On probabilistic analysis of disagreement in synchronous consensus protocols

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Abstract—This paper focuses on probabilistic analysis of disagreement of a 1-of-n selection algorithm among processes of a system with unbounded number of communication failures. In a 1-of-n selection problem, a set of \( n \) nodes are to select one common value among a set of \( n \) proposed values. There are two possible outcomes of each node’s selection process: it can decide either to select a value, or to abort. Agreement implies that all nodes select the same value, or all nodes decide to abort. Previous research has shown that it is impossible to guarantee agreement among the nodes in a synchronous system subjected to an unbounded number of message losses. Our aim is to find decision algorithms for which the probability of disagreement is as low as possible. Previously, we presented a probabilistic analysis of a 1-of-n selection algorithm using an optimistic and a moderately pessimistic decision criterion. In this paper, we study how the probability of disagreement varies for two different decision criteria, one optimistic and one pessimistic. We assume two communication failure models, symmetric and asymmetric message losses. For symmetric communication failures, we present the details on the closed-form expressions we derived to calculate the probabilities of each outcome of the algorithm for each of the two decision criteria. Also in this paper, we study the sensibility of the asymmetric failure model for which the probability of message loss is assumed to be equal for all the communication links, against to a more realistic failure model for which the probability of message loss increases with the increase of the physical distances among the nodes in a system.

Keywords—distributed algorithms; consensus; reliability analysis; symmetric and asymmetric communication failures.

I. INTRODUCTION

The problem of reaching consensus on a value among a set of cooperating distributed computing units has been studied extensively over the last thirty years. Despite this, we still lack definite answers to how consensus is best solved in distributed systems that rely on wireless communication. A main challenge in solving the consensus problem in wireless distributed systems is that the communication channel can be subjected to disturbances of varying duration and magnitude. Hence, in wireless applications we cannot make any assumptions about the number of messages that can be lost during the execution of a distributed consensus algorithm. We know from previous research [1], [2] that it is impossible to construct an algorithm that guarantees consensus in the face of unrestricted communication failures.

We are interested in the design of simple consensus algorithms that minimize the probability of disagreement in the presence of an arbitrary number of communication failures. Examples of the application of such protocols are in the area of cooperative systems including autonomous and semi-autonomous cooperative systems for improving traffic safety and fuel-efficiency of road vehicles, such as vehicle platooning, virtual traffic lights\(^1\) and coordinated lane change [3]. In [4], consensus is used to enforce the security of safety critical applications and to avoid false warning injection attacks in road vehicles. Another set of similar applications are related to the coordination of unmanned aerial vehicles (UAVs), such as the use of consensus to provide reconfigurable control to UAVs in a flight formation [5], or adjust UAVs’ trajectories in the presence of collision avoiding manoeuvres that cannot be foreseen [6].

Our goal is to investigate the possibility of using simple deterministic synchronous algorithms for fast consensus in safety-critical cooperative systems. To this end, we investigate a family of synchronous round-based consensus algorithms to solve the 1-of-n selection problem in the presence of symmetric and asymmetric communication failures. Since we know that we cannot construct an algorithm that solves this problem perfectly, our aim is to find algorithms for which the probability of disagreement is as low as possible.

In a 1-of-n selection problem, each node (in a system of \( n \) nodes) proposes a value and then all nodes must either select the same value, which has to be one of the proposed values, or decide to abort. Disagreement occurs if some nodes decide to select a value, while the remaining nodes decide to abort. Note that the algorithms we study in this paper are safe in the sense that they guarantee that the nodes will never decide on different values. As we will show later, the probability of disagreement for a given algorithm depends in general on three parameters: i) the number of nodes in the system, ii) the number of rounds of message exchange, and iii) the probability of message loss. In addition, it also depends on the decision criterion that determines whether a node should decide to abort or to select a value.

The main contributions of the paper are as follows. We investigate two practical decision criteria called the optimistic

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\(^1\)In a virtual traffic light, road vehicles approaching an intersection interact via wireless communication to form a virtual, or imaginary, traffic light.
and pessimistic decision criterion. For these, we present closed-form expressions for calculating the probability of disagreement in the presence of symmetric communication failures. For asymmetric communication failures, we use a probabilistic model checking tool to calculate the probability of disagreement. We use this tool and the closed-form formulas to illustrate how the probability of disagreement varies for different system configurations.

