THE UNIVALENCE AXIOM IN CUBICAL SETS

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Abstract. In this note we show that Voevodsky's univalence axiom holds in the model of type theory based on cubical sets as described in [1, 4]. We will also discuss Swan's construction of the identity type in this variation of cubical sets. This proves that we have a model of type theory supporting dependent products, dependent sums, univalent universes, and identity types with the usual judgmental equality, and this model is formulated in a constructive metatheory.

1. Review of the cubical set model

We give a brief overview of the cubical set model, introducing some different notations, but will otherwise assume the reader is familiar with [1, 4].

As opposed to [1, 4] let us define cubical sets as contravariant presheaves on the opposite of the category used there, that is, the category of cubes $\mathcal{C}$ contains as objects finite sets $I = \{i_1, \dots, i_n\}$ $(n \geq 0)$ of names and a morphism $f\colon J \to I$ is given by a set-theoretic map $I \to J \cup \{0, 1\}$ which is injective when restricted to the preimage of J; we will write compositions in applicative order. The category of cubical sets is the category of presheaves $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$. For $i \notin I$, the face morphisms are denoted by $(i/0), (i/1)\colon I \to I, i$ and are induced by setting i to 0 and 1, respectively; degenerating along $i \notin I$ is denoted by $s_i\colon I, i \to I$ and is induced by the inclusion $I \subseteq I, i$.

If Γ is a cubical set, we write $\Gamma \vdash A$ to mean that A is a presheaf on the category of element of Γ [1, 4]. Let $\Gamma \vdash A$ be a type, and let $\rho \in \Gamma(I)$ and $J \subseteq I$. A *J-tube* in $A\rho$ is given by a family $\vec{u}$ of elements $u_{jc} \in A\rho(j/c)$ for $(j, c) \in J \times \{0, 1\}$ which is adjacent compatible, that is, $u_{jc}(k/d) = u_{kd}(j/c)$ for $(j, c), (k, d) \in J \times \{0, 1\}$. For $(i, a) \in (I - J) \times \{0, 1\}$ we say that an element $v \in A\rho(i/a)$ *fits* such a tube $\vec{u}$ if $u_{jc}(i/a) = v(j/c)$ for all $(j, c) \in J \times \{0, 1\}$.

Recall from [1, Section 4] that a *(uniform) Kan structure* for a type $\Gamma \vdash A$ is given by operations which (uniformly) fill open boxes: for any $\rho \in \Gamma(I)$, $J \subseteq I$, a J-tube $\vec{u}$ in $A\rho$ and $i \in I - J$ we are given fillings (with notation for *compositions* on the right)

$$\mathsf{fill}^{+i}_{A\rho}\,[J \mapsto \vec{u}]\,u_{i0} \in A\rho \qquad \mathsf{comp}^{+i}_{A\rho}\,[J \mapsto \vec{u}]\,u_{i0} := (\mathsf{fill}^{+i}_{A\rho}\,[J \mapsto \vec{u}]\,u_{i0})(i/1)$$

whenever $u_{i0} \in A\rho(i/0)$ fits $\vec{u}$ at the side $(i, 0)$, and, in the other direction,

$$\mathsf{fill}^{-i}_{A\rho}\,[J \mapsto \vec{u}]\,u_{i1} \in A\rho \qquad \mathsf{comp}^{-i}_{A\rho}\,[J \mapsto \vec{u}]\,u_{i1} := (\mathsf{fill}^{-i}_{A\rho}\,[J \mapsto \vec{u}]\,u_{i1})(i/0)$$

whenever $u_{i1} \in A\rho(i/1)$ fits $\vec{u}$ at side $(i, 1)$. We will usually omit the "+" in the superscript. These filling operations are subject to the equations

$$(\mathsf{fill}^{i}_{A\rho}\,[J \mapsto \vec{u}]\,u_0)(j/c) = u_{jc} \quad \text{for } (j, c) \in \{(i, 0)\} \cup (J \times \{0, 1\}),$$
$$(\mathsf{fill}^{i}_{A\rho}\,[J \mapsto \vec{u}]\,u_0)f = \mathsf{fill}^{f(i)}_{A\rho f}\,[Jf \mapsto \vec{u}f]\,(u_0(f - i))$$

for $f\colon K \to I$ defined on i, J, where $f - i\colon K - f(i) \to I - i$ is like f but skips i, and Jf is the image of J under f. And likewise for the filling in the other direction.

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When J is empty and $\vec{u}$ the empty tube, we simply write $[]$ instead of $[J \mapsto \vec{u}]$. By *Kan type* we refer to a type together with a Kan structure on the type.

