# On Higher Inductive Types in Cubical Type Theory

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**Abstract**

Cubical type theory provides a constructive justification to certain aspects of homotopy type theory such as Voevodsky's univalence axiom. This makes many extensionality principles, like function and propositional extensionality, directly provable in the theory. This paper describes a constructive semantics, expressed in a presheaf topos with suitable structure inspired by cubical sets, of some higher inductive types. It also extends cubical type theory by a syntax for the higher inductive types of spheres, torus, suspensions, truncations, and pushouts. All of these types are justified by the semantics and have judgmental computation rules for all constructors, including the higher dimensional ones, and the universes are closed under these type formers.

## Contents



# 1 Introduction

Homotopy type theory [23] provides a new and promising approach to equality in type theory where types are thought of as abstract spaces and equality as paths in these spaces [4]. Iterated equality proofs then correspond to *homotopies* between paths. This intuition is motivated by homotopy theoretic models, in particular by the Kan simplicial set model [13] due to Voevodsky. This allows one to find new principles in type theory inspired by homotopy theory. Prime examples of this are Voevodsky's *univalence axiom* [24], which generalizes the principle of propositional extensionality to dependent type theory, and the stratification of types by the complexity of their equality (i.e., by their homotopy level or "h-level" [25]).

In the homotopical interpretation of type theory inductive types are represented as discrete spaces with only points in them. Higher inductive types are a natural generalization where types may also be generated by paths (potentially higher dimensional). This notion of types, combined with universes and the univalence axiom, is an important extension of dependent type theory, which allows for an elegant and original synthetic development of algebraic topology, using in a key way type-theoretic ideas (such as the encode-decode method [23]). Impressive examples of such development are, among others, the definition of the Hopf fibration, the Freudenthal suspension theorem and the Blakers-Massey theorem [5, 11]. However, and somewhat surprisingly, despite several efforts (e.g., [17]), the *consistency* of such an extension, which would justify these impressive developments, has not yet been established. The simplicial set model [13] provides (in a classical framework) a model for the univalence axiom, but it only provides a model for some very particular higher inductive types (such as the spheres, and the propositional truncation via an impredicative encoding [25]), and, as explained in [17], it is not clear how to extend this model to a model of parametrized higher inductive types like the suspension or pushouts (expressed as operations on a given universe).

**Contributions** The first contribution of the present paper is to provide such a semantics, starting in an essential way not from the simplicial set model, but from a cubical set model [7, 18]. This semantics is furthermore carried out in a constructive meta-theory. Our second contribution is to extend cubical type theory with a syntax for higher inductive types, exemplified by: spheres, the torus, suspensions, truncations, and pushouts. These types illustrate many of the difficulties in giving a computational justification for a general class of higher inductive types, in particular: the spheres and torus have higher dimensional constructors, furthermore one version of the torus has "fibrant" structure in its endpoints, the suspension has a parameter type, the truncations are recursive, and the pushouts have function applications in the endpoints of the path constructor. We show how to overcome all of these difficulties in a uniform way which suggests an approach to the problem of defining a schema for higher inductive types in cubical type theory.

Furthermore, all of the higher inductive types we consider have the following good properties justified by our semantics:

1. judgmental computation rules for all constructors,
2. strict stability under substitution, and
3. closure under universe levels (the higher inductive types live in the same universe as their parameters).

We have also implemented a variation of the system presented in this paper and performed multiple experiments with it.[1]

**Outline** The paper begins by describing the semantics, expressed in a presheaf topos with suitable structure, of the circle (Section 2.1), suspension (Section 2.2), and pushouts (Section 2.3). The next section starts with a short background on cubical type theory (Section 3.1) followed by the extension to the theory with: circle and spheres (Section 3.3.1), the torus (Section 3.3.2), suspensions (Section 3.3.3), propositional truncation, (Section 3.3.4) and pushouts (Section 3.3.5). The paper ends with conclusions and discussions on future and related work (Section 4).

[1] See: `https://github.com/mortberg/cubicaltt/tree/hcomptrans`

## 2 Semantics of higher inductive types

As shown in [18, 16, 1], the presentation of the semantics of cubical type theory can be both simplified and clarified by using the language of extensional type theory (with universes). This language can be given meaning in any presheaf topos, so long as we assume that the ambient set theory has a hierarchy of Grothendieck universes. In particular, we are going to show that the justification of higher inductive types can be done internally, using the existence of suitable initial algebras as the only extra assumption. We then justify the existence of these initial algebras for our presheaf topos externally. The key idea will be a decomposition of the notion of composition structure [18, 16] in a *transport* and a *homogeneous composition* operation.[2] This decomposition can be described internally.

We will work here in the presheaf topos over the Lawvere theory of De Morgan algebras [7, 16] (but, following [18], our results are valid in a more general setting). The presentation we use in [7] of this category is the following: we fix a countable set of names/symbols and the objects of the category $I, J, \ldots$ are finite sets of symbols. A map $J \to I$ is then a set theoretic map from $I$ to the free De Morgan algebra $\mathsf{dM}(J)$ on $J$. The corresponding presheaf model has then a generic De Morgan algebra $\mathbb{I}$, taking $\mathbb{I}(J)$ to be $\mathsf{dM}(J)$. (To have such a structure on $\mathbb{I}$ is not strictly necessary [18], but it simplifies the presentation.)

This type $\mathbb{I}$ is used as an abstract representation of the unit interval, so that a path in a type $A$ is represented by an element of the exponential $A^{\mathbb{I}}$. The extra data needed to define a cubical sets model is a notion of *cofibration*, which specifies the shape of filling problems that can be solved in a dependent type. We represent this by a type of *cofibrant* propositions $\mathbb{F}$ (denoted by `Cof` in [18]). In [7], this is represented by the *face lattice* (see Section 3.1), but other choices are possible. (Classically, this type $\mathbb{F}$ is a subtype of the subobject classifier of the presheaf topos, but, as stressed in [16], we can avoid mentioning the impredicative type of propositions altogether, and work in a predicative meta-theory.) We write $[\varphi]$ for the type associated to the proposition $\varphi : \mathbb{F}$. So $[\varphi]$ is a sub-singleton, and any element of $[\varphi]$ is equal to a fixed element $\mathtt{tt}$.

A *partial element* of a type $T$ is given by an element $\varphi$ in $\mathbb{F}$ and a function $[\varphi] \to T$. We say that a total element $v$ of $T$ extends such a partial element $\varphi, u$ if we have $\varphi \Rightarrow u\,\mathtt{tt} = v$, where $\Rightarrow$ denotes implication between propositions.

In this extensional type theory, we can think of a dependent type $A$ over a given type $\Gamma$ as a family of types $A\rho$ indexed by elements $\rho$ of $\Gamma$.

We now recall the notions of composition and filling structures [7, 18]. Let $A$ be a dependent type over a type $\Gamma$.

**Definition 1.** A *composition structure* $\mathsf{c}_A$ on $A$ is an operation taking as inputs $\gamma$ in $\Gamma^{\mathbb{I}}$, a proposition $\varphi$ in $\mathbb{F}$, a partial element $u$ in $[\varphi] \to \Pi(i : \mathbb{I})\, A\gamma(i)$, and an element $u_0$ in $A\gamma(0)$ such that $\varphi \Rightarrow u\,\mathtt{tt}\,0 = u_0$. This operation produces an element $u_1 = \mathsf{c}_A\ \gamma\,\varphi\,u\,u_0$ in $A\gamma(1)$ such that $\varphi \Rightarrow u\,\mathtt{tt}\,1 = u_1$.

The type of all such operations is written $\mathsf{Comp}(\Gamma, A)$ (see [18] for an explicit internal definition).

**Definition 2.** A *filling structure* $\mathsf{f}_A$ on $A$ is an operation taking the same input as $\mathsf{c}_A$ above, but producing an element $v = \mathsf{f}_A\ \gamma\,\varphi\,u\,u_0$ in $\Pi(i : \mathbb{I})\, A\gamma(i)$ such that $v$ extends $u$, i.e., $\varphi \Rightarrow u\,\mathtt{tt} = v$.

We write $\mathsf{Fill}(\Gamma, A)$ for the type of filling structures on $A$.

This notion of filling structure is an internal form of the homotopy extension property, which was recognized very early (see, e.g., [10]) as a key for an abstract development of algebraic topology.

[2] As explained in [1] this decomposition was first introduced in an early version of [7], precisely to address the problem of the semantics of propositional truncation and this decomposition is also present in [3, 2, 6].

As explained in [7, 18] we have that $\mathsf{Comp}(\Gamma, A)$ is a retract of $\mathsf{Fill}(\Gamma, A)$.

In the particular case where $\Gamma$ is the unit type, then $A$ is a "global" type, and $\mathsf{Comp}(\Gamma, A)$ becomes the type $\mathsf{Fib}(A)$ of *fibrancy* structures. Such an operation $h_A$ takes as argument $u_0$ in $A$ and a partial element $\varphi, u$ of $A^{\mathbb{I}}$ such that $\varphi \Rightarrow u \mathbin{\texttt{tt}} 0 = u_0$ and produces an element $u_1 = h_A\ \varphi\ u\ u_0$ such that $\varphi \Rightarrow u \mathbin{\texttt{tt}} 1 = u_1$.

In general, if $A$ is a family of types over $\Gamma$, to give a composition structure for each fiber, that is an element in $\Pi(\rho : \Gamma)\, \mathsf{Fib}(A\rho)$, is not enough to get a global composition structure, that is an element in $\mathsf{Comp}(\Gamma, A)$ (see [18] for an explicit counterexample). An element in $\Pi(\rho : \Gamma)\, \mathsf{Fib}(A\rho)$ is called a *homogeneous composition structure*.

We now describe the notion of *transport* operation, which is the missing component to get a composition structure from a homogeneous composition structure. This decomposition of the composition operation into a transport and homogeneous composition operation plays a crucial role for interpreting higher inductive types depending on parameters (like suspension, pushouts, or propositional truncation).

**Definition 3.** A *transport structure* $t_A$ on $A$ is an operation taking as arguments a path $\gamma$ in $\Gamma^{\mathbb{I}}$, a proposition $\varphi$ in $\mathbb{F}$ such that $\varphi \Rightarrow \forall(i : \mathbb{I})\, \gamma(0) = \gamma(i)$, and an element $u_0$ in $A\gamma(0)$. This operation produces an element $u_1 = t_A\, \gamma\, \varphi\, u_0$ in $A\gamma(1)$ such that $\varphi \Rightarrow u_0 \mathbin{\texttt{tt}} = u_1$.

The condition $\varphi \Rightarrow \forall(i : \mathbb{I})\, \gamma(0) = \gamma(i)$ expresses that the path $\gamma$ is *constant on* $\varphi$

Clearly we obtain a homogeneous composition structure from any composition structure. We also get:

**Lemma 4.** *If a family of types $A$ over $\Gamma$ has a composition structure $\mathsf{c}_A$, then it has a transport structure $t_A$.*

*Proof.* We can take $t_A\, \gamma\, \varphi\, u_0 = \mathsf{c}_A\ \gamma\, \varphi\, (\lambda(x : [\varphi])(i : \mathbb{I})\, u_0)\, u_0$. ☐

**Lemma 5.** *If a family of types $A$ over $\Gamma$ has a homogeneous composition structure $h_A$ and a transport structure $t_A$, then it has a composition structure $\mathsf{c}_A$.*

*Proof.* We can define $\mathsf{c}_A\ \gamma\, \varphi\, u\, u_0$ as

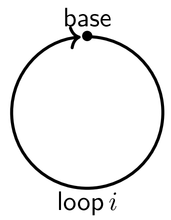
$$h_A\, \gamma(1)\, \varphi\, (\lambda(x : [\varphi])(i : \mathbb{I})\, t_A\, \gamma'(i)\, (i = 1)\, (u\, x\, i))\, (t_A\, \gamma'(i)\, 0_{\mathbb{F}}\, u_0)$$

where $\gamma'(i) = \lambda(j : \mathbb{I})\, \gamma(i \vee j)$. ☐

We are now going to develop some universal algebra internally in the presheaf model. The operations will involve the interval object $\mathbb{I}$ and the type $\mathbb{F}$ of cofibrant propositions, and can be seen as a generalization of the usual notion of operations in universal algebra.