Another contribution of this paper is to evaluate the sensibility of our asymmetric failure model for which the probability of message loss is assumed to be equal for all the communication links, against to an asymmetric failure model where the probability of message loss increases with increasing the physical distances among the nodes in a system. In our previous works [7], [8], we assumed that the probability of message loss among the nodes is equal for all communications links. However, we know that in vehicle-to-vehicle communication (V2V), which is an application of the 1-of-n selection algorithm (e.g., [7], [9]), the probability of a communication failure depends on a number of factors, including the distance among vehicles. For example, the results presented in [10] show that under certain scenarios the probability of receiving a data packet can vary from 1 to almost 0 when the distance varies between 0 and 150 m. So, we aim to evaluate the sensibility of our asymmetric communication failure assumptions, to a more realistic failure model which considers the effect of the physical distance among vehicles in a V2V communication, on the quality of their corresponding communication link.

The remainder of the paper is organized as follows. Section II describes our system and failure model and the 1-of-n selection algorithm with two decision criteria. In Section III, we show our main observations of the behaviour of the 1-of-n selection algorithm assuming symmetric failures only. Then, we present the details on the closed-form expressions we derived for calculating the probability of disagreement for symmetric communication failures for both optimistic and pessimistic decision criteria. Section IV presents a number of diagrams that illustrates how the probability of disagreement varies for different protocol configurations under asymmetric failure model. Additionally, in Section IV, we present briefly the PRISM models we used to calculate the probabilities of each outcome of the algorithm. Then in Section IV-C, we compare the behaviour of the consensus protocol for two different asymmetric failure models, one when the probability of message loss is equal for all links and one for which the probability of asymmetric message loss varies depending on the distance among the nodes. We discuss the related work in Section V, and finally in Section VI we conclude and outline some directions for future research.

II. PROTOCOL DESCRIPTION

We know that it is impossible to construct an algorithm which guarantees consensus among the processes of a distributed system in presence of unbounded communication failures. As mentioned before, our focus therefore is to design simple synchronous consensus algorithms for safety-critical systems with unreliable underlying communication links, with the main aim of minimizing the probability of disagreement.

A. System Model and Failure Assumptions

We define a 1-of-n selection algorithm based on the classical round-based computational model which is used by many researchers, such as in [11], [12] and [13]. We consider a synchronous system consisting of a set of n processes. The processes are indexed respectively with their identifiers as: Π = {p₁, . . . , pₙ}. We assume that processes are fully connected to one another via wireless broadcast links and that they execute a deterministic protocol in R rounds of message exchange. Each round consists of three phases: send, receive and compute.

Failures occur as message losses and can occur on any communication link at any time during the execution of the protocol. In other words, there are no restrictions on the number, timing or pattern of the lost messages. We consider two different scenarios for a lost message: (i) when all the intended receivers of the message fail to receive the message (symmetric message loss), and (ii) when only a subset of the intended receivers fail to receive the message (asymmetric message loss).

For simplicity, we assume that the nodes are fault-free. Note, however, that a send omission failure of a node is equivalent to a symmetric message loss, and that a receive omission failure is equivalent to asymmetric message loss where only one node fails to receive a message.

B. The Consensus Algorithm

Alg. 1 shows the pseudocode of the 1-of-n selection algorithm executed by each process pᵢ ∈ Π. We assume that all processes initially construct a message, denoted as msgᵢ, that contains a proposed value (proposedᵢ) and a bit-vector (vᵢ) of length n that represents the view of process pᵢ. Initially, vᵢ[j] = 0 for all j (i.e., at this point pᵢ has not received any message from other processes in the system). We define vᵢ as complete if all elements of this vector are set to 1. Similarly, we define vᵢ as incomplete if at least one element of vᵢ is 0.