In [1] we showed that Kan types are closed under *weak* identity types, that is, identity types where the usual judgmental equality for the J-eliminator only holds up to propositional equality; we will denote this type by $\Gamma \vdash \mathsf{Path}_A\, u\, v$ (where $\Gamma \vdash A$, $\Gamma \vdash u : A$, and $\Gamma \vdash v : A$) and refer to it as *path type*. We will reserve Id_A for the identity type with the usual judgmental equality defined in Section 4.

2. Path types

It will be convenient below to introduce paths using separated products.

Definition 1. Given cubical sets Γ and Δ, we say that $u \in \Gamma(I)$ and $v \in \Delta(I)$ are *separated*, denoted by $u \# v$, if they come through degeneration from cubes with disjoint sets of directions. More precisely, if there are $J \subseteq I$, $K \subseteq I$ with $J \cap K = \emptyset$ and $u' \in \Gamma(J), v' \in \Delta(K)$ such that $u = u's$ and $v = v's'$ with s and s' induced by the inclusion $J \subseteq I$ and $K \subseteq I$, respectively.

The *separated product* $\Gamma * \Delta$ of Γ and Δ is defined by

$$(\Gamma * \Delta)(I) = \{(u, v) \in \Gamma(I) \times \Delta(I) \mid u \# v\} \subseteq (\Gamma \times \Delta)(I).$$

The restrictions are inherited from $\Gamma \times \Delta$, that is, they are defined component wise. It can be shown that $- * -$ extends to a functor, and that $- * \Delta$ has a right adjoint.

Of particular interest is $\Gamma * \mathbb{I}$ where $\mathbb{I}$ is the interval defined by $\mathbb{I}(J) = J \cup \{0, 1\}$ (see [1, Section 6.1]). Then

$$(\Gamma * \mathbb{I})(I) = (\Gamma(I) \times \{0, 1\}) \cup \{(\rho s_i, i) \mid i \in I \wedge \rho \in \Gamma(I - i)\}.$$

If $(\rho, i) \in (\Gamma * \mathbb{I})(I)$ with $i \in \mathbb{I}$, then $\rho = \rho' s_i$ for a uniquely determined ρ' which we denote by $\rho - i$.

We can use $\Gamma * \mathbb{I}$ to formulate the following introduction rule[1] for path types

$$\frac{\Gamma \vdash A \qquad \Gamma * \mathbb{I} \vdash w : A\mathsf{p}}{\Gamma \vdash \langle\rangle w : \mathsf{Path}_A\, w[0]\, w[1]}$$

where $[0], [1] \colon \Gamma \to \Gamma * \mathbb{I}$ are induced by the global elements 0 and 1 of $\mathbb{I}$, respectively, and $\mathsf{p} \colon \Gamma * \mathbb{I} \to \Gamma$ is the first projection. The binding operation is interpreted by $(\langle\rangle w)\rho = \langle i \rangle\, w(\rho s_i, i)$ with i a fresh name (see [1, Section 8.2]).

We can also interpret an application rule

$$\frac{\Gamma \vdash v : \mathsf{Path}_A\, u_0\, u_1 \qquad \sigma \colon \Delta \to \Gamma \qquad \Delta \vdash r : \mathbb{I} \qquad \Delta \vdash \sigma \# r}{\Delta \vdash v\sigma \mathbin{@} r : A\sigma}$$

where $\Delta \vdash \sigma \# r$ means $\sigma\rho \# r\rho$ for each $\rho \in \Delta(I)$. To define $(v\sigma \mathbin{@} r)\rho \in A\sigma\rho$ for $\rho \in \Delta(I)$ let us write $v\sigma\rho = \langle i \rangle \omega$ (for i fresh, $\omega \in A\sigma\rho s_i$). If $r\rho \in \{0, 1\}$, then $(v\sigma \mathbin{@} r)\rho := (\langle i \rangle \omega) \mathbin{@} r\rho := \omega(i/r\rho)$; otherwise $r\rho = j$ is a name and $\rho = (\rho - j)s_j$, so also $\omega = \omega' s_j$ for $\omega' \in A\sigma(\rho - j)s_i$ and we set $(v\sigma \mathbin{@} r)\rho = \omega'(i/j)$.