## 2.1 Semantics of the circle

The circle, denoted $\mathbb{S}^1$, is represented as a higher inductive type with a path $\mathsf{loop}$ in direction $i : \mathbb{I}$ connecting a point $\mathsf{base}$ to itself:



An $S^1$*-algebra structure* on a type $A$ consists of a fibrancy structure $h_A$ together with a base point $b_A$ and a loop $l_A$ in $A^{\mathbb{I}}$ connecting $b_A$ to itself (i.e., $l_A\ 0 = l_A\ 1 = b_A$). Given two $S^1$-algebras $A, h_A, b_A, l_A$ and $B, h_B, b_B, l_B$ a function $\alpha : A \to B$ is a map of $S^1$-algebras if it satisfies $\alpha\ b_A = b_B$ and $\alpha\ (l_A\ i) = l_B\ i$ and

$$\alpha\ (h_A\ \varphi\ u\ u_0) = h_B\ \varphi\ (\lambda(x : [\varphi])(i : \mathbb{I})\ \alpha\,(u\ x\ i))\ (\alpha\ u_0).$$

We will show below using *external* reasoning:

**Proposition 6.** *There exists an initial $S^1$-algebra, which we denote by $\mathbb{S}^1$, $\mathsf{hcomp}, \mathsf{base}, \mathsf{loop}$.*

So $\mathbb{S}^1$ has a structure of an $S^1$-algebra and the fact that it is initial means that, for any $S^1$-algebra $A, h_A, b_A, l_A$ there exists a unique $S^1$-algebra map $\mathbb{S}^1 \to A$.

By definition, the type $\mathbb{S}^1$ is fibrant since it has a fibrancy structure $\mathsf{hcomp}$. Furthermore, we can prove that initiality implies the dependent elimination rule.[3]

**Proposition 7.** $\mathbb{S}^1$ *satisfies the dependent elimination rule for the circle: given a family of type $P$ over $\mathbb{S}^1$ with a composition structure, and $a$ in $P$ $\mathsf{base}$ and $l\ i$ in $P$ $(\mathsf{loop}\ i)$ such that $l\ 0 = l\ 1 = a$ there exists a map $\mathsf{elim} : \Pi(x : \mathbb{S}^1) P\ x$ such that $\mathsf{elim}\ \mathsf{base} = a$ and $\mathsf{elim}\ (\mathsf{loop}\ i) = l\ i$.*

*Proof.* We know by [7, 18] that $A = \Sigma(x : \mathbb{S}^1) P\ x$ has a composition structure. It has then a natural $\mathbb{S}^1$-algebra structure, taking $b_A = \mathsf{base}, a$ and $l_A\ i = \mathsf{loop}\ i, l\ i$. This structure is such that the first projection $\pi_1 : A \to \mathbb{S}^1$ is a map of $S^1$-algebras. We have a unique $S^1$-algebra map $\alpha : \mathbb{S}^1 \to A$ and $\pi_1 \circ \alpha$ is the identity on $\mathbb{S}^1$. We can then define $\mathsf{elim}\ x = \pi_2\ (\alpha\ x)$ in $P\ x$. ☐

## 2.2 Semantics of the suspension operation

The suspension $\Sigma A$ of a type $A$ has constructors $\mathsf{N}$ and $\mathsf{S}$ (two poles) and a path between them for any element of A. This enables us to give a direct definition of $\mathbb{S}^{n+1}$ as $\Sigma^n \mathbb{S}^1$. Compared to the case of the circle, this higher inductive type presents the extra complexity of having parameters and the decomposition of the composition operation will be the key for providing its semantics.

Given a type $X$, a $\Sigma X$*-algebra structure* on a type $A$ consists of a fibrancy structure $h_A$ together with two points $n_A, s_A$, and a family of paths $l_A$ in $X \to A^{\mathbb{I}}$ connecting $n_A$ to $s_A$ (i.e., $l_A\ x\ 0 = n_A$ and $l_A\ x\ 1 = s_A$ for all $x$ in $X$). Given two $\Sigma X$-algebras $A, h_A, n_A, s_A, l_A$ and $B, h_B, n_B, s_B, l_B$ a function $\alpha\colon A \to B$ is a map of $\Sigma X$-algebras if it satisfies $\alpha\ n_A = n_B$ and $\alpha\ s_A = s_B$ and $\alpha\ (l_A\ i) = l_B\ i$ and

$$\alpha\ (h_A\ \varphi\ u\ u_0) = h_B\ \varphi\ (\lambda(x : [\varphi])(i : \mathbb{I})\ \alpha\,(u\ x\ i))\ (\alpha\ u_0).$$

As for the circle we can show using external reasoning:

**Proposition 8.** *There exists an initial $\Sigma X$-algebra, which we denote by $\Sigma X$,$\mathsf{hcomp}$,$\mathsf{N}$,$\mathsf{S}$, $\mathsf{merid}$.*

By definition, the type $\Sigma X$ is fibrant since it has a fibrancy structure $\mathsf{hcomp}$. Using this filling structure, we prove as above:

**Proposition 9.** $\Sigma X$ *satisfies the dependent elimination rule for the suspension: given a family of type $P$ over $\Sigma X$ with a composition structure, and $n$ in $P$ $\mathsf{N}$ and $s$ in $P$ $\mathsf{S}$ and $l\ x\ i$ in $P$ $(\mathsf{merid}\ x\ i)$ such that $l\ x\ 0 = n$ and $l\ x\ 1 = s$ there exists a map $\mathsf{elim} : \Pi(x : \Sigma X) P\ x$ such that $\mathsf{elim}\ \mathsf{N} = n$ and $\mathsf{elim}\ \mathsf{S} = s$ and $\mathsf{elim}\ (\mathsf{merid}\ x\ i) = l\ x\ i$.*

[3] This is a direct generalization of the usual argument that a natural number object satisfies the dependent elimination rule.

The operation $\Sigma X$ is functorial in $X$. Given a map $u : X \to Y$ we get a $\Sigma X$-structure on $\Sigma Y$ by taking $l_{\Sigma Y}\ x\ i = \mathsf{merid}_Y\ (u\ x)\ i$ and hence a map $\Sigma u : \Sigma X \to \Sigma Y$.

Let now $A$ be a dependent family of types over a given type $\Gamma$, so that $A\rho$ is a type for any $\rho$ in $\Gamma$. We define a new family of types $\Sigma A$ over $\Gamma$ by taking $(\Sigma A)\rho = \Sigma(A\rho)$. By construction, this new family *always* has a homogeneous composition structure (without any hypothesis on $A$). Furthermore, we have the following proposition, which motivates the introduction of the notion of transport structure.

**Proposition 10.** *If $A$ has a transport structure $t_A$, then $\Sigma A$ has a transport structure, and hence (since it has a homogeneous composition structure) also a composition structure by Lemma 5.*

*Proof.* Given $\gamma$ in $\Gamma^{\mathbb{I}}$ and $\varphi$ such that $\gamma$ is constant on $\varphi$ (i.e., $\varphi \Rightarrow \forall (i : \mathbb{I})\ \gamma(0) = \gamma(i)$), we have a map $t_A\ \gamma\ \varphi : A\gamma(0) \to A\gamma(1)$ which is the identity on $\varphi$ and hence the map $\Sigma(t_A\ \gamma\ \varphi)$ is a transport map $\Sigma A\gamma(0) \to \Sigma A\gamma(1)$ which is the identity on $\varphi$. ☐

This example motivates the decomposition of the composition operation into a transport and homogeneous composition operations. In a context, we could only build an initial algebra for the *homogeneous* composition operation (by doing it pointwise) and it does not seem possible to do it for the composition operation directly. The problem does not appear for a type like the circle which has no parameters, for which homogeneous and general compositions coincide. For the suspension, we have to argue further that we also get a transport operation. (This problem seems connected to the problem of size blow-up for parametrized higher inductive types due to fibrant replacement in the simplicial set model discussed in [17].)

The same argument would apply to the propositional truncation $\|X\|$ of a type $X$. We would instead consider the following notion of algebra: a type $A$ with a fibrancy structure, a map $i_A : X \to A$ and a map $sq_A : A \to A \to A^{\mathbb{I}}$ such that $sq\ a_0\ a_1$ is a path connecting $a_0$ to $a_1$.

## 2.3 Pushouts

Many higher inductive types can be encoded as (homotopy) pushouts of spans of other types. In particular (homotopy) coequalizers, which together with coproducts (which are encoded using $\Sigma$-types), can be used to compute general colimits of diagrams of types. This has been used to encode many known higher inductive types, including recursive ones like propositional [8, 14] and higher truncations [19].

The semantics of pushouts involves the same problem with parameters as in the previous example, but the definition of the transport function is more complex and we will need to introduce some auxiliary operations definable from transport.

A *span* $D = (C, A, B, u, v)$ consists of two maps $u \colon C \to A$ and $v \colon C \to B$. Given such a span, we define a $D$-algebra to be a type $X$ with a fibrancy structure $h_X$ and maps $i_X \colon A \to X$ and $j_X \colon B \to X$ and $p_X \colon C \to X^{\mathbb{I}}$ such that $p_X\ z\ 0 = i_X\ (u\ z)$ and $p_X\ z\ 1 = j_X\ (v\ z)$. As above, there is a canonical notion of $D$-algebra maps, and (in suitable presheaf models) we have an initial $D$-algebra, which we write $\mathsf{po}(D) = A \sqcup_C B, \mathsf{hcomp}, \mathsf{inl}, \mathsf{inr}, \mathsf{push}$.

We can relativize this situation over a type $\Gamma$. If $A, B, C$ are families of types over $\Gamma$ and $u$ (resp. $v$) is a family of maps $u\rho \colon C\rho \to A\rho$ (resp. $v\rho \colon C\rho \to A\rho$) we consider $D = (C, A, B, u, v)$ to be a span over $\Gamma$, with $D\rho = (C\rho, A\rho, B\rho, u\rho, v\rho)$. If the span $D$ is given over $\Gamma$, we define $\mathsf{po}(D)$ in a pointwise way as for the suspensions, taking $\mathsf{po}(D)\rho$ to be $\mathsf{po}(D\rho)$.

We want to prove that if $C, A, B$ have transport structures, then so does $\mathsf{po}(D)$. In order to do that, we first show how to define further operations from a given transport structure.

