf{P}}) \\
\Sigma\beta_{1\mathsf{P}} \quad &: \mathsf{projl}_{\mathsf{P}}\,(u_{\mathsf{P}},_{\mathsf{P}}\,v_{\mathsf{P}}) = \mathsf{projl}\,(u_{\mathsf{P}}, v_{\mathsf{P}}) = u_{\mathsf{P}} \\
\Sigma\beta_{2\mathsf{P}} \quad &: \mathsf{projr}_{\mathsf{P}}\,(u_{\mathsf{P}},_{\mathsf{P}}\,v_{\mathsf{P}}) = \mathsf{projr}\,(u_{\mathsf{P}}, v_{\mathsf{P}}) = v_{\mathsf{P}} \\
\Sigma\eta_{\mathsf{P}} \quad &: (\mathsf{projl}_{\mathsf{P}}\,t_{\mathsf{P}},_{\mathsf{P}}\,\mathsf{projr}_{\mathsf{P}}\,t_{\mathsf{P}}) = (\mathsf{projl}\,t_{\mathsf{P}}, \mathsf{projr}\,t_{\mathsf{P}}) = t_{\mathsf{P}}
\end{aligned}$$

$$\begin{aligned}
\Sigma[]_{\mathsf{P}} \quad &: (\Sigma_{\mathsf{P}}\,A_{\mathsf{P}}\,B_{\mathsf{P}})[\sigma_{\mathsf{P}}]_{\mathsf{P}} = \\
&\Sigma\,\big(A_{\mathsf{P}}\big[F\,\sigma\circ\mathsf{p}^2, \sigma_{\mathsf{P}}[\mathsf{p}], F\,(\mathsf{projl}\,\mathsf{q})[F_{\triangleright.2}\circ(F\,\sigma\circ\mathsf{p}^2, \mathsf{q})]\big]\big) \\
&\quad\big(B_{\mathsf{P}}\big[F_{\triangleright.2}\circ\big(F\,\sigma\circ\mathsf{p}^3, F\,(\mathsf{projl}\,\mathsf{q})[F_{\triangleright.2}\circ(F\,\sigma\circ\mathsf{p}^3, 1)]\big), (\sigma_{\mathsf{P}}[\mathsf{p}^2], 0), \\
&\qquad\quad F\,(\mathsf{projr}\,\mathsf{q})[F_{\triangleright.2}\circ(F\,\sigma\circ\mathsf{p}^3, 1)]\big]\big) = \\
&\Sigma\,\big(A_{\mathsf{P}}\big[F\,\sigma\circ\mathsf{p}^2, \sigma_{\mathsf{P}}[\mathsf{p}], F\,(\mathsf{projl}\,\mathsf{q})[F_{\triangleright.2}\circ(\mathsf{p}^2, \mathsf{q})]\big]\big) \\
&\quad\big(B_{\mathsf{P}}\big[F_{\triangleright.2}\circ\big(F\,\sigma\circ\mathsf{p}^3, F\,(\mathsf{projl}\,\mathsf{q})[F_{\triangleright.2}\circ(\mathsf{p}^3, 1)]\big), (\sigma_{\mathsf{P}}[\mathsf{p}^2], 0), \\
&\qquad\quad F\,(\mathsf{projr}\,\mathsf{q})[F_{\triangleright.2}\circ(\mathsf{p}^3, 1)]\big]\big) = \\
&\Sigma_{\mathsf{P}}\,(A_{\mathsf{P}}[\sigma_{\mathsf{P}}]_{\mathsf{P}})\,(B_{\mathsf{P}}[\sigma_{\mathsf{P}}{}^{\uparrow_{\mathsf{P}}}]_{\mathsf{P}})
\end{aligned}$$

$$\begin{aligned}
,[]_{\mathsf{P}} \quad &: (u_{\mathsf{P}},_{\mathsf{P}}\,v_{\mathsf{P}})[\sigma_{\mathsf{P}}]_{\mathsf{P}} = (u_{\mathsf{P}}[F\,\sigma\circ\mathsf{p}, \sigma_{\mathsf{P}}], v_{\mathsf{P}}[F\,\sigma\circ\mathsf{p}, \sigma_{\mathsf{P}}]) = (u_{\mathsf{P}}[\sigma_{\mathsf{P}}]_{\mathsf{P}},_{\mathsf{P}}\,v_{\mathsf{P}}[\sigma_{\mathsf{P}}]_{\mathsf{P}}) \\