Clearly, with v as above, $\Gamma \vdash v \mathbin{@} 0 = u_0 : A$ and $\Gamma \vdash v \mathbin{@} 1 = u_1 : A$. Moreover we have the following β- and η-rules:

$$\Delta \vdash (\langle\rangle w)\sigma \mathbin{@} r = w(\sigma, r) : A\sigma$$
$$\Gamma \vdash \langle\rangle(v\mathsf{p} \mathbin{@} \mathsf{q}) = v : \mathsf{Path}_A\, u_0\, u_1$$

[1] Here and below such rules should be read semantically.

3. Equivalences

We will now recall the definition of an equivalence as a map having contractible fibers and then derive an operation for contractible Kan types. To enhance readability we define the following types using variable names:

$$\begin{aligned}\mathsf{isContr}\, A &= \Sigma(x : A)\, \Pi(y : A)\, \mathsf{Path}_A\, x\, y\\ \mathsf{fib}\, h\, v &= \Sigma(x : A)\, \mathsf{Path}_B\, v\, (h\, x)\\ \mathsf{isEquiv}\, h &= \Pi(y : B)\ \mathsf{isContr}(\mathsf{fib}\, h\, y)\\ \mathsf{Equiv}\, A\, B &= \Sigma(h : A \to B)\ \mathsf{isEquiv}\, h\end{aligned}$$

where A and B are types, $h : A \to B$, and $v : B$ (all in an ambient context Γ). This can of course also be formally written name-free: for example, the first type can be written as $\Gamma \vdash \Sigma A \Pi A\mathsf{p}\, (\mathsf{Path}_{A\mathsf{pp}} \mathsf{qp}\ \mathsf{q})$.

Definition 2. A *(uniform) acyclic-fibration structure* on a type $\Gamma \vdash A$ is given by operations uniformly filling any tube, that is, given $\rho \in \Gamma(I)$, $J \subseteq I$, a J-tube $\vec{u}$ in $A\rho$, we have operations

$$\mathsf{ext}_{A\rho}[J \mapsto \vec{u}] \in A\rho$$

extending $\vec{u}$ (so $(\mathsf{ext}_{A\rho}[J \mapsto \vec{u}])(i/a) = u_{ia}$ for $(i, a) \in J \times \{0, 1\}$) and for $f \colon K \to I$ defined on J we have

$$\mathsf{ext}_{A\rho}[J \mapsto \vec{u}]\, f = \mathsf{ext}_{A(\rho f)}[Jf \mapsto \vec{u}f].$$

Lemma 3. *Any Kan type $\Gamma \vdash A$ which is contractible, that is, where we have a $\Gamma \vdash p : \mathsf{isContr}\, A$, has an acyclic-fibration structure.*

Proof. Given $\rho \in \Gamma(I)$ and a tube $\vec{u}$ in $A\rho$ we can set with i fresh:

$$\mathsf{ext}_{A\rho}[J \mapsto \vec{u}] = \mathsf{comp}^i_{A\rho s_i}[J \mapsto (p\rho.2\, \vec{u})\, @\, i]\, (p\rho.1),$$

where $(p\rho.2\, \vec{u})\, @\, i$ is the J-tube given by $(p(\rho(j/c)).2\, u_{jc})\, @\, i$ at side $(j, c) \in J \times \{0, 1\}$. ☐

Next we will define an operation G which allows us to transform an equivalence into a "path"[2]. First we state the following two rules that G will satisfy:

(1) $$\frac{\Gamma \vdash A \qquad \Gamma \vdash B \qquad \Gamma \vdash w : \mathsf{Equiv}\, A\, B}{\Gamma * \mathbb{I} \vdash \mathsf{G}\, w} \qquad \begin{aligned}\Gamma &\vdash (\mathsf{G}\, w)[0] = A\\ \Gamma &\vdash (\mathsf{G}\, w)[1] = B\end{aligned}$$

(2) $$\frac{\sigma \colon \Delta \to \Gamma \qquad \Gamma \vdash A \qquad \Gamma \vdash B \qquad \Gamma \vdash w : \mathsf{Equiv}\, A\, B}{\Delta * \mathbb{I} \vdash (\mathsf{G}\, w)(\sigma * \mathrm{id}) = \mathsf{G}(w\sigma)}$$

The latter rule expresses stability under substitutions. Here and below G (and friends g, ug) have A and B as implicit arguments.