**Lemma 11.** *Given a family of types $A$ over $\Gamma$ with a transport structure $t_A$ we can define a new operation $f_A$ such that $f_A\ \varphi\ \gamma\ a_0$ is a path in $\Pi(i : \mathbb{I})A\gamma(i)$ constant on $\varphi$ and connecting $a_0$ to $t_A\ \varphi\ \gamma\ a_0$ for any $\gamma$ in $\Gamma^{\mathbb{I}}$ constant on $\varphi$ and $a_0$ in $A\gamma(0)$. Furthermore given any section $a$ in $\Pi(i : \mathbb{I})A\gamma(i)$ we can define an operation $sq_A\ \varphi\ \gamma\ a$ which is a path in $(A\gamma(1))^{\mathbb{I}}$ connecting $t_A\ \varphi\ \gamma\ a(0)$ to $a(1)$, and which is constant on $\varphi$.*

*Proof.* We define

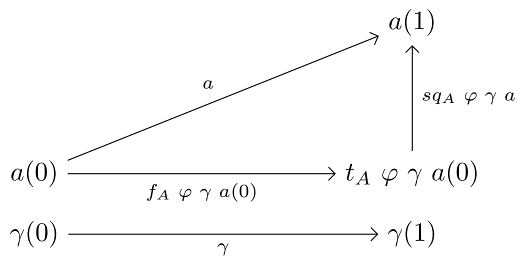
$$f_A\ \varphi\ \gamma\ a_0 = \lambda(i : \mathbb{I})\, t_A\ (\varphi \vee (i = 0))\ (\lambda(j : \mathbb{I})\gamma(i \wedge j))\ a_0$$

which connects $a_0$ to $t_A\ \varphi\ \gamma\ a_0$ and is constant on $\varphi$, and

$$sq_A\ \varphi\ \gamma\ a = \lambda(i : \mathbb{I})\, t_A\ (\varphi \vee (i = 1))\ (\lambda(j : \mathbb{I})\gamma(i \vee j))\ a(i)$$

which connects $t_A\ \varphi\ \gamma\ a(0)$ to $a(1)$ and is constant on $\varphi$. □

The relationship between these operations can be displayed as:




so that $sq_A$ can be though of as an operation which "squeezes" the path $a$ into the fiber over $\gamma(1)$.

**Corollary 12.** *Given two families of types $C$ and $A$ over $\Gamma$ with transport structures $t_C$ and $t_A$ respectively, and a map $u : C \to A$ over $\Gamma$, there exists an operation $l\ \varphi\ \gamma\ c_0$ which is a path in $(A\gamma(1))^{\mathbb{I}}$ constant over $\varphi$ and connecting $t_A\ \varphi\ \gamma\ (u\gamma(0)\ c_0)$ and $u\gamma(1)\ (t_C\ \varphi\ \gamma\ c_0)$, given $\gamma$ in $\Gamma^{\mathbb{I}}$ constant over $\varphi$ and $c_0$ in $C\gamma(0)$.*

*Proof.* We apply the $sq_A$ operation and the $f_C$ operation from Lemma 11 to the path $\lambda(i : \mathbb{I})\, u\gamma(i)\, (f_C\ \varphi\ \gamma\ c_0\ i)$. □

**Proposition 13.** *Given a family of spans $D = (C, A, B, u, v)$ over a type $\Gamma$ such that $A$, $B$, and $C$ have transport structures then the family $\mathsf{po}(D)$ also has a transport structure, and hence also a composition structure by Lemma 5.*

*Proof.* We use the previous corollary to provide a structure of $D\gamma(0)$-algebra on $\mathsf{po}(D)\gamma(1)$, structure which coincides with the one of $\mathsf{po}(D)\gamma(0)$ on $\varphi$. By initiality we get a map $\mathsf{po}(D)\gamma(0) \to \mathsf{po}(D)\gamma(1)$ which is the identity on $\varphi$, and is the desired transport function. (For a more detailed explanation see the syntactic presentation in Section 3.3.5.) □

## 2.4 Existence of initial algebras

We now explain the proof of Proposition 6 asserting the existence of a suitable initial algebra. We cannot prove this in an abstract way, but we need to use the fact that we are working with presheaf models over a small base category, in our case the Lawvere theory of the theory of De Morgan algebras. We write $I, J, K, \ldots$ for the objects in the base category. The only thing we are going to use about this category is that it has a canonical cylinder functors $I \mapsto I^+$ (where $I^+$ is obtained from $I$ by adding a fresh symbol) with natural transformations

$e_0, e_1 : I \to I^+$ and $\sigma : I^+ \to I$. We only describe the case of $S^1$-algebra here, but all other cases follow the same pattern. The interested reader may consult Appendix A for the proofs for the other higher inductive types. The argument we give can be seen as a constructive version of the small object argument [22], and it crucially uses the fact that each $\mathbb{F}(I)$ has decidable equality.

We first define a family of sets $\mathbb{S}^1_{\text{pre}}(I)$ which is an "upper approximation" of the circle, together with maps $\mathbb{S}^1_{\text{pre}}(I) \to \mathbb{S}^1_{\text{pre}}(J)$, $u \mapsto uf$ for $f : J \to I$. An element of $\mathbb{S}^1_{\text{pre}}(I)$ is of the form:

- $\mathsf{base}$, or
- $\mathsf{loop}\ r$ with $r \neq 0, 1$ in $\mathbb{I}(I)$, or
- $\mathsf{hcomp}\ [\varphi \mapsto u]\ u_0$ with $\varphi \neq 1$ in $\mathbb{F}(I)$ and $u_0$ in $\mathbb{S}^1_{\text{pre}}(I)$ and $u$ a family of elements $u_f$ in $\mathbb{S}^1_{\text{pre}}(J)$ for $f : J \to I^+$ such that $\varphi\sigma f = 1$.

In this way an element of $\mathbb{S}^1_{\text{pre}}(I)$ can be seen as a well-founded tree. Note that we do not yet require that the sides in $\mathsf{hcomp}$ match up with the base. In order to express this we first define $uf$ in $\mathbb{S}^1_{\text{pre}}(J)$ for $f : J \to I$ by induction on $u$:

$$\mathsf{base} f = \mathsf{base}$$

$$(\mathsf{loop}\ r)f = \begin{cases} \mathsf{loop}\ (rf) & \text{if } rf \neq 0 \text{ and } rf \neq 1 \\ \mathsf{base} & \text{otherwise} \end{cases}$$

$$(\mathsf{hcomp}\ [\varphi \mapsto u]\ u_0)f = \begin{cases} u_{f^+ e_1} & \text{if } \varphi f = 1 \\ \mathsf{hcomp}\ [\varphi f \mapsto u f^+]\ (u_0 f) & \text{otherwise} \end{cases}$$

where $uf^+$ is the family $(uf^+)_g = u_{f^+ g}$ for $g : K \to J^+$.

Note that we may not have in general $(vf)g = v(fg)$ for $v$ in $\mathbb{S}^1_{\text{pre}}(I)$ and $f : J \to I$ and $g : K \to J$. We then define the subset $\mathbb{S}^1(I) \subseteq \mathbb{S}^1_{\text{pre}}(I)$ by taking the elements $\mathsf{base}$, $\mathsf{loop}\ r$ and $\mathsf{hcomp}\ [\varphi \mapsto u]\ u_0$ such that $u_0 \in \mathbb{S}^1(I)$, $u_f \in \mathbb{S}^1(J)$, for $f : J \to I^+$ satisfying $u_0 g = u_{g^+ e_0}$ for $g : J \to I$ and $u_f g = u_{fg}$ for $f : J \to I^+$ and $g : K \to J$. This defines a cubical set $\mathbb{S}^1$, such that $\mathbb{S}^1(I)$ is a subset of $\mathbb{S}^1_{\text{pre}}(I)$ for each $I$.

As defined $\mathbb{S}^1$ has a structure of an $S^1$-algebra. Since furthermore the elements of $\mathbb{S}^1(I)$ are well-founded trees built from $\mathsf{base}$, $\mathsf{loop}$ and $\mathsf{hcomp}$ it is also the initial $S^1$-algebra in this presheaf model.

### 2.5 Universes

As shown externally in [7, 18] (and internally in [16]) we can define in the presheaf model a cumulative hierarchy of (univalent and fibrant) universes $U_n$ which classify families of types of a given size with a composition structure. Since the way we build initial algebras preserves the universe level, our definition, e.g., of the suspension can be seen as an operation $\Sigma : U_n \to U_n$. Thus, we have presented a semantics of a large class of higher inductive types with univalent universes. (As shown in [23], the univalence axiom is essential for any non trivial use of higher inductive types.)

## 3 Higher inductive types in cubical type theory

In this section we discuss the extensions to cubical type theory by higher inductive types. We begin by repeating the basic notions of cubical type theory.

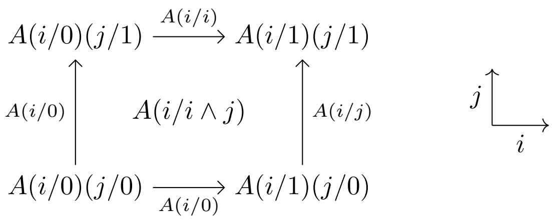
## 3.1 Background: cubical type theory

Cubical type theory extends a dependent type theory with one universe $\mathsf{U}$ closed under $\Pi$- and $\Sigma$-types with $\mathsf{Path}$-types, composition operations and $\mathsf{Glue}$-types.

The $\mathsf{Path}$-types internalize the idea from homotopy type theory that equalities should correspond to paths. We write $\mathsf{Path}\,A\,a\,b$ for the type of paths in $A$ with endpoints $a$ and $b$. These types behave like function types and have both abstraction (written $\langle i \rangle\, t$ for $t$ with $i$ abstracted) and application (written using juxtaposition). The path abstraction binds "dimension variables" ranging over an abstract interval $\mathbb{I}$ specified by the grammar:

$$r, s \quad ::= \quad 0 \mid 1 \mid i \mid 1 - r \mid r \wedge s \mid r \vee s$$

The set $\mathbb{I}$ is a De Morgan algebra with the $1 - r$ operation as De Morgan involution. A type in a context with variables $i_1, \ldots, i_n : \mathbb{I}$ should be thought of as an $n$-dimensional cube and the substitutions $(i/0)$ and $(i/1)$ give the faces of this cube. A substitution $(i/j)$ renames one of the dimensions of $A$ and as there are no injectivity constraints on these renaming substitutions one can perform substitutions which gives a "diagonal" of a cube (i.e., if $A$ is a square depending on $i, j : \mathbb{I}$ then $A(i/j)$ is a diagonal). The $\wedge$ and $\vee$ operations are called *connections* and provide convenient ways of building higher dimensional cubes from lower dimensional ones. For instance, if $A$ is a line depending on $i$, then $A(i/i \wedge j)$ is the interior of the square:




The face lattice $\mathbb{F}$ is a distributive lattice generated by formal symbols $(i = 0)$ and $(i = 1)$ with the relation $(i = 0) \wedge (i = 1) = 0_{\mathbb{F}}$. The elements of the face lattice can be described by the grammar

$$\varphi, \psi \quad ::= \quad 0_{\mathbb{F}} \mid 1_{\mathbb{F}} \mid (i = 0) \mid (i = 1) \mid \varphi \wedge \psi \mid \varphi \vee \psi$$

There is a canonical lattice map $\mathbb{I} \to \mathbb{F}$ sending $i$ to $(i = 1)$ and $1 - i$ to $(i = 0)$. We write $(r = 1)$ for the image of $r : \mathbb{I}$ in $\mathbb{F}$ and we write $(r = 0)$ for $((1 - r) = 1)$.