Algorithm 1 Generic algorithm for 1-of-n selection (pᵢ)

\[
\text{msg}_i \leftarrow \{v_i, \text{proposed}_i\};
\]

for \( r = 1 \) to \( R \) do

\[
\text{begin\_round}
\]

\[
\text{send\_to\_all}(\text{msg}_i);
\]

\[
\text{receive\_from\_all}();
\]

\[
\text{compute}(	ext{msg}_i);
\]

\[
\text{end\_round}
\]

execute decision algorithm();

After initialization, each process iterates the send, receive and compute phases in each of the \( R \) rounds. These phases work as follows:

**Send**: Process \( p_i \) ∈ Π broadcasts \( \text{msg}_i \) to all other processes. **Receive**: Process \( p_i \) receives messages from the other \( n - 1 \) processes. If \( p_i \) does not receive a message from process \( p_j \) within a bounded time, it assumes that message to be lost.

2The processes are indexed respectively with their identifiers as: Π = \{p₁, . . . , pₙ\}.
Compute: Each process $p_i$ performs the computations specified for each decision algorithm and updates its local state including $msg_i$, if necessary. After a process finishes the send, receive and compute phases for all $R$ rounds, it executes the decision algorithm. At the end of the execution of the algorithm each process either decides to select a value or decides to abort. We consider two different decision criteria for the consensus algorithm given in Alg. 1; namely the optimistic decision criterion, pessimistic decision criterion. Alg. 2 shows the description of the optimistic decision criterion. Executing the optimistic decision criterion, if the view of a process $p_i$ is complete at the end of the $R^{th}$ round it selects its $proposed_i$ as the highest value. Indeed a process with complete view optimistically assumes that all the other processes have also complete views and select a value. A process with incomplete view at the end of the $R^{th}$ round decides to abort. Alg. 3 shows the compute phase for the pessimistic decision criterion. At the end of all rounds except for the last round, process $p_i$ updates its proposed value and its view vector in the same way as in the compute phase of the optimistic criterion. In addition at the end of all rounds $p_i$ updates its $C_i$, its confirmation vector, by definition.

III. SYMMETRIC COMMUNICATION FAILURES

In this section, we present several graphs showing how the probabilities of different outcomes of the 1-of-$n$ selection algorithm varies for different configurations of a system under the symmetric communication failures only. We discuss and analyse the results in Section III-A. Then in Section III-B, we present the closed-form expressions we derived to compute the probabilities of agreement on a value, $P_{AB}$, agreement to abort, $P_{AB}$, and disagreement, $P_{DG}$, among $n$ processes executing the 1-of-$n$ selection algorithm in $R$ rounds. The corresponding probabilities are calculated as a function of the probability of a symmetric communication failure denoted as $q$, which is the probability of failure of a send operation by a process.

A. Observations for Symmetric Failures

Fig. 1 shows the probabilities of agreement, abort and disagreement respectively denoted as $AG$, $AB$ and $DG$, as a function of $q$ for 1-of-3 selection algorithm with symmetric failures in two rounds for both optimistic and pessimistic decision criterion. As expected, with increasing the probability of message loss, for both decision criteria, the probability of
agreement decreases from 1 to 0 while the probability of abort increases from 0 to 1. With higher probability of message loss there are more executions of the algorithm that all processes have incomplete views at the end of round $R$ and therefore we have lower number of agreement cases and larger number of cases of abort. We see from Fig. 1 that each of the curves for the probability of disagreement shows a distinct peak which occurs at a higher value of $q \simeq 0.63$ for the optimistic decision criterion compared to the pessimistic decision criterion which occurs at $q \simeq 0.15$. Also, the maximum probability of disagreement is considerably higher for the optimistic decision criterion than for the pessimistic decision criterion. Fig. 2 shows $P_{DG}$ for a 1-of-3 consensus algorithm as a function of $q$. The solid curves show the results for the optimistic decision criterion with $R = 2, 3, 4$ and 6. The dotted curves show the corresponding results for the pessimistic criterion. We see that the peak values of $P_{DG}$ for the optimistic criterion are considerably higher than those for the pessimistic criterion. Also as expected from the results shown in Fig. 1, the $P_{DG}$ for the optimistic decision criterion peaks at higher values of $q$ compare to the pessimistic approach. For both decision criteria, with increasing the number of rounds, the peaks of $P_{DG}$ move to the right w.r.t the x-axis. This is expected since with more rounds of execution the probability that all processes have a complete view at the last round is higher. Fig. 3 shows the probability of disagreement, $P_{DG}$, for a 1-of-$n$ selection algorithm with $R = 3$ rounds. The different curves represent the results for systems with different number of processes ($n = 2, 3, 4$ and 6). For both decision criteria, with increasing $n$, the peak of the curves move to the left w.r.t the x-axis. Hence, for systems with higher number of processes, if we assume fixed number of rounds of execution, the peak of $P_{DG}$ occurs at lower values of $q$. This implies that the probability of agreement to abort increases when the number of processes increases.