Definition 4. Assume the premiss of (1) and define for every $\rho \in \Gamma(I)$:

(3) $$\begin{aligned}&(\mathsf{G}\, w)(\rho, 0) = A\rho, \text{ with restrictions as in } A,\\ &(\mathsf{G}\, w)(\rho, 1) = B\rho, \text{ with restrictions as in } B, \text{ and}\\ &(\mathsf{G}\, w)(\rho, i) = \{(u, v) \mid u \in A(\rho - i) \wedge v \in B\rho \wedge v(i/0) = w(\rho - i).1\, u\}.\end{aligned}$$

In the last case $\rho \# i$, so $\rho = (\rho - i)s_i$, and the elements of $\mathsf{G}\, w$ are given by an element u of A and a line v in B starting in $w.1\, u$. The restrictions in the latter case are a little involved. We need $(u, v)f \in (\mathsf{G}\, w)(\rho f, f(i))$ for $f \colon J \to I$. If $f(i) = 0$, we take $(u, v)f = us_i f$, indeed in $A\rho f$. If $f(i) = 1$, we take $(u, v)f = vf$, indeed in

[2] We will see later that this is indeed induces a path in a universe whenever both types A and B are small.

$B\rho f$. Finally, if f is defined on i, we have $f - i\colon J - f(i) \to I - i$ and we define $(u, v)f = (u(f - i), vf)$, which is indeed correct as $(\rho - i)(f - i) = \rho f - f(i)$ under the given assumptions. This concludes the definition of $\mathsf{G}\, w$; Theorem 5 states the properties we need.

Similar to the glueing operation of [2] there is an operation g and a "global" introduction rule for G:

$$\frac{\Gamma \vdash u : A \qquad \Gamma * \mathbb{I} \vdash v : B\mathsf{p} \qquad \Gamma \vdash v[0] = w.1\, u : B \qquad \Gamma \vdash w : \mathsf{Equiv}\, A\, B}{\Gamma * \mathbb{I} \vdash \mathsf{g}\, u\, v : \mathsf{G}\, w}$$

Moreover, we have a map $\Gamma * \mathbb{I}.\, \mathsf{G}\, w \vdash \mathsf{ug} : B\mathsf{pp}$; g and ug are interpreted as follows:

$$\begin{aligned}
(\mathsf{g}\, u\, v)(\rho, 0) &= u\rho, & \mathsf{ug}((\rho, 0), u) &= w\rho.1\ u,\\
(\mathsf{g}\, u\, v)(\rho, 1) &= v(\rho, 1), & \mathsf{ug}((\rho, 1), v) &= v,\\
(\mathsf{g}\, u\, v)(\rho, i) &= (u(\rho - i), v(\rho, i)), & \mathsf{ug}((\rho, i), (u, v)) &= v.
\end{aligned}$$

Note that the definition of $\mathsf{G}\, w$ only refers to $w.1\colon A \to B$ and does not use that w is an equivalence. This is however used to prove:

Theorem 5. *Assume the premiss of* (1)*, then $\Gamma * \mathbb{I} \vdash \mathsf{G}\ w$ is a Kan type whenever $\Gamma \vdash A$ and $\Gamma \vdash B$ are. Moreover, the equations in* (1) *and* (2) *also hold for the Kan structure on the type $\mathsf{G}\ w$.*

Proof. We write h for $w.1$. To define the Kan structure on $(\mathsf{G}\, w)(\rho, r)$ for $(\rho, r) \in (\Gamma * \mathbb{I})(I)$ we argue by cases: if $r = 0$, then the Kan structure is induced by $A\rho$; if $r = 1$, the Kan structure is induced by $B\rho$. This is clearly necessary to satisfy the equations in (1). Let us now consider the main case where $r = i \in I$ is a name and thus $\rho \# i$, $\rho = (\rho - i)s_i$. We are given j (the name along which we fill), $a \in \{0, 1\}$, a J-tube $\vec{u}$ in $(\mathsf{G}\, w)(\rho, i)$ (with $J \subseteq I - j$), and $u_{ja} \in (\mathsf{G}\, w)(\rho, i)(j/a)$ which fits $\vec{u}$; we want to define the Kan filling operation

$$u := \mathsf{fill}^{\pm j}_{(\mathsf{G}\, w)(\rho, i)}\, [J \mapsto \vec{u}]\, u_{ja} \in (\mathsf{G}\, w)(\rho, i)$$

where $\pm$ is "+" if $a = 0$, and "$-$" otherwise. For this we have to construct $u = (u', u'')$ with $u' \in A(\rho - i)$ and $u'' \in B\rho$ such that $u''(i/0) = h(\rho - i)\, u'$.