The judgment $\Gamma \vdash \varphi : \mathbb{F}$ says that $\varphi$ is a face formula involving only the dimension variables declared in $\Gamma$. Given a formula $\varphi$ we can *restrict* a context $\Gamma$ and obtain a new context written $\Gamma, \varphi$ (assuming that $\varphi$ only depends on the dimension variables in $\Gamma$). We call terms and types in such a restricted context *partial.* These restricted contexts are used for specifying the boundary of higher dimensional cubes, for example if $A$ is a line depending on $i$ the partial type $i : \mathbb{I}, (i = 0) \vee (i = 1) \vdash A$ is only the two endpoints of $A$. If $\Gamma, \varphi \vdash v : A$, we write $\Gamma \vdash u : A[\varphi \mapsto v]$ to denote the two judgments:

$$\Gamma \vdash u : A \qquad\qquad \Gamma, \varphi \vdash u = v : A$$

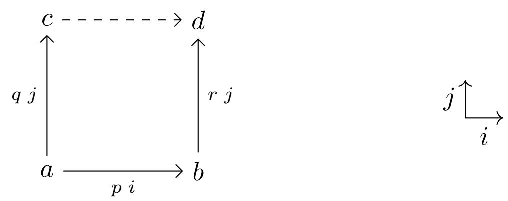
Using this we can express the typing rule for the composition operations:

$$\frac{\Gamma, i : \mathbb{I} \vdash A \qquad \Gamma \vdash \varphi : \mathbb{F} \qquad \Gamma, \varphi, i : \mathbb{I} \vdash u : A \qquad \Gamma \vdash u_0 : A(i/0)[\varphi \mapsto u(i/0)]}{\Gamma \vdash \mathsf{comp}^i\ A\ [\varphi \mapsto u]\ u_0 : A(i/1)[\varphi \mapsto u(i/1)]}$$

This operation takes a line type $A$, a formula $\varphi$, a partial line term $u$ and a term $u_0$ of type $A(i/0)$ (note that $i$ may occur freely in $A$ and $u$). Furthermore we require that $\Gamma, \varphi \vdash u_0 = u(i/0) : A(i/0)$. The result is a term in $A(i/1)$ such that $\mathsf{comp}^i\ A\ [\varphi \mapsto u]\ u_0 = u(i/1)$

on $\Gamma, \varphi$. The computation rules for the composition operations are given as judgmental equalities defined by cases on the type $A$.

The intuition is that $u$ specifies the sides of an open box while $u_0$ specifies the bottom of the box and the fact that the sides have to be connected to the bottom is expressed by the equation relating $u_0$ and $u(i/0)$. The result of the composition operation is then the lid of this open box. For example given a dimension variable $i : \mathbb{I}$ and paths $p$, $q$ and $r$ as indicated in:




the composition $\mathsf{comp}^j\ A\ [(i = 0) \mapsto q\ j, (i = 1) \mapsto r\ j]\ (p\ i)$ is the dashed line at the top of the square.[4] Here $p$ is a line in $A(i/0)$ while $q$ and $r$ are lines in $A$ (so these can be thought of as heterogeneous equalities where the left endpoint is in $A(i/0)$ and the right is in $A(i/1)$). The resulting composition is then a line in $A(i/1)$.

The composition operations allows us to define transport from a line type:

$$\frac{\Gamma, i : \mathbb{I} \vdash A \qquad \Gamma \vdash u_0 : A(i/0)}{\Gamma \vdash \mathsf{transport}^i\ A\ u_0 = \mathsf{comp}^i\ A\ []\ u_0 : A(i/1)}$$

Combined with "contractibility of singletons" (which is directly provable using a connection) we get the induction principle for $\mathsf{Path}$-types, which means that they behave like Martin-Löf's identity types (modulo the computation rule for the induction principle which only holds up to a $\mathsf{Path}$).

The $\mathsf{Glue}$-types allow us to prove both the univalence axiom and that the universe has a composition operation, however as they do not play an important role in this paper we omit them from this introduction to cubical type theory.

## 3.2 A common pattern for higher inductive types

All of the examples of higher inductive types that we consider in this paper follow a common pattern. Each such higher inductive type $\mathsf{D}(\vec{z} : \vec{P})$ is specified by a telescope[5] of parameters $\vec{z} : \vec{P}$ (over an ambient context $\Gamma$) and a *list* of constructors $\vec{\mathsf{c}}$. Each $\mathsf{c}$ in $\vec{\mathsf{c}}$ is specified by the data:

$$\mathsf{c} : (\vec{x} : \vec{A}(\vec{z}))\ [\vec{i}]\ \mathsf{D}(\vec{z})[\varphi(\vec{i}) \mapsto e(\vec{z}, \vec{x}, \vec{i})]$$

Here the telescope $\vec{x} : \vec{A}$ specifies the types of the arguments to $\mathsf{c}$, and in the case of *recursive* higher inductive types, as in, e.g., propositional truncation, $\mathsf{D}$ might itself appear in $\vec{A}$. The length of the list of names $\vec{i}$ specifies the dimension of the cube $\mathsf{c}$ introduces: we say that $\mathsf{c}$ is a *point*, *path*, or *square* constructor according to whether the length of $\vec{i}$ is $0, 1$, or $2$, respectively. The data $\varphi \mapsto e$ specifies the *endpoints* of the constructor $\mathsf{c}$, with $\varphi$ an element of the face lattice $\mathbb{F}$ whose free variables are among $\vec{i}$, and $e$ is a partial element

$$\vec{z} : \vec{P}, \vec{x} : \vec{A}(\vec{z}), \vec{i} : \mathbb{I}, \varphi(\vec{i}) \vdash e(\vec{z}, \vec{x}, \vec{i}) : \mathsf{D}(\vec{z})$$

[4] Note that we are using a notation for the "system" $[i = 0) \mapsto q\ j, (i = 1) \mapsto r\ j]$. Formally this is given by the formula $(i = 0) \vee (i = 1)$ and a partial element with endpoints $q\ j$ and $r\ j$.

[5] A *telescope* $x_1 : A_1, \ldots, x_n : A_n$ (written as $\vec{x} : \vec{A}$) over a context $\Gamma$ is a (possibly empty) list of object variable declarations such that $\Gamma, \vec{x} : \vec{A}$ is a well-formed context, so $\vec{x} : \vec{A}$ neither contains context restrictions $\Delta, \varphi$ nor dimension variables $i : \mathbb{I}$.

mentioning only previous constructors in the list $\vec{\mathsf{c}}$ and possibly $\mathsf{hcomp}$'s (see below).

For each instance $\vec{u} : \vec{P}$ of the telescope $\vec{z} : \vec{P}$ we say that $\mathsf{D}(\vec{u})$ is a type and we will have an introduction rule for a constructors $\mathsf{c}$ specified as above

$$\frac{\vec{v} : \vec{A}(\vec{u}) \qquad \vec{r} : \mathbb{I}}{\mathsf{c}\,\vec{v}\,\vec{r} : \mathsf{D}(\vec{u})}$$

and a judgmental equality $\mathsf{c}\,\vec{v}\,\vec{r} = e(\vec{u}, \vec{v}, \vec{r}) : \mathsf{D}(\vec{u})$ in case we additionally have $\varphi(\vec{r}) = 1 : \mathbb{F}$ (all in an ambient context). Note that this judgmental equality for $\mathsf{c}$ requires us to make sure that whenever we define a function $\mathsf{f} : \Pi(x : \mathsf{D}(\vec{u}))\, P(x)$ that its semantics preserve this equality, so that

$$\varphi(\vec{r}) \vdash \mathsf{f}(\mathsf{c}\,\vec{v}\,\vec{r}) = \mathsf{f}(e(\vec{u}, \vec{v}, \vec{r})) : P(\mathsf{c}\,\vec{v}\,\vec{r}).$$

This requirement has to be taken care in particular in the typing rules for the eliminator for $\mathsf{D}(\vec{u})$.

Recall from Section 2 on semantics that we decomposed the composition structure for higher inductive types into a homogeneous composition structure and a transport structure. The homogeneous composition structure was introduced as constructors and the same is reflected in the syntax by adding a rule

$$\frac{\Gamma \vdash \vec{u} : \vec{P} \qquad \Gamma \vdash \varphi : \mathbb{F} \qquad \Gamma, i : \mathbb{I}, \varphi \vdash v : \mathsf{D}(\vec{u}) \qquad \Gamma \vdash v_0 : \mathsf{D}(\vec{u})[\varphi \mapsto v(i/0)]}{\Gamma \vdash \mathsf{hcomp}^i_{\mathsf{D}(\vec{u})}\,[\varphi \mapsto v]\, v_0 : \mathsf{D}(\vec{u})[\varphi \mapsto v(i/1)]}$$

where the key point is that $i$ may be free in $v$, but *not* in $D(\vec{u})$, as opposed to the composition operations where $i$ may be free in both $v$ and $D(\vec{u})$. In the examples we will not repeat these homogeneous composition constructors for every higher inductive type we consider and they are always assumed to be included as part of the definition of the higher inductive type under consideration.

We could do the same for traditional inductive types like the natural numbers and have a constructor $\mathsf{hcomp}^i_{\mathsf{N}}$ instead of explaining composition by recursion. We can prove that this "weaker" form of natural numbers type is equivalent, and hence equal (by univalence) to the regular one.

To reflect the transport structure in the syntax we specify a $\mathsf{trans}$ operation for higher inductive types $A := \mathsf{D}(\vec{u})$ given $\Gamma, i : \mathbb{I} \vdash \vec{u} : \vec{P}$ by the rule:

$$\frac{\Gamma \vdash \varphi : \mathbb{F} \qquad \Gamma, i : \mathbb{I}, \varphi \vdash A = A(i/0) \qquad \Gamma \vdash u_0 : A(i/0)}{\Gamma \vdash \mathsf{trans}^i\, A\, \varphi\, u_0 : A(i/1)[\varphi \mapsto u_0]}$$

Note that since $\Gamma, i : \mathbb{I}, \varphi \vdash A = A(i/0)$ also $\Gamma, \varphi \vdash A(i/0) = A(i/1)$ (and hence this equation also holds in context $\Gamma, i : \mathbb{I}, \varphi$).

Similar to how the transport structure is explained in the semantics by recursion on the argument we will add a judgmental equality for each of the possible shapes of $u_0$: one for each constructor $\mathsf{c}$ and one for the $\mathsf{hcomp}$ constructor:

$$\mathsf{trans}^i A\,\varphi\,(\mathsf{hcomp}^j_{A(i/0)}\,[\psi \;\mapsto\; u]\, u_0) \;=\; \mathsf{hcomp}^j_{A(i/1)}\,[\psi \;\mapsto\; \mathsf{trans}^i A\,\varphi\, u]\,(\mathsf{trans}^i A\,\varphi\, u_0)$$

(Note that we can assume that $i \neq j$ as we can always rename one of them as they are both bound.) As the $\mathsf{hcomp}$ case is the same for all examples we omit it from the definition of $\mathsf{trans}$ for the higher inductive types considered in Section 3.3.

We can define a derived "$\mathsf{squeeze}$" operation:

$$\frac{\Gamma \vdash \varphi : \mathbb{F} \qquad \Gamma, i : \mathbb{I}, \varphi \vdash A = A(i/0) \qquad \Gamma, i : \mathbb{I} \vdash a : A}{\Gamma, i : \mathbb{I} \vdash \mathsf{squeeze}^i\, A\,\varphi\, a := \mathsf{trans}^j\, A(i/i \vee j)\,(\varphi \vee (i = 1))\, a : A(i/1)}$$

This operation satisfies

$$
\begin{aligned}
(\mathsf{squeeze}^i\, A\, \varphi\, a)(i/0) &= \mathsf{trans}^j\, A(i/j)\, \varphi\, a(i/0)\\
(\mathsf{squeeze}^i\, A\, \varphi\, a)(i/1) &= a(i/1)
\end{aligned}
$$

and the induced path is constantly $a$ on $\varphi$.

Assuming that we have defined trans for a higher inductive type $\Gamma, i : \mathbb{I} \vdash A$ we can define the composition operation:

$$
\frac{\Gamma \vdash \varphi : \mathbb{F} \qquad \Gamma, i : \mathbb{I}, \varphi \vdash u : A \qquad \Gamma \vdash u_0 : A(i/0)[\varphi \mapsto u(i/0)]}{\Gamma \vdash \mathsf{comp}^i\, A\, [\varphi \mapsto u]\, u_0 := \quad \mathsf{hcomp}^i_{A(i/1)}\, [\varphi \mapsto \mathsf{squeeze}^i\, A\, 0_{\mathbb{F}}\, u]\, (\mathsf{trans}^i\, A\, 0_{\mathbb{F}}\, u_0) : A(i/1)}
$$

This satisfies the required judgmental computation rule for comp because of the computation rules for hcomp and squeeze. This means that in order to define the composition operation for a higher inductive type we only need to define the trans operation when applied to constructors.