\top\eta_{\mathsf{P}} \quad &: (t_{\mathsf{P}} : \mathsf{Tm}_{\mathsf{P}}\,\Gamma_{\mathsf{P}}\,\top_{\mathsf{P}}) = (t_{\mathsf{P}} : \mathsf{Tm}\,(F\,\Gamma \triangleright \Gamma_{\mathsf{P}})\,\top) \overset{\top\eta}{=} \mathsf{tt} = \mathsf{tt}_{\mathsf{P}} \\
\top[]_{\mathsf{P}} \quad &: \top_{\mathsf{P}}[\sigma_{\mathsf{P}}]_{\mathsf{P}} = \top = \top_{\mathsf{P}} \\
\mathsf{tt}[]_{\mathsf{P}} \quad &: \mathsf{tt}_{\mathsf{P}}[\sigma_{\mathsf{P}}]_{\mathsf{P}} = \mathsf{tt} = \mathsf{tt}_{\mathsf{P}} \\
\mathsf{U}\beta_{\mathsf{P}} \quad &: \mathsf{El}_{\mathsf{P}}\,(\mathsf{c}_{\mathsf{P}}\,A_{\mathsf{P}}) = \mathsf{El}\,(\mathsf{app}\,(\mathsf{lam}\,(\mathsf{c}\,A_{\mathsf{P}}))) \overset{\Pi\beta}{=} \mathsf{El}\,(\mathsf{c}\,A_{\mathsf{P}}) \overset{\mathsf{U}\beta}{=} A_{\mathsf{P}} \\
\mathsf{U}\eta_{\mathsf{P}} \quad &: \mathsf{c}_{\mathsf{P}}\,(\mathsf{El}_{\mathsf{P}}\,a_{\mathsf{P}}) = \mathsf{lam}\,(\mathsf{c}\,(\mathsf{El}\,(\mathsf{app}\,a_{\mathsf{P}}))) \overset{\mathsf{El}\eta}{=} \mathsf{lam}\,(\mathsf{app}\,a_{\mathsf{P}}) \overset{\Pi\eta}{=} a_{\mathsf{P}} \\
\mathsf{U}[]_{\mathsf{P}} \quad &: (\mathsf{U}_{\mathsf{P}}\,i)[\sigma_{\mathsf{P}}]_{\mathsf{P}} = F\,(\mathsf{El}_S\,\mathsf{q})[F_{\triangleright.2}\circ(F\,\sigma\circ\mathsf{p}^2, \mathsf{q})] \Rightarrow \mathsf{U}\,i = \\
&F\,(\mathsf{El}_S\,\mathsf{q})[F_{\triangleright.2}\circ(F\,\sigma)^{\uparrow}\circ(\mathsf{p}^2, \mathsf{q})] \Rightarrow \mathsf{U}\,i =
\end{aligned}$$

$$F\,(\mathsf{El}_S\,\mathsf{q})[F\,(\sigma^{\uparrow})\circ F_{\triangleright.2}\circ(\mathsf{p}^2,\mathsf{q})]\Rightarrow\mathsf{U}\,i = F\,(\mathsf{El}_S\,\mathsf{q})[F_{\triangleright.2}\circ(\mathsf{p}^2,\mathsf{q})]\Rightarrow\mathsf{U}\,i=\mathsf{U}_{\mathsf{P}}\,i$$

$$\begin{aligned}
\mathsf{El}[]_{\mathsf{P}} \quad &: (\mathsf{El}_{\mathsf{P}}\,a_{\mathsf{P}})[\sigma_{\mathsf{P}}]_{\mathsf{P}} = (\mathsf{El}\,(\mathsf{app}\,a_{\mathsf{P}}))[F\,\sigma\circ\mathsf{p}^2,\sigma_{\mathsf{P}}[\mathsf{p}],\mathsf{q}] \overset{\mathsf{El}[]}{=} \\
&\quad \mathsf{El}\,((\mathsf{app}\,a_{\mathsf{P}})[F\,\sigma\circ\mathsf{p}^2,\sigma[\mathsf{p}],\mathsf{q}]) \overset{\mathsf{app}[]}{=} \mathsf{El}\,(\mathsf{app}\,(a_{\mathsf{P}}[F\,\sigma\circ\mathsf{p},\sigma_{\mathsf{P}}])) = \mathsf{El}_{\mathsf{P}}\,(a_{\mathsf{P}}[\sigma_{\mathsf{P}}]_{\mathsf{P}}) \\