B. Closed-form Expressions

We present our approach in derivation of closed-form expressions to calculate the probabilities for each outcome of the 1-of-$n$ selection algorithm in $R$ rounds, assuming symmetric communication failures only. First, we count the number of different possibilities that all of the $n$ processes agree or disagree. The number of different possibilities for each outcome, should be counted out of the total $2^{n-R}$ possibilities. In particular, we compute $AG$, $AB$, and $DG$ for a given $n$ and $R$, where $AG$ is the total number of executions that all processes select the same value. $AB$ is the total number of executions that all processes decide to abort and $DG$ is the total number of executions that some processes select a value while others decide to abort. In addition, by considering the probability of a message loss, $q$, we also compute $P_{AG}$: the probability of agreement on a value; $P_{AB}$: the probability of agreement to abort and $P_{DG}$: the probability of disagreement for a system of $n$ processes where $P_{AG} + P_{AB} + P_{DG} = 1$.

According to Algorithm 1, each process executes the send operation in each round. Each send operation can either be successfully received by all processes or no process receives the message. As a result, there are $2^{n-R}$ possible combinations of the views of the $n$ processes at the end of the $R^{th}$ round. For large values of $n$ and $R$ the number of different possible executions of the algorithm can be exponential. For such big systems, it can be computationally prohibitive to calculate $AG$, $AB$, and $DG$, when $n \cdot R$ is very large (e.g., $n = 20$ cars execute the 1-of-$n$ selection algorithm for $R = 5$ rounds in a road intersection). Therefore, finding an efficient way to compute $AG$, $AB$, and $DG$, is a non-trivial and challenging problem. Our endeavour is to address this challenge and efficiently

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3The messages sent from $n$ number of processes can be lost or delivered in any round among $R$ rounds of execution.
compute $AG$, $AB$, and $DG$ for the optimistic and pessimistic decision criteria. We show that it is possible to compute $AG$, $AB$ and $DG$ in linear time by conducting elegant analysis of each decision criterion for the $I$-of-$n$ selection algorithm. The following propositions are useful to show how we derive the closed-form expressions. For details on the proofs for the given expressions we refer the reader to Appendix.

We use Proposition 1—3 in finding $AG$, $AB$ and $DG$ for a decision criterion. In addition to the propositions, for the ease of presentation, we define predicates $C_0$, $C_1$ and $C_2$ each representing an IF condition given in Algorithm 2 and Algorithm 4. Table I shows the conditions under which the given predicates can be true or false for a given process as $p_i$ executing the $I$-of-$n$ selection algorithm.

Proposition 1. Two or more processes fail to send their messages in all the $1...K$ rounds, if and only if, all $n$ processes have incomplete views at the end of $K$th round.

Proposition 2. All processes have complete views at the end of $K$th round, if and only if, each process successfully broadcasts its message in at least one of the $K$ rounds.

Proposition 3. One process has complete view and the remaining $(n-1)$ nodes have incomplete views at the end of $K$th round, if and only if, exactly one process fails to broadcast its message in all of the $K$ rounds.

1) Analysis of the Optimistic Decision Criterion: According to Alg. 2, if the view of a process is complete (i.e., $C_0$ is true) at the end of the $R$th round, the process selects a value; otherwise, it must decide to abort.