Note, that we can always define a trans operation for any type $\Gamma, i : \mathbb{I} \vdash A$ that already has a composition operation:

$$
\frac{\Gamma \vdash \varphi : \mathbb{F} \qquad \Gamma, i : \mathbb{I}, \varphi \vdash A = A(i/0) \qquad \Gamma \vdash u_0 : A(i/0)}{\Gamma \vdash \mathsf{ctrans}^i\, A\, \varphi\, u_0 := \mathsf{comp}^i\, A\, [\varphi \mapsto u_0]\, u_0 : A(i/1)[\varphi \mapsto u_0]}
$$

In line with Lemma 11 a corresponding "filling" operation which connects the input of trans to its output can also be derived:

$$
\frac{\Gamma \vdash \varphi : \mathbb{F} \qquad \Gamma, i : \mathbb{I}, \varphi \vdash A = A(i/0) \qquad \Gamma \vdash u_0 : A(i/0)}{\Gamma, i : \mathbb{I} \vdash \mathsf{transFill}^i\, A\, \varphi\, u_0 := \mathsf{trans}^j\, A(i/i \wedge j)\, (\varphi \vee (i = 0))\, u_0 : A}
$$

Note that $\Gamma, i : \mathbb{I}, \varphi \vdash A = A(i/0)$ entails

$$
\Gamma, i : \mathbb{I}, j : \mathbb{I}, \varphi \vee (i = 0) \vdash A(i/i \wedge j) = A(i/i \wedge j)(j/0).
$$

This operation satisfies

$$
\begin{aligned}
(\mathsf{transFill}^i\, A\, \varphi\, u_0)(i/0) &= u_0\\
(\mathsf{transFill}^i\, A\, \varphi\, u_0)(i/1) &= \mathsf{trans}^j\, A(i/j)\, \varphi\, u_0
\end{aligned}
$$

and the induced path is constantly $u_0$ on $\varphi$. We write ctransFill for the corresponding operation defined using ctrans.

## 3.3 Examples of higher inductive types

In this section we describe how to extend cubical type theory with the circle and spheres, torus, suspensions, propositional truncation, and pushouts. As with all the other type formers we have to explain their formation, introduction, elimination, and computation rules, as well as how composition computes. All of these examples follow the common pattern presented in the previous section.

### 3.3.1 The circle and spheres

The extension of cubical type theory with the circle and spheres was sketched in [7, Section 9.2].

**Formation** In order to extend the theory with the circle we first add it as a type by:

$$\frac{}{\vdash \mathbb{S}^1} \qquad\qquad \frac{}{\mathbb{S}^1 : \mathsf{U}}$$

**Introduction** The circle is generated by a point and a path constructor:

$$\frac{}{\mathsf{base} : \mathbb{S}^1} \qquad\qquad \frac{r : \mathbb{I}}{\mathsf{loop}\, r : \mathbb{S}^1}$$

with the judgmental equalities $\mathsf{loop}\, 0 = \mathsf{loop}\, 1 = \mathsf{base}$ so that $\mathsf{loop}$ connects the point to itself.

**Elimination** Given a dependent type $x : \mathbb{S}^1 \vdash P(x)$, a term $b : P(\mathsf{base})$ and a path $i : \mathbb{I} \vdash l : P(\mathsf{loop}\, i)[(i = 0) \vee (i = 1) \mapsto b]$ we can define $\mathsf{f} : \Pi(x : \mathbb{S}^1)\, P(x)$ by cases:

$$\mathsf{f}\, \mathsf{base} = b \qquad\qquad \mathsf{f}\, (\mathsf{loop}\, r) = l\, r$$

and for the $\mathsf{hcomp}$ constructor:

$$\mathsf{f}\, (\mathsf{hcomp}^i_{\mathbb{S}^1}\, [\varphi \mapsto u]\, u_0) = \mathsf{comp}^i\, P(v)\, [\varphi \mapsto \mathsf{f}\, u]\, (\mathsf{f}\, u_0)$$

where w.l.o.g. $i$ is fresh and:

$$\begin{aligned} v &:= \mathsf{hfill}^i\, \mathbb{S}^1\, [\varphi \mapsto u]\, u_0 \\ &= \mathsf{hcomp}^j_{\mathbb{S}^1}\, [\varphi \mapsto u(i/i \wedge j), (i = 0) \mapsto u_0]\, u_0 \end{aligned}$$

As the equation for the eliminator applied to an $\mathsf{hcomp}$ is analogous for all the other higher inductive considered here we will omit it in the sequel.

Using this we can directly define the eliminator:

$$\frac{x : \mathbb{S}^1 \vdash P(x) \qquad b : P(\mathsf{base}) \qquad l : \mathsf{Path}^i\, P(\mathsf{loop}\, i)\, b\, b \qquad u : \mathbb{S}^1}{\mathbb{S}^1\text{-}\mathsf{elim}_{x.P}\ b\, l\, u : P(u)}$$

where $\mathsf{Path}^i$ denotes the heterogeneous path type (see [7, Section 9.2]). The judgmental computation rules then follow from the definition above. As we have dependent $\mathsf{Path}$-types (which behave like heterogeneous equalities) the $\mathsf{loop}$ case of $\mathsf{f}$ can be expressed directly by an equation without "$\mathsf{apd}$" and $l$ does not involve any $\mathsf{transport}$ as opposed to [23, Section 6.4].

**Composition** As $\mathbb{S}^1$ has no parameters we let $\mathsf{trans}^i\, \mathbb{S}^1\, \varphi\, u_0 = u_0$. This means that the composition $\mathsf{comp}^i\, \mathbb{S}^1\, [\varphi \mapsto u]\, u_0$ computes directly to the constructor $\mathsf{hcomp}^i_{\mathbb{S}^1}\, [\varphi \mapsto u]\, u_0$.

The higher dimensional spheres, $\mathbb{S}^n$, can directly be defined by generalizing the definition $\mathbb{S}^1$ so that $\mathsf{loop}$ takes $r_1, \dots, r_n : \mathbb{I}$. It is trivial to define $\mathsf{trans}^i\, \mathbb{S}^n\, \varphi\, u_0$ in analogy with $\mathbb{S}^1$. The elimination is also analogous to that of $\mathbb{S}^1$ using an $n$-dimensional cube in $P(\mathsf{loop}\, r_1\, \dots\, r_n)$ for the $\mathsf{loop}$ case.

### 3.3.2 The torus; two equivalent formulations

We define the torus in two ways, the first one (written $\mathbb{T}$) is analogous to $\mathbb{S}^2$ and the second (written $\mathbb{T}_\mathsf{F}$) is the cubical analogue of the torus as defined in [23, Section 6.6]. The $\mathbb{T}_\mathsf{F}$ torus involves the fibrant structure of the 1-dimensional cells in the 2-dimensional cell. Higher inductive types of this kind are not supported by [17] and we make crucial use of the fact that we have homogeneous composition as a constructor in order to represent them.

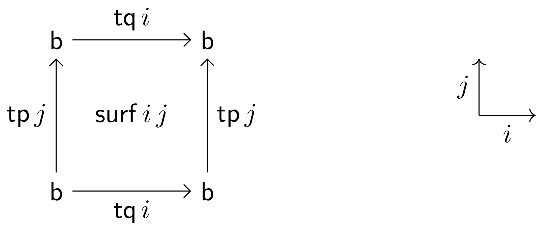
**Formation** The formation rules for the torus types are given by:

$$\frac{}{\vdash \mathbb{T}} \qquad \frac{}{\mathbb{T} : \mathsf{U}} \qquad \frac{}{\vdash \mathbb{T}_\mathsf{F}} \qquad \frac{}{\mathbb{T}_\mathsf{F} : \mathsf{U}}$$

**Introduction** The point, lines and square constructors for $\mathbb{T}$ are given by:

$$\frac{}{\mathsf{b} : \mathbb{T}} \qquad \frac{r : \mathbb{I}}{\mathsf{tp}\, r : \mathbb{T}} \qquad \frac{r : \mathbb{I}}{\mathsf{tq}\, r : \mathbb{T}} \qquad \frac{r : \mathbb{I} \qquad s : \mathbb{I}}{\mathsf{surf}\, r\, s : \mathbb{T}}$$

satisfying $\mathsf{tp}\, 0 = \mathsf{tp}\, 1 = \mathsf{tq}\, 0 = \mathsf{tq}\, 1 = \mathsf{b}$. The constructors for $\mathbb{T}_\mathsf{F}$ are defined by the same rules as for $\mathbb{T}$ and we write them with $\mathsf{F}$ in subscript. The square constructor for $\mathbb{T}$ satisfies $\mathsf{surf}\, 0\, s = \mathsf{surf}\, 1\, s = \mathsf{tp}\, s$ and $\mathsf{surf}\, r\, 0 = \mathsf{surf}\, r\, 1 = \mathsf{tq}\, r$ so that we get the square representing the gluing diagram used in the topological definition of the torus:

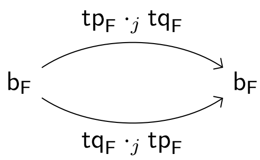



Given $s : \mathbb{I}$ we define the composition of $\mathsf{tp}_\mathsf{F}$ and $\mathsf{tq}_\mathsf{F}$ by:

$$\mathsf{tp}_\mathsf{F} \cdot_s \mathsf{tq}_\mathsf{F} := \mathsf{hcomp}^i_{\mathbb{T}_\mathsf{F}}\, [(s = 0) \mapsto \mathsf{b}_\mathsf{F}, (s = 1) \mapsto \mathsf{tq}_\mathsf{F}\, i]\, (\mathsf{tp}_\mathsf{F}\, s)$$

The composition $\mathsf{tq}_\mathsf{F} \cdot_s \mathsf{tp}_\mathsf{F}$ is defined analogously.