We can map $u_{ja}, \vec{u}$ to B using ug and obtain an open box $v_{ja}, \vec{v}$ given by

$$v_{kb} := \mathsf{ug}((\rho(k/b), i(k/b)), u_{kb}) \in B\rho(k/b).$$

Note that h maps $\vec{u}(i/0)$ to $\vec{v}(i/0)$. There are four cases to consider depending on how the open box relates to the direction i.

Each case will be illustrated afterwards with simplified low-dimensional J and omitting ρ where the left component is in A and on the right we have the open box $\vec{v}$ in B; the direction i extends to the right.

Case $i \neq j$ and $i \notin J$. We extend the J-tube $\vec{u}$ to J, i-tube by constructing u_{i0} and u_{i1} and then proceed as in the next case with the tube $\vec{u}, u_{i0}, u_{i1}$. Note that we want

$$\begin{aligned}
u_{i0} &\in (\mathsf{G}\, w)(\rho, i)(i/0) = A(\rho - i), \text{ and}\\
u_{i1} &\in (\mathsf{G}\, w)(\rho, i)(i/1) = B(\rho - i),
\end{aligned}$$

so we can take

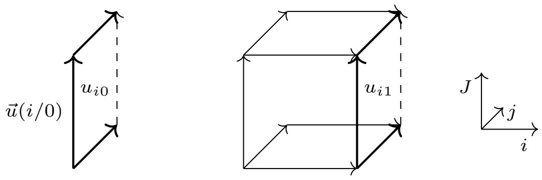
$$u_{i0} = \mathsf{fill}^{\pm j}_{A(\rho - i)}\, [J \mapsto \vec{u}(i/0)]\, (u_{ja}(i/0)), \text{ and} \tag{4}$$

$$u_{i1} = \mathsf{fill}^{\pm j}_{B(\rho - i)}\, [J \mapsto \vec{u}(i/1)]\, (u_{ja}(i/1)). \tag{5}$$

The resulting open box is compatible by construction. Note that this (together with the cases for $r = 0$ and $r = 1$) also ensures that the Kan structure satisfies the equations in (1).

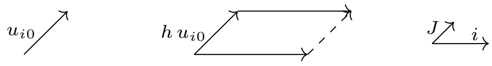
We illustrate this case in the picture below. We construct u_{i0} and u_{i1} by filling the open boxes indicated by thicker lines on the left and on the right, respectively.

(6)



Case $i \neq j$ and $i \in J$. In this case $v_{i0} = \mathsf{ug}((\rho(i/0), 0), u_{i0}) = w\rho(i/0).1\, u_{i0} = h(\rho - i)\, u_{i0}$ since $\rho \# i$. We can therefore take $u = (u_{i0}, v) \in (\mathsf{G}\, w)(\rho, i)$ where $v = \mathsf{fill}^{\pm j}_{B(\rho - i)}\, [J \mapsto \vec{v}]\, v_{ja}$. This can be illustrated as in picture (6) but now we are also given u_{i0} and u_{i1}, and correspondingly there is also the missing side v_{i0} (which is equal to $h\, u_{i0}$) to get an open box in B which we fill to obtain v.

Case $j = i$ and $a = 0$. Like in the previous case we can take $u = (u_{i0}, v) \in (\mathsf{G}\, w)(\rho, i)$ where $v = \mathsf{fill}^{\pm j}_{B(\rho - i)}\, [J \mapsto \vec{v}]\, v_{ja}$. This case is illustrated in the following picture where one should note that $v_{i0} = h\, u_{i0}$.



Case $j = i$ and $a = 1$. In this case the direction of the filling is opposite to h, and therefore we have to use the information in $w.2$ about $\mathsf{fib}\, h$. The family $\vec{w}$ defined by

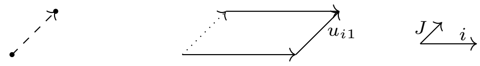
$$w_{kb} := (u_{kb}, \langle i \rangle\, v_{kb}) \in (\mathsf{fib}\, h)((\rho - i)(k/b), u_{i1}(k/b))$$

for $(k, b) \in J \times \{0, 1\}$ constitutes a J-tube over $(\rho - i, u_{i1})$ in the contractible type $\Gamma.B \vdash \mathsf{fib}\, h$. Using Lemma 3 we can extend this to obtain

$$(u', \omega) = \mathsf{ext}_{(\mathsf{fib}\, h)(\rho - i, u_{i1})}[J \mapsto \vec{w}] \in (\mathsf{fib}\, h)(\rho - i, u_{i1})$$

so we can take $u := (u', \omega \,@\, i) \in (\mathsf{G}\, w)(\rho, i)$.