\mathsf{Bool}[]_{\mathsf{P}} \quad &: \mathsf{Bool}_{\mathsf{P}}[\sigma_{\mathsf{P}}]_{\mathsf{P}} = \\
&\quad \Sigma\,\mathsf{Bool}\,\big(\mathsf{Id}\,(F\,\mathsf{Bool}[F\,\sigma\circ\mathsf{p}^3]) \\
&\qquad\qquad \big(\mathsf{if}\,(F\,\mathsf{Bool}[F\,\sigma\circ\mathsf{p}^4])\,0\,(F\,\mathsf{true}[F\,\sigma\circ\mathsf{p}^3])\,(F\,\mathsf{false}[F\,\sigma\circ\mathsf{p}^3])\big)\,1\big) = \\
&\quad \Sigma\,\mathsf{Bool}\,\big(\mathsf{Id}\,(F\,\mathsf{Bool}[\mathsf{p}^3])\,\big(\mathsf{if}\,(F\,\mathsf{Bool}[\mathsf{p}^4])\,0\,(F\,\mathsf{true}[\mathsf{p}^3])\,(F\,\mathsf{false}[\mathsf{p}^3])\big)\,1\big) = \\
&\quad \mathsf{Bool}_{\mathsf{P}} \\
\mathsf{true}[]_{\mathsf{P}} \quad &: \mathsf{true}_{\mathsf{P}}[\sigma_{\mathsf{P}}]_{\mathsf{P}} = \big(\mathsf{true},\mathsf{refl}\,(F\,(\mathsf{true}_S[\sigma])[\mathsf{p}])\big) \overset{\mathsf{true}[]_S}{=} (\mathsf{true},\mathsf{refl}\,(F\,\mathsf{true}_S[\mathsf{p}])) = \mathsf{true}_{\mathsf{P}} \\
\mathsf{false}[]_{\mathsf{P}} \quad &: \mathsf{false}_{\mathsf{P}}[\sigma_{\mathsf{P}}]_{\mathsf{P}} = \big(\mathsf{false},\mathsf{refl}\,(F\,(\mathsf{false}_S[\sigma])[\mathsf{p}])\big) \overset{\mathsf{false}[]_S}{=} (\mathsf{false},\mathsf{refl}\,(F\,\mathsf{false}_S[\mathsf{p}])) = \mathsf{false}_{\mathsf{P}} \\
\mathsf{if}[]_{\mathsf{P}} \quad &: (\mathsf{if}_{\mathsf{P}}\,_\,t_{\mathsf{P}}\,u_{\mathsf{P}}\,v_{\mathsf{P}})[\sigma_{\mathsf{P}}]_{\mathsf{P}} = \\
&\quad \big(\mathsf{J}\,_\,(\mathsf{if}\,_\,(\mathsf{projl}\,t_{\mathsf{P}})\,u_{\mathsf{P}}\,v_{\mathsf{P}})\,(\mathsf{projr}\,t_{\mathsf{P}})\big)[F\,\sigma\circ\mathsf{p},\sigma_{\mathsf{P}}] \overset{\mathsf{J}[],\mathsf{if}[],\mathsf{projl}[],\mathsf{projr}[]}{=} \\
&\quad \big(\mathsf{J}\,_\,(\mathsf{if}\,_\,(\mathsf{projl}\,(t_{\mathsf{P}}[\sigma_{\mathsf{P}}]_{\mathsf{P}}))\,(u_{\mathsf{P}}[\sigma_{\mathsf{P}}]_{\mathsf{P}})\,(v_{\mathsf{P}}[\sigma_{\mathsf{P}}]_{\mathsf{P}}))\,(\mathsf{projr}\,(t_{\mathsf{P}}[\sigma_{\mathsf{P}}]_{\mathsf{P}}))\big) = \\
&\quad \mathsf{if}_{\mathsf{P}}\,_\,(t_{\mathsf{P}}[\sigma_{\mathsf{P}}]_{\mathsf{P}})\,(u_{\mathsf{P}}[\sigma_{\mathsf{P}}]_{\mathsf{P}})\,(v_{\mathsf{P}}[\sigma_{\mathsf{P}}]_{\mathsf{P}})
\end{aligned}$$