The square constructor for $\mathbb{T}_\mathsf{F}$ satisfies $\mathsf{surf}_\mathsf{F}\, 0\, s = \mathsf{tp}_\mathsf{F} \cdot_s \mathsf{tq}_\mathsf{F}$, $\mathsf{surf}_\mathsf{F}\, 1\, s = \mathsf{tq}_\mathsf{F} \cdot_s \mathsf{tp}_\mathsf{F}$ and $\mathsf{surf}_\mathsf{F}\, r\, 0 = \mathsf{surf}_\mathsf{F}\, r\, 1 = \mathsf{b}_\mathsf{F}$. This way the 2-cell $\langle i\, j \rangle\, \mathsf{surf}_\mathsf{F}\, i\, j$ corresponds to a cubical version of the globe (which can be turned into a square with reflexivity at $\mathsf{b}_\mathsf{F}$ as sides):




**Elimination** We write $(i = 0/1)$ for $(i = 0) \vee (i = 1)$. Given a dependent type $x : \mathbb{T} \vdash P(x)$, a term $b : P(\mathsf{b})$, paths $i : \mathbb{I} \vdash l_p : P(\mathsf{tp}\, i)[(i = 0/1) \mapsto b]$ and $i : \mathbb{I} \vdash l_q : P(\mathsf{tq}\, i)[(i = 0/1) \mapsto b]$ and a square $i, j : \mathbb{I} \vdash s_{pq} : P(\mathsf{surf}\, i\, j)\, [(i = 0/1) \mapsto l_p\, j, (j = 0/1) \mapsto l_q\, i]$ we can define $\mathsf{f} : \Pi(x : \mathbb{T})\, P(x)$ by cases:

$$\begin{aligned} \mathsf{f}\, \mathsf{b} &= b \\ \mathsf{f}\, (\mathsf{tp}\, r) &= l_p\, r \\ \mathsf{f}\, (\mathsf{tq}\, r) &= l_q\, r \\ \mathsf{f}\, (\mathsf{surf}\, r\, s) &= s_{pq}\, r\, s \end{aligned}$$

Similarly for a dependent type $x : \mathbb{T}_\mathsf{F} \vdash P(x)$, a term $b : P(\mathsf{b}_\mathsf{F})$, paths $i : \mathbb{I} \vdash l_p : P(\mathsf{tp}_\mathsf{F}\, i)[(i = 0/1) \mapsto b]$ and $i : \mathbb{I} \vdash l_q : P(\mathsf{tq}_\mathsf{F}\, i)[(i = 0/1) \mapsto b]$ we define:

$$l_p \cdot_j l_q := \mathsf{comp}^i\, P(v)\, [(j = 0) \mapsto b, (j = 1) \mapsto l_q\, i]\, (l_p\, j)$$

where $v := \mathsf{hfill}^i_{\mathbb{T}_\mathsf{F}}\ [(j=0) \mapsto \mathsf{b}_\mathsf{F}, (j=1) \mapsto \mathsf{tq}_\mathsf{F}\ i]\ (\mathsf{tp}_\mathsf{F}\ j)$. We define $l_q \cdot_j l_p$ analogously and we can then require a square $i, j : \mathbb{I} \vdash s_{pq} : P(\mathsf{surf}_\mathsf{F}\ i\ j)\ [(i=0) \mapsto l_p \cdot_j l_q, (i=1) \mapsto l_q \cdot_j l_p, (j = 0/1) \mapsto b]$. Using this we can define $\mathsf{f} : \Pi(x : \mathbb{T}_\mathsf{F})\ P(x)$ by cases like for $\mathbb{T}$.

Working with $\mathbb{T}$ is easier than $\mathbb{T}_\mathsf{F}$ and the proof that $\mathbb{T} \simeq \mathbb{S}^1 \times \mathbb{S}^1$ has been formalized in CUBICALTT by Dan Licata.[6] The proof of this is very direct and a lot shorter than the existing proofs in the literature [21, 15]. One first defines a maps $f_1 : \mathbb{T} \to \mathbb{S}^1 \times \mathbb{S}^1$ and $f_2 : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{T}$ by:

$$\begin{aligned} f_1\ \mathsf{b} &= (\mathsf{base}, \mathsf{base}) & f_2\ (\mathsf{base}, \mathsf{base}) &= \mathsf{b} \\ f_1\ (\mathsf{tp}\ r) &= (\mathsf{loop}\ r, \mathsf{base}) & f_2\ (\mathsf{loop}\ r, \mathsf{base}) &= \mathsf{tp}\ r \\ f_1\ (\mathsf{tq}\ r) &= (\mathsf{base}, \mathsf{loop}\ r) & f_2\ (\mathsf{base}, \mathsf{loop}\ r) &= \mathsf{tq}\ r \\ f_1\ (\mathsf{surf}\ r\ s) &= (\mathsf{loop}\ r, \mathsf{loop}\ s) & f_2\ (\mathsf{loop}\ r, \mathsf{loop}\ s) &= \mathsf{surf}\ r\ s \end{aligned}$$

These are obviously inverses and the equivalence can be established. The formal proof in CUBICALTT is slightly more complicated as it is not possible to directly do the double recursion in $f_2$, but the basic idea is the same. This example shows how having a system in which higher inductive types which compute also for higher constructors makes it possible to simplify formal proofs in synthetic homotopy theory.

**Composition** As neither $\mathbb{T}$ or $\mathbb{T}_\mathsf{F}$ have any parameters the transport operation is trivial just like for $\mathbb{S}^n$, so the composition operations reduces to the $\mathsf{hcomp}$ constructors.

### 3.3.3 Suspension

The suspension of a type $A$, written $\Sigma A$, is more involved than the higher inductive types considered so far as it has a parameter and just as in the semantics we have to explain the transport operation.

**Formation** In order to extend the theory with suspensions we add the rules:

$$\frac{\vdash A}{\vdash \Sigma A} \qquad\qquad \frac{A : \mathsf{U}}{\Sigma A : \mathsf{U}}$$

Note that we allow $\Sigma A$ to be in the same universe as $A$, this is justified by the semantics as explained in Section 2.5.

**Introduction** The suspensions are generated by:

$$\frac{}{\mathsf{N} : \Sigma A} \qquad\qquad \frac{}{\mathsf{S} : \Sigma A} \qquad\qquad \frac{a : A \qquad r : \mathbb{I}}{\mathsf{merid}\ a\ r : \Sigma A}$$

satisfying $\mathsf{merid}\ a\ 0 = \mathsf{N}$ and $\mathsf{merid}\ a\ 1 = \mathsf{S}$.

**Elimination** Given a dependent type $x : \Sigma A \vdash P(x)$, terms $n : P(\mathsf{N})$ and $s : P(\mathsf{S})$ and a family of paths $x : A, i : \mathbb{I} \vdash m(x, i) : P(\mathsf{merid}\ x\ i)[(i=0) \mapsto n, (i=1) \mapsto s]$ we can define a function $\mathsf{f} : \Pi(x : \Sigma A)\ P(x)$ by cases:

$$\begin{aligned} \mathsf{f}\ \mathsf{N} &= n \\ \mathsf{f}\ \mathsf{S} &= s \\ \mathsf{f}\ (\mathsf{merid}\ a\ r) &= m(a, r) \end{aligned}$$

[6]See: `https://github.com/mortberg/cubicaltt/blob/hcomptrans/examples/torus.ctt`

**Composition** The $\mathsf{trans}^i\,(\Sigma A)\,\varphi\,u_0$ operation is defined by cases

$$\mathsf{trans}^i\,(\Sigma A)\,\varphi\,\mathsf{N} = \mathsf{N}$$
$$\mathsf{trans}^i\,(\Sigma A)\,\varphi\,\mathsf{S} = \mathsf{S}$$
$$\mathsf{trans}^i\,(\Sigma A)\,\varphi\,(\mathsf{merid}\,a\,r) = \mathsf{merid}\,(\mathsf{ctrans}^i\,A\,\varphi\,a)\,r$$

### 3.3.4 Propositional truncations

Another class of interesting higher inductive types are the truncations, these introduce some new complications as they are recursive in the sense that the higher constructors quantify over elements of the type. The propositional truncation takes a type $A$ and "squashes" it to a 0-type $\|A\|$ (in the sense that the equality type of $\|A\|$ has no interesting structure).

**Formation** In order to extend the theory with propositional truncation we add the rules:

$$\frac{\vdash A}{\vdash \|A\|} \qquad \frac{A : \mathsf{U}}{\|A\| : \mathsf{U}}$$

**Introduction** The propositional truncation of $A$ is generated by:

$$\frac{a : A}{\mathsf{inc}\,a : \|A\|} \qquad \frac{v : \|A\| \qquad w : \|A\| \qquad r : \mathbb{I}}{\mathsf{sq}\,v\,w\,r : \|A\|}$$

satisfying $\mathsf{sq}\,v\,w\,0 = v$ and $\mathsf{sq}\,v\,w\,1 = w$.

**Elimination** Given a dependent type $x : \|A\| \vdash P(x)$, a family of terms $x : A \vdash t(x) : P(\mathsf{inc}\;x)$ and family of paths $v, w : \|A\|, x : P(v), y : P(w), i : \mathbb{I} \vdash p(v,w,x,y,i) : P(\mathsf{sq}\,v\,w\,i)[(i=0) \mapsto x, (i=1) \mapsto y]$ we can define $\mathsf{f} : \Pi(x : \|A\|)\,P(x)$ by cases:

$$\mathsf{f}\,(\mathsf{inc}\,a) = t(a)$$
$$\mathsf{f}\,(\mathsf{sq}\,v\,w\,r) = p(v, w, \mathsf{f}\,v, \mathsf{f}\,w, r)$$

This is directly structurally recursive and the only difference compared to $\Sigma A$ is that we have to make a recursive call for each recursive argument.

**Composition** We define $\mathsf{trans}^i\|A\|\,\varphi\,u_0$ by cases on $u_0$:

$$\mathsf{trans}^i\|A\|\,\varphi\,(\mathsf{inc}\;a) = \mathsf{inc}\,(\mathsf{ctrans}^i\,A\,\varphi\,a)$$
$$\mathsf{trans}^i\|A\|\,\varphi\,(\mathsf{sq}\,v\,w\,r) = \mathsf{sq}\,(\mathsf{trans}^i\|A\|\,\varphi\,v)\,(\mathsf{trans}^i\|A\|\,\varphi\,w)\,r$$

The explanation of propositional truncation in [7, Section 9.2] used a similar decomposition, but the introduction of the trans operation allows a much simpler formulation.

### 3.3.5 Pushouts

The definition of pushouts in cubical type theory is similar to the other parametrized higher inductive types, but special care has to be taken when defining trans as the endpoints of the path constructors involve the parameters to the pushout.

**Formation** We extend the theory with:

$$\frac{\vdash A \qquad \vdash B \qquad \vdash C \qquad u : C \to A \qquad v : C \to B}{\vdash A \sqcup_C B}$$

$$\frac{A : \mathsf{U} \qquad B : \mathsf{U} \qquad C : \mathsf{U} \qquad u : C \to A \qquad v : C \to B}{A \sqcup_C B : \mathsf{U}}$$

**Introduction** Given $u : C \to A$ and $v : C \to B$ the pushout is generated by:

$$\frac{a : A}{\mathsf{inl}\, a : A \sqcup_C B} \qquad \frac{b : B}{\mathsf{inr}\, b : A \sqcup_C B} \qquad \frac{c : C \qquad r : \mathbb{I}}{\mathsf{push}\, c\, r : A \sqcup_C B}$$

satisfying $\mathsf{push}\, c\, 0 = \mathsf{inl}(u\, c)$ and $\mathsf{push}\, c\, 1 = \mathsf{inr}(v\, c)$. Note that $\langle i \rangle\, \mathsf{push}\, c\, i$ gives a path between $\mathsf{inl}(u\, c)$ and $\mathsf{inr}(v\, c)$ for all $c : C$ as desired. The reason we do not call the path constructor "glue" like in [23, Section 6.8] is that this is already used for the constructor for the Glue-types.