Let us illustrate this case: we are given the two dots on the left and the solid lines on the right in the picture below, and we want to construct the dashed line and a square on the right such that the dashed line is mapped to the dotted line via h, that is, we basically want to construct an element in the fiber of u_{i0} under h.



This concludes the definition of the filling operations of $\mathsf{G}\, w$. To see that this filling operation is uniform, note that for an $f \colon K \to I$ defined on j, J and on i the case which defines the filling of $(u_{ja}, \vec{u})f$ coincides with the case used to defined $(u_{ja}, \vec{u})$ by the injectivity requirement on f—uniformity then follows for each case separately since we only used operations that suitably commute with f in the definition of the filling. If f is only defined on j, J but not on i, the first case has to apply—to simplify notation assume f is (i/c)—then by construction (equations (4) and (5))

$$\left(\mathsf{fill}^{\pm j}_{(\mathsf{G}\, w)(\rho, i)}\, [J \mapsto \vec{u}]\, u_{ja}\right)(i/c) = u_{ic} = \mathsf{fill}^{\pm j}_{(\mathsf{G}\, w)(\rho - i, c)}\, [J \mapsto \vec{u}(i/c)]\, (u_{ja}(i/c)).$$

□

Theorem 6. *We can refine the Kan structure given in Theorem 5 such that it satisfies*

$$\mathsf{comp}^{i}_{(\mathsf{G}\, w)(\rho, i)}\, []\, u = h(\rho - i)\ u,$$
$$\mathsf{comp}^{-i}_{(\mathsf{G}\, w)(\rho, i)}\, []\, u = h^{-1}(\rho - i)\ u,$$

where we have written h for $w.1$ and h^{-1} for the inverse extracted from $w.2$.

Proof. We will now explain how to modify the Kan structure given in the proof of Theorem 5 to obtain the above equations. The last two cases in the proof above where $i = j$ are modified by an additional case distinction on whether J is empty or not. If J is not empty, proceed as before. In case J is empty and $a = 0$, then we are given $u_{i0} \in A(\rho - i)$ and an empty tube and can define $\mathsf{fill}^i_{(\mathsf{G}\,w)(\rho,i)}\,[]\,u_{i0} = (u_{i0}, h\rho\; u_{i0})$. In case J is empty and $a = 1$, we can set $\mathsf{fill}^i_{(\mathsf{G}\,w)(\rho,i)}\,[]\,u_{i0} = (h^{-1}\,u_{i0}, \eta\,(\rho - i)\,u_{i0}\,@\,i)$ where η is easily defined in terms of w and of type $\Pi(x : A)\,\mathsf{Path}_B(h^{-1}\,(h\,x))\,x$.

That definition remains uniform is proved as in Theorem 5 using the observation that $|J| = |Jf|$ for f defined on J. □

Let us recall the definition of a universe U of small Kan types (assuming a Grothendieck universe of small sets in the ambient set theory). We denote the Yoneda embedding by $\mathbf{y}$. The I-cubes of U are given by small Kan types $\mathbf{y}\,I \vdash A$, that is, a Kan type $\mathbf{y}\,I \vdash A$ such that for all $f\colon J \to I$ the set A_f is small. Restrictions $\mathsf{U}(I) \to \mathsf{U}(J)$ along $f\colon J \to I$ are given by substituting along $\mathbf{y}\,f : \mathbf{y}\,J \to \mathbf{y}\,I$.

A small Kan type $\Gamma \vdash A$ has a code $\Gamma \vdash \ulcorner A \urcorner : \mathsf{U}$ given by $(\ulcorner A \urcorner \rho)_f = A(\rho f)$. The decoding of $\Gamma \vdash a : \mathsf{U}$ into a small Kan type $\Gamma \vdash \mathsf{El}\,a$ is given by $(\mathsf{El}\,a)\rho = (a\rho)_{\mathrm{id}}$. One can check that $\mathsf{El}\,\ulcorner A \urcorner = A$ and $\ulcorner \mathsf{El}\,a \urcorner = a$.