Finding closed-form expressions for $P_{AG}$: To derive the closed-form expression to calculate $P_{AG}$, first, we determine $AG$, the number of cases that the views of all processes are complete at the end of the $R$th round (i.e., $C_0$ is true for all processes). According to Proposition 2, the views of all of the $n$ processes are complete, if and only if each process successfully sends its message at least once during the $R$ rounds. Considering that each send operation in a round, can either be successful or unsuccessful, there are $\sum_{i=1}^{\binom{R}{i}}$ possible executions of the algorithm in which at least once in $R$ rounds, the message broadcast by a process is successfully delivered to the other processes. Since there are $n$ processes in a system, we can calculate the total number of executions resulting in agreement on a value among $n$ processes in $R$ rounds using the given in Eq. 1.

$$ AG = \left( \sum_{i=1}^{R} \binom{R}{i} \right)^n = (2^R - 1)^n \quad (1) $$

The expression given in Eq. 2 calculates the probability that all processes decide to select a value as a function of $q$, the probability of a message loss. It is assumed that $i$ send operations of each process are successful with the probability of $1-q$ while $R-i$ send operations are unsuccessful for each process with the probability of $q$. Note that $i$ starts from one since according to Proposition 2, we assume that each process broadcasts its message at least once with the probability of 1. So the probability that $n$ processes decide to select a value (i.e., agreement on value) is given as follows:

$$ P_{AG} = \left( \sum_{i=1}^{R} \binom{R}{i} \cdot (1-q)^i \cdot q^{R-i} \right)^n = (1-q^R)^n \quad (2) $$

Finding closed-form expressions for $P_{AB}$: To calculate $AB$, we need to find the number of ways in which the view of each of the $n$ processes is incomplete at the end of $R$th round (i.e., $C_0$ is false for all processes). According to Proposition 1, $C_0$ is false for all processes if and only if at least two processes fail to broadcast their messages during all $R$ rounds. If there are $i$ processes that fail to broadcast their message in all $R$ rounds, where $2 \leq i \leq n$, then each of the remaining $n-i$ processes successfully broadcast their message in at least one of the $R$ rounds. For a given $i$ provided that $2 \leq i \leq n$, there are $\binom{R}{i}$ possible cases in which all $n-i$ processes successfully broadcast their message in at least one of the $R$ rounds. Moreover, $i$ number of processes can be selected out of $n$ processes in $\binom{n}{i}$ number of ways. Therefore, the closed-form expression to calculate the number of cases that all processes decide to abort is given in Eq. 3:

$$ AB = \sum_{i=2}^{n} \binom{n}{i} \cdot (2^R - 1)^{n-i} \quad (3) $$

Given that the probability of a message loss is $q$, the probability that exactly $i$ processes fail to send in all $R$ rounds is $(q^R)^i$. On the other hand, the probability that all $n-i$ processes successfully broadcast their message in at least one of the $R$ rounds is \[ \sum_{j=1}^{R} \binom{R}{j} \cdot (1-q)^j \cdot q^{R-j} \] \[ = (1-q^R)^{n-i}, \] where $2 \leq i \leq n$. Consequently, the closed-form expression to calculate the probability that all processes agree to abort is given in Eq. 4:

$$ P_{AB} = \sum_{i=2}^{n} \binom{n}{i} \cdot (q^R)^i \cdot (1-q^R)^{n-i} \quad (4) $$

Finding closed-form expressions for $P_{DG}$: We know that the total number of possible executions of a $I$-of-$n$ selection algorithm in $R$ rounds is $AG + AB + DG = 2^nnR$. So we can calculate the value of $DG$ using the expression given in Eq. 5.

$$ DG = 2^nnR - AG - AB \quad (5) $$

where $AG$ and $AB$ are computed in Eq. (1) and Eq. (3), respectively. Similarly, the probability of disagreement can be calculated from the formula 6 follows:

$$ P_{DG} = 1 - P_{AG} - P_{AB} \quad (6) $$

where $P_{AG}$ and $P_{AB}$ are computed in Eq. (2) and Eq. (4),
respectively. As mentioned before details on the proofs for the given expression are given in Appendix A