**Elimination** Given a dependent type $x : A \sqcup_C B \vdash P(x)$, families of terms $x : A \vdash l(x) : P(\mathsf{inl}\ x)$ and $x : B \vdash r(x) : P(\mathsf{inr}\ x)$ and a family of paths $x : C, i : \mathbb{I} \vdash p(x, i) : P(\mathsf{push}\, x\, i)[(i = 0) \mapsto l(u\, x), (i = 1) \mapsto r(v\, x)]$ we can define $\mathsf{f} : \Pi(x : A \sqcup_C B)\, P(x)$ by cases:

$$\begin{aligned} \mathsf{f}\,(\mathsf{inl}\, a) &= l(a) \\ \mathsf{f}\,(\mathsf{inr}\, b) &= r(b) \\ \mathsf{f}\,(\mathsf{push}\, c\, r) &= p(c, r) \end{aligned}$$

**Composition** We write $P$ for $A \sqcup_C B$ and the judgmental computation rules for trans are defined by cases:

$$\begin{aligned} \mathsf{trans}^i\, P\, \varphi\, (\mathsf{inl}\, a) &= \mathsf{inl}\, (\mathsf{ctrans}^i\, A\, \varphi\ a) \\ \mathsf{trans}^i\, P\, \varphi\, (\mathsf{inr}\, b) &= \mathsf{inr}\, (\mathsf{ctrans}^i\, B\, \varphi\ b) \\ \mathsf{trans}^i\, P\, \varphi\, (\mathsf{push}\, c\, r) &= \mathsf{hcomp}^i_{P(i/1)}\, S\, (\mathsf{push}\, (\mathsf{ctrans}^i\, C\, \varphi\, c)\, r) \end{aligned}$$

where $S$ is the system:

$$\begin{aligned} [&(r = 0) \mapsto \mathsf{squeeze}^i\, P\, \varphi\, (\mathsf{inl}(u\, (\mathsf{ctransFill}^i\, C\, \varphi\, c)))\, (i/1 - i), \\ &(r = 1) \mapsto \mathsf{squeeze}^i\, P\, \varphi\, (\mathsf{inr}(v\, (\mathsf{ctransFill}^i\, C\, \varphi\, c)))\, (i/1 - i), \\ &(\varphi = 1) \mapsto \mathsf{push}\, c\, r] \end{aligned}$$

Note that the recursive call to squeeze is justified as it is applied to a point constructor already explained.

Furthermore, note that the endpoint correction for $\mathsf{push}\, c\, r$ is necessary as, for example, in the case where $r$ is a dimension variable $j$ the path constructor $\mathsf{push}\, (\mathsf{ctrans}^i\, C\, \varphi\, c)\, j$ connects

$$\mathsf{inl}(u(i/1)\, (\mathsf{ctrans}^i\, C\, \varphi\, c)) \qquad \text{to} \qquad \mathsf{inr}(v(i/1)\, (\mathsf{ctrans}^i\, C\, \varphi\, c))$$

in direction $j$, but we require something that connects

$$\mathsf{inl}(\mathsf{ctrans}^i\, A\, \varphi\, (u(i/0)\, c)) \qquad \text{to} \qquad \mathsf{inr}(\mathsf{ctrans}^i\, B\, \varphi\, (v(i/0)\, c))$$

since the definition of trans should be stable under the substitutions $(j/0)$ and $(j/1)$. To see that the correction is correct at $(r = 0)$ note that $\mathsf{squeeze}^i\, P\, \varphi\, (\mathsf{inl}(u\, (\mathsf{ctransFill}^i\, C\, \varphi\, c)))\, (i/1 - i)$ connects

$$\mathsf{inl}(u(i/1)\, (\mathsf{ctrans}^i\, C\, \varphi\, c)) \qquad \text{to} \qquad \mathsf{inl}(\mathsf{ctrans}^i\, A\, \varphi\, (u(i/0)\, c))$$

as required.

## 3.4 A variation on cubical type theory

In the previous section we have seen that the equations to define $\mathsf{trans}^i\,A$ for a higher inductive type $A$ applied to a constructor involves $\mathsf{trans}^i\,A$ for the recursive arguments to the constructor (see the equation for $\mathsf{sq}\,v\,w\,r$ for propositional truncation in Section 3.3.4), and involves the derived operations $\mathsf{ctrans}$ for non-recursive arguments (e.g., in the equation for $\mathsf{merid}\,a\,r$ in Section 3.3.3). In general, $\mathsf{trans}$ and $\mathsf{ctrans}$ which are available for $A$ do not coincide definitionally, making it impossible to treat the recursive and non-recursive arguments to a constructor uniformly.

This mismatch suggests a variant of cubical type theory where the operations $\mathsf{trans}$ and $\mathsf{hcomp}$ are taken as primitives and $\mathsf{comp}$ is instead a derived operation as we did here for higher inductive types. We can then explain $\mathsf{trans}$ and $\mathsf{hcomp}$ by cases on the shape of the type. In this variation of cubical type theory the algorithm for $\mathsf{trans}$ in a higher inductive type applied to a constructor can be uniformly described as follows.

Given a higher inductive type $\mathsf{D}(\vec{z} : \vec{P})$ specified as in Section 3.2 and a constructor $\mathsf{c}$ specified by:

$$\mathsf{c} : (\vec{x} : \vec{A}(\vec{z}))\,[\vec{i}]\,\mathsf{D}(\vec{z})[\varphi(\vec{i}) \mapsto e(\vec{z}, \vec{x}, \vec{i})] \tag{1}$$

Further, assume parameters $\Gamma, i : \mathbb{I} \vdash \vec{u} : \vec{P}$ of the higher inductive type $\mathsf{D}(\vec{z} : \vec{P})$ such that $\Gamma, i : \mathbb{I}, \psi \vdash \vec{u} = \vec{u}(i/0) : \vec{P}$ for $\Gamma \vdash \psi : \mathbb{F}$. We now explain the judgmental equalities of

$$w_1 := \mathsf{trans}^i\,\mathsf{D}(\vec{u})\,\psi\,(\mathsf{c}\,\vec{v}\,\vec{r})$$

for $\Gamma \vdash \vec{v} : \vec{A}(\vec{u}(i/0))$ and $\Gamma \vdash \vec{r} : \mathbb{I}$. This $\mathsf{c}\,\vec{v}\,\vec{r}$ restricts to $\varphi(\vec{r}) \mapsto e(\vec{u}(i/0), \vec{v}, \vec{r})$. We want to define $\Gamma \vdash w_1 : \mathsf{D}(\vec{u}(i/1))[\psi \mapsto \mathsf{c}\,\vec{v}\,\vec{r}$ such that $w_1$ restricts to

$$\varphi(\vec{r}) \mapsto \mathsf{trans}^i\,\mathsf{D}(\vec{u})\,\psi\,e(\vec{u}(i/0), \vec{v}, \vec{r}). \tag{2}$$

We get a line in $\vec{x} : \vec{A}(\vec{u})$ in the context $\Gamma, i : \mathbb{I}$

$$\vec{v} \xrightarrow{\vec{\theta} := \mathsf{transFill}^i\,(\vec{x} : \vec{A}(\vec{u}))\,\psi\,\vec{v}} \mathsf{trans}^i\,(\vec{x} : \vec{A}(\vec{u}))\,\psi\,\vec{v}$$

along $i$, where $\mathsf{transFill}^i\,(\vec{x} : \vec{A})\,\psi\,\vec{v}$ is the extension of $\mathsf{transFill}$ to telescopes, mapping the empty telescope to itself, and

$$\mathsf{transFill}^i\,(x : A, \vec{x} : \vec{A}(x))\,\psi\,(v, \vec{v}) = \tilde{v}, \mathsf{transFill}^i\,(\vec{x} : \vec{A}(\tilde{v}))\,\psi\,\vec{v}$$

with $\tilde{v} = \mathsf{transFill}^i\,A\,\psi\,v$. The extension of $\mathsf{trans}$ to telescopes is the $(i/1)$ face of the corresponding $\mathsf{transFill}$.

We start with $\Gamma \vdash w_1' : \mathsf{D}(\vec{u}(i/1))$ given by

$$w_1' := \mathsf{c}\,(\mathsf{trans}^i\,(\vec{x} : \vec{A})\,\psi\,\vec{v})\,\vec{r}$$

which restricts to $\varphi(\vec{r}) \mapsto e(\vec{u}(i/1), \mathsf{trans}^i\,(\vec{x} : \vec{A})\,\psi\,\vec{v}, \vec{r})$ and which we have to correct to match (2). To make this correction, consider the line $\Gamma, \varphi(\vec{r}), i : \mathbb{I} \vdash \alpha(i) : \mathsf{D}(\vec{u}(i/1))$ given by

$$\alpha(i) := \mathsf{squeeze}^i\,\mathsf{D}(\vec{u})\,\psi\,e(\vec{u}, \vec{\theta}, \vec{r})$$

connecting the element in (2) to $e(\vec{u}(i/1), \mathsf{trans}^i\,(\vec{x} : \vec{A})\,\psi\,\vec{v}, \vec{r})$. Note that $\alpha(i)$ coincides with $e(\vec{u}(i/0), \vec{v}, \vec{r})$ (and hence with $\mathsf{c}\,\vec{v}\,\vec{r}$) on $\psi$.

We now add the judgmental equality

$$w_1 = \mathsf{hcomp}^i_{\mathsf{D}(\vec{u}(i/1))}\,[\varphi(\vec{r}) \mapsto \alpha(1-i), \psi \mapsto \mathsf{c}\,\vec{v}\,rs]\,w_1'.$$

Note that in the definition $\alpha$ we recursively call trans for D on $e$. To ensure that this call is well-founded it is crucial to have restrictions on how $e$ may look like.

Also note that this algorithm might not be optimal: For a higher inductive type without any parameters (e.g., $\mathbb{S}^1$) we could have simply defined trans to be the identity as we did in the previous section. For a type where the endpoints of constructors are suitably simple, like suspensions and propositional truncation, but not pushouts, we could have directly taken $w_1'$ above.

Our general pattern of constructors (1) suggests to formulate a schema. Such a schema would have to ensure that $\mathsf{D}(\vec{z})$ only appears strictly positive in $\vec{A}$ and would have to restrict what possible endpoints $e$ are allowed. We leave the detailed formulation of the semantics of such a schema as future work.

# 4 Conclusions and related work

In this paper we constructed the semantics of some important higher inductive types in cubical sets. A crucial ingredient was the decomposition of the composition structure into a homogeneous composition structure and a transport structure. Using this decomposition we could define higher inductive type formers such that they preserve the universe level and are strictly stable under substitution.

We also extended cubical type theory with some higher inductive types. While [12] only proves canonicity for cubical type theory extended with the circle and propositional truncation, it should be straightforward to extend this result to the higher inductive types presented in this paper using the obvious operational semantics obtained by orienting the judgmental equalities given here. It also remains to prove normalization and decidability of type-checking for cubical type theory and in particular also for our extension with higher inductive types.

As mentioned in Section 3.4, it is more natural for a general treatment of higher inductive types to formulate a variation of cubical type theory based homogeneous compositions and transport as primitive instead of heterogeneous compositions. It seems that our description of transport for higher inductive types also works for a more general schema, but its details and semantics still have to be worked out.

Using the experimental implementation of the system presented in this paper we have formalized the "Brunerie number"[7], i.e., $n$ such that $\pi_4(\mathbb{S}^3) \simeq \mathbb{Z}/n\mathbb{Z}$. The formalization closely follows [5, Appendix B] and the definition involves multiple higher inductive types (the spheres, truncations, and join construction) together with many uses of the univalence axiom. By the classical definition of this homotopy group we know that the expected value for $n$ is $\pm 2$ and this also is proved to be the case in [5]. But as we have a constructive justification for all of the notions involved in the definition we can in principle directly obtain this numeral by computation. However, this computation so far has been unfeasible.

Further future work is to relate our semantics to other models of homotopy type theory. In particular, clarify the connection of the model structure on cubical sets [20] and the usual model structure on simplicial sets. It is also of interest to investigate to what extent the techniques developed in this paper can be adapted to the simplicial set model.[8]

**Related work** The papers [1, 3, 2, 6] presents cubical type theories inspired by an alternative cubical set category with different fibrancy structure, but with the same decomposition of the composition operation in a homogeneous composition and a transport operation. This decomposition was introduced in an early version of [7] precisely to solve the problem of the

[7]The complete self-contained formalization can be found at: `https://github.com/mortberg/cubicaltt/blob/hcomptrans/examples/brunerie.ctt`

[8]See the following discussion for more details: `https://groups.google.com/d/msg/homotopytypetheory/bNHRnGiF5R4/3RYz1YFmBQAJ`

interpretation of higher inductive types with parameters [16]. The suspensions are covered in [1], and [6] defines a schema for higher inductive types formulated in this setting. The papers [3, 2, 6] describe computational type theories in the style of Nuprl with a semantics where types are interpreted as partial equivalence relations which gives canonicity for booleans. The schema presented in [6] covers all of the examples of higher inductive types considered in this paper.