Theorem 7. U *has a Kan structure.*

Proof. [4, Theorem 4.2]. □

Because our operation G preserves smallness we obtain an operation turning an equivalence between small Kan types into a path in U: given $\Gamma \vdash a : \mathsf{U}$ and $\Gamma \vdash b : \mathsf{U}$ with $\Gamma \vdash w : \mathsf{Equiv}\,(\mathsf{El}\,a)\,(\mathsf{El}\;b)$ we get a small type $\Gamma * \mathbb{I} \vdash \mathsf{G}\,w$ which is a (small) Kan type by Theorem 5, so $\Gamma * \mathbb{I} \vdash \ulcorner \mathsf{G}\,w \urcorner : \mathsf{U}$ with $\ulcorner \mathsf{G}\,w \urcorner[0] = \ulcorner (\mathsf{G}\,w)[0] \urcorner = \ulcorner \mathsf{El}\,a \urcorner = a$ and likewise $\ulcorner \mathsf{G}\,w \urcorner[1] = b$. Finally, abstracting gives a path $\Gamma \vdash \langle\rangle \ulcorner \mathsf{G}\,w \urcorner : \mathsf{Path}_{\mathsf{U}}\,a\,b$.

So we derived an operation

$$\mathsf{ua} : \Pi(a\,b : \mathsf{U})\big(\mathsf{Equiv}\,(\mathsf{El}\,a)\,(\mathsf{El}\,b) \to \mathsf{Path}_{\mathsf{U}}\,a\,b\big)$$

or written name-free and uncurried as:

$$\Gamma.\mathsf{U}.\mathsf{Up}.\mathsf{Equiv}\,(\mathsf{El}\,\mathsf{qp})\,(\mathsf{El}\,\mathsf{q}) \vdash \mathsf{ua} : \mathsf{Path}_{\mathsf{U}}\,(\mathsf{qpp})\,(\mathsf{qp})$$

given by $\mathsf{ua} = \langle\rangle \ulcorner \mathsf{G}\,\mathsf{q} \urcorner$.

Corollary 8 (Univalence)**.** *The Kan type*

$$\vdash \Pi(a : \mathsf{U})\ \mathsf{isContr}\big(\Sigma(b : \mathsf{U})\,\mathsf{Equiv}\,(\mathsf{El}\,a)\,(\mathsf{El}\,b)\big) \tag{7}$$

has a section.

Proof. In addition to ua we obtain a term ua_β of type

$$\Pi(a\,b : \mathsf{U})\,\Pi(w : \mathsf{Equiv}\,(\mathsf{El}\,a)\,(\mathsf{El}\,b))\,\Pi(x : \mathsf{El}\,a)\,\mathsf{Path}_{\mathsf{El}\,b}\,(\mathsf{transp}\,(\mathsf{ua}\,w)\,x)\,(w.1\,x)$$

where transp is the transport operation for paths for the type $\mathsf{U} \vdash \mathsf{El}$ (see the operation T in [1, Section 8.2]). Indeed, the path to justify ua_β is given by reflexivity using our refined Kan structure from Theorem 6 and that transp is given in terms of composition with an empty tube.

The rest of the proof can be given internally in type theory from ua and ua_β, using that the path type satisfies function extensionality [4, Theorem 3.20] and we have the usual eliminator but with propositional equality. From ua we get a map from $\mathsf{Equiv}\,(\mathsf{El}\,a)\,(\mathsf{El}\,b)$ to $\mathsf{Path}_{\mathsf{U}}\,a\,b$; there is also a map in the other direction namely the equivalence induced by transporting along the given path. Now ua_β and function extensionality imply that these maps constitute a section-retraction pair, and hence also $\Sigma(b : \mathsf{U})\,\mathsf{Equiv}\,(\mathsf{El}\,a)\,(\mathsf{El}\,b)$ is a retract of $\Sigma(b : \mathsf{U})\,\mathsf{Path}_{\mathsf{U}}\,a\,b$. But the latter type is contractible and thus so is the former, concluding the proof. □

4. Identity types

We will now describe the identity type which justifies the usual judgmental equality for its eliminator following Swan [5].

We start from a natural transformation $\alpha : A \to B$ between two presheafs A and B on the same category of element of a cubical set Γ. We are going to define a *factorization* of this map α: we define a new type $\Gamma \vdash M_\alpha$ with two maps $\Gamma \vdash i_\alpha : A \to M_\alpha$ and $\Gamma \vdash p_\alpha : M_\alpha \to B$ such that $\alpha = p_\alpha i_\alpha$. Furthermore, i_α will be a *cofibration* (i.e. has the lifting property w.r.t. any fibration) and p_α will be a *acyclic fibration*. This factorization corresponds to Garner's factorization using the refined small object argument [3] specialized to cubical sets.