The paper [17] presents a semantics of higher inductive types in a general framework of "sufficiently nice" Quillen model categories. However as it is now, it models a type theory which does not contain any universes (see [17, pp. 5–6] for a discussion of this point).

A schema with point, path, and square constructors expressed in the style of [23] is presented in [9]. This paper also contains a semantics for these higher inductive types in the groupoid model.

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## A Appendix: construction of initial algebras

In this appendix we sketch how to construct the semantic versions of the examples of higher inductive types $T$ that we consider. With suitable definitions of $T$-algebra structures these proofs can be seen as constructions of initial $T$-algebras.

**Torus** The semantic version of $\mathbb{T}$ is very similar to that of $\mathbb{S}^1$, so we only give the semantics of $\mathbb{T}_{\mathsf{F}}$ as it is more interesting. Just as for the circle we first define an upper approximation of sets $\mathbb{T}_{\mathsf{F}}^{\text{pre}}(I)$, together with maps $\mathbb{T}_{\mathsf{F}}^{\text{pre}}(I) \to \mathbb{T}_{\mathsf{F}}^{\text{pre}}(J)$ for $f : J \to I$. An element of $\mathbb{T}_{\mathsf{F}}^{\text{pre}}(I)$ is of the form:

- $\mathsf{b}_{\mathsf{F}}$, or
- $\mathsf{tp}_{\mathsf{F}}\, r$ or $\mathsf{tq}_{\mathsf{F}}\, r$ with $r \neq 0, 1$ in $\mathbb{I}(I)$, or
- $\mathsf{surf}_{\mathsf{F}}\, r\, s$ with $r, s \neq 0, 1$ in $\mathbb{I}(I)$, or
- $\mathsf{hcomp}\ [\varphi \mapsto u]\ u_0$ with $\varphi \neq 1$ in $\mathbb{F}(I)$ and $u_0$ in $\mathbb{T}_{\mathsf{F}}^{\text{pre}}(I)$ and $u$ a family of elements $u_f$ in $\mathbb{T}_{\mathsf{F}}^{\text{pre}}(J)$ for $f : J \to I^+$ such that $\varphi\sigma f = 1$.

We write $\mathsf{tp}_{\mathsf{F}} \cdot_r \mathsf{tq}_{\mathsf{F}}$ for

$$\mathsf{hcomp}\ [(r = 0) \mapsto \mathsf{b}_{\mathsf{F}}\, \mathsf{p}, (r = 1) \mapsto \mathsf{tq}_{\mathsf{F}}\, \mathsf{q}]\ (\mathsf{tp}_{\mathsf{F}}\, r)$$

and similarly for $\mathsf{tq}_{\mathsf{F}} \cdot_r \mathsf{tp}_{\mathsf{F}}$. We define $uf$ in $\mathbb{T}_{\mathsf{F}}^{\text{pre}}(J)$ for $f : J \to I$ by induction on $u$ just like for $\mathbb{S}^1_{\text{pre}}$, the interesting case is:

$$(\mathsf{surf}_{\mathsf{F}}\, r\, s)f = \begin{cases} \mathsf{surf}_{\mathsf{F}}\, (rf)\, (sf) & \text{if } rf \neq 0, 1 \text{ and } sf \neq 0, 1 \\ \mathsf{tp}_{\mathsf{F}} \cdot_{sf} \mathsf{tq}_{\mathsf{F}} & \text{if } rf = 0 \text{ and } sf \neq 0, 1 \\ \mathsf{tq}_{\mathsf{F}} \cdot_{sf} \mathsf{tp}_{\mathsf{F}} & \text{if } rf = 1 \text{ and } sf \neq 0, 1 \\ \mathsf{b}_{\mathsf{F}} & \text{otherwise} \end{cases}$$

$$(\mathsf{hcomp}\ [\varphi \mapsto u]\ u_0)f = \begin{cases} u_{f^+ e_1} & \text{if } \varphi f = 1 \\ \mathsf{hcomp}\ [\varphi f \mapsto uf^+]\ (u_0 f) & \text{otherwise} \end{cases}$$

where $uf^+$ is the family $(uf^+)_g = u_{f^+ g}$ for $g : K \to J^+$.

We then define the subset $\mathbb{T}_{\mathsf{F}}(I) \subseteq \mathbb{T}_{\mathsf{F}}^{\text{pre}}(I)$ by taking the elements $\mathsf{b}_{\mathsf{F}}$, $\mathsf{tp}_{\mathsf{F}}\, r$, $\mathsf{tq}_{\mathsf{F}}\, r$, $\mathsf{surf}_{\mathsf{F}}\, r\, s$ and $\mathsf{hcomp}\ [\varphi \mapsto u]\ u_0$ such that $u_0 \in \mathbb{T}_{\mathsf{F}}(I)$, $u_f \in \mathbb{T}_{\mathsf{F}}(J)$ for $f : J \to I^+$ satisfying $u_0 f = u_{f^+ e_0}$ for $f : J \to I$ and $u_f g = u_{fg}$ for $f : J \to I^+$ and $g : K \to J$. This defines a cubical set $\mathbb{T}_{\mathsf{F}}$, such that $\mathbb{T}_{\mathsf{F}}(I)$ is a subset of $\mathbb{T}_{\mathsf{F}}^{\text{pre}}(I)$ for each $I$.

**Suspension** Given presheaf $\Gamma$ and $A$ a dependent presheaf over $\Gamma$ (which is a presheaf on the category of elements of $\Gamma$) we explain how to build the suspension of $A$, written $\Sigma A$, which is an initial $\Sigma A$-algebra. Just like for the parameter-free higher inductive types we first define a family of sets $X(I, \rho)$, for $\rho \in \Gamma(I)$ which is an upper approximation of the suspension. An element of $X(I, \rho)$ is of the form:

- $\mathsf{N}$, $\mathsf{S}$, or
- $\mathsf{merid}\ a\ r$ with $a \in A(I, \rho)$ and $r \neq 0, 1$ in $\mathbb{I}(I)$, or
- $\mathsf{hcomp}\ [\varphi \mapsto u]\ u_0$ with $\varphi \neq 1$ in $\mathbb{F}(I)$ and $u_0$ in $X\rho$ and $u$ a family of elements $u_f$ in $X(J, \rho\sigma f)$ for $f \colon J \to I^+$ such that $\varphi\sigma f = 1$.

In this way an element of $X(I,\rho)$ can be seen as a well-founded tree. We now define the tentative restriction maps $X(I,\rho) \to X(J,\rho f)$, $u \mapsto uf$ for $f\colon J \to I$ by induction on $u$:

$$\begin{aligned}
\mathsf{N}f &= \mathsf{N}\\
\mathsf{S}f &= \mathsf{S}\\
(\mathsf{merid}\, a\, r)f &= \begin{cases} \mathsf{N} & \text{if } rf = 0\\ \mathsf{S} & \text{if } rf = 1\\ \mathsf{merid}\,(af)\,(rf) & \text{otherwise}\end{cases}\\
(\mathsf{hcomp}\ [\varphi \mapsto u]\ u_0)f &= \begin{cases} u_{f+e_1} & \text{if } \varphi f = 1\\ \mathsf{hcomp}\ [\varphi f \mapsto uf^+]\ (u_0 f) & \text{otherwise}\end{cases}
\end{aligned}$$

where $uf^+$ is the family $(uf^+)_g = u_{fg}$ for $g : K \to J^+$.

We define $(\Sigma A)(I,\rho)$ as the subset of $X(I,\rho)$ of elements $\mathsf{N}$, $\mathsf{S}$ or $\mathsf{merid}\ a\ r$ with $a \in A\rho$ and $\mathsf{hcomp}\ [\varphi \mapsto u]\ u_0$ with $u_0$ in $(\Sigma A)\rho$ and $u_{f+e_0} = u_0 f$ for $f : J \to I$ and each $u_f$ in $(\Sigma A)(\rho f)$ for $f : J \to I^+$ and $u_f g = u_{fg}$ for $g : K \to J$ and $f : J \to I^+$.

This defines the initial $\Sigma A$-algebra relative to a context $\Gamma$. Since this operation commutes with substitution $\Delta \to \Gamma$, it is an external description of the operation which takes an arbitrary type $A$ and produces the free $\Sigma A$-algebra.

**Pushouts** Given $D = A, B, C, u : C \to A, v : C \to B$ a diagram over $\Gamma$ we explain how to define $A \sqcup_C B$, initial $D$-algebra over $\Gamma$. We first define a family of sets $X(I,\rho)$, for $\rho \in \Gamma(I)$ which is an upper approximation of the pushout. An element of $X(I,\rho)$ is of the form:

- $\mathsf{inl}\ a$ for $a \in A(I,\rho)$, or
- $\mathsf{inr}\ b$ for $b \in B(I,\rho)$, or
- $\mathsf{push}\ c\ r$ with $c \in C(I,\rho)$ and $r \in \mathbb{I}(I)$ such that $r \neq 0, 1$, or
- $\mathsf{hcomp}\ [\varphi \mapsto u]\ u_0$ with $\varphi \neq 1$ in $\mathbb{F}(I)$ and $u_0$ in $X(I,\rho)$ and $u$ a family of elements $u_f$ in $X(J,\rho\sigma f)$ for $f\colon J \to I^+$ such that $\varphi\sigma f = 1$.

The maps $X(I,\rho) \to X(J,\rho f)$ for $f\colon J \to I$ are defined by induction:

$$\begin{aligned}
(\mathsf{inl}\ a)f &= \mathsf{inl}\,(af)\\
(\mathsf{inr}\ a)f &= \mathsf{inr}\,(bf)\\
(\mathsf{push}\, c\, r)f &= \begin{cases} \mathsf{inl}\,(\mathrm{app}(u, cf)) & \text{if } rf = 0\\ \mathsf{inr}\,(\mathrm{app}(v, cf)) & \text{if } rf = 1\\ \mathsf{push}\,(cf)\,(rf) & \text{otherwise}\end{cases}\\
(\mathsf{hcomp}\ [\varphi \mapsto u]\ u_0)f &= \begin{cases} u_{f+e_1} & \text{if } \varphi f = 1\\ \mathsf{hcomp}\ [\varphi f \mapsto uf^+]\ (u_0 f) & \text{otherwise}\end{cases}
\end{aligned}$$

where $uf^+$ is the family $(uf^+)_g = u_{f+g}$ for $g : K \to J^+$.

We define $(A \sqcup_C B)(I,\rho)$ for $\rho \in \Gamma(I)$ as the subset of $X(I,\rho)$ with elements

- $\mathsf{inl}\ \ a$ with $a \in A(I,\rho)$, or
- $\mathsf{inr}\ \ b$ with $b \in B(I,\rho)$, or
- $\mathsf{push}\ c\ r$ with $c \in C(I,\rho)$ and $r \in \mathbb{I}(I)$ such that $r \neq 0, 1$, or
- $\mathsf{hcomp}\ [\varphi \mapsto u]\ u_0$ with $u_0$ in $(A \sqcup_C B)(I,\rho)$ and $u_{f+e_0} = u_0 f$ if $f\colon J \to I$ and each $u_f$ in $(A \sqcup_C B)(J,\rho\sigma f)$ for $f\colon J \to I^+$ and $u_f g = u_{fg}$ for $g\colon K \to J$ and $f\colon J \to I^+$.