For ρ in $\Gamma(I)$ we define $M_\alpha\rho$ and $p_\alpha\ w$ for w in $M_\alpha\rho$ by induction. An element $M_\alpha\rho$ is either of the form $i\ u$ with u in $A\rho$ (where i is a constructor) and $p_\alpha\ (i\ u) = \alpha\ u$ in $B\rho$, or of the form $(v, [J \mapsto \vec{u}])$ with $p_\alpha\ (v, [J \mapsto \vec{u}]) = v$ and v in $B\rho$ and u_{jb} in $M_\alpha\rho(j/b)$ such that $p_\alpha\ u_{jb} = v(j/b)$. We furthermore define $(i\ u)(k/b) = i\ u(k/b)$ and $(v, [J \mapsto \vec{u}])(j/b) = u_{jb}$ for j in J and $(v, [J \mapsto \vec{u}])(k/b) = (v(k/b), [J \mapsto \vec{u}(k/b)])$ if k is not in J.

Lemma 9. *If $\Gamma \vdash B$ has a fibration structure, then so has $\Gamma \vdash M_\alpha$.*

Proof. Given ρ in $\Gamma(I, i)$ and a $J \subseteq I$ and a J-tube $\vec{m}$ in $M_\alpha\rho$ and m_0 in $M_\alpha\rho(i0)$ which fits $\vec{m}$ we define

$$\mathsf{fill}^i_{M_\alpha\rho}\ [J \mapsto \vec{m}]\ m_0 = (v, [J, i \mapsto \vec{w}])$$

with $v = \mathsf{fill}^i_{B\rho}\ [J \mapsto \vec{v}]\ v_0$ and $w_0 = p_\alpha\ v_0$ and $v_{jb} = p_\alpha\ m_{jb}$ and $w_{jb} = m_{jb}$ if j is in J and $w_{i0} = m_0$ and $w_{i1} = (\mathsf{comp}^i_{B\rho}\ [J \mapsto \vec{v}]\ v_0, [J \mapsto \vec{m}(i/1)])$. □

Lemma 10. *Given a family $D(w)$ $(w : M_\alpha)$ with a fibration structure and given a partial section $s\ a : D(i_\alpha\ a)$ $(a : A)$ and a total section $s'\ w : D(w)$ $(w : M_\alpha)$ with a path $e\ a : \mathsf{Path}\ D(i_\alpha\ a)\ (s\ a)\ (s'\ (i_\alpha\ a))$, it is possible to define a total section $\tilde{s}\ w : D(w)$ such that we have a judgemental equality $\tilde{s}\ (i_\alpha\ a) = s\ a : D(i_\alpha\ a)$.*

Proof. For ρ in $\Gamma(I)$ we define $\tilde{s}\rho\ w$ and a path $\tilde{e}\rho\ w : \mathsf{Path}\ D(w)\ (\tilde{s}\rho\ w)\ (s'\rho\ w)$ by induction on w in $M_\alpha\rho$. We define first $\tilde{s}\rho\ (i\ u) = s\rho\ u$ and $\tilde{e}\rho\ (i\ u) = e\rho\ u$. If $w = (v, [J \mapsto \vec{u}])$ we define $\tilde{s}\rho\ w = \mathsf{comp}^i_{D\rho\ w}\ [J \mapsto \vec{s}]\ (s'\rho\ w)$ and $\tilde{e}\rho\ w = \mathsf{fill}^i_{D\rho\ w}\ [J \mapsto \vec{s}]\ (s'\rho\ w)$ where $s_{jb} = \tilde{s}\rho\ u_{jb}$. □

This implies the following result, which expresses that $i_\alpha : A \to M_\alpha$ is a *cofibration.*

Corollary 11. *Given a family $D(w)$ $(w : M_\alpha)$ with an acyclic fibration structure and given a partial section $s\ a : D(i_\alpha\ a)$ $(a : A)$ it is possible to define a total section $\tilde{s}\ w : D(w)$ such that we have a judgemental equality $\tilde{s}\ (i_\alpha\ a) = s\ a : D(i_\alpha\ a)$.*

This also implies the following result, which expresses that $i_\alpha : A \to M_\alpha$ is a *trivial* cofibration as soon as α has a homotopy inverse.

Corollary 12. *If α has a homotopy inverse, given a family $D(w)$ $(w : M_\alpha)$ with a fibration structure and given a partial section $s\ a : D(i_\alpha\ a)$ $(a : A)$ it is possible to define a total section $\tilde{s}\ w : D(w)$ such that we have a judgemental equality $\tilde{s}\ (i_\alpha\ a) = s\ a : D(i_\alpha\ a)$.*

The representation of the identity type with the usual judgemental equality for its eliminator follows from these results by considering the case where B is the type of paths over a type A and $\alpha\ a$ is the constant path a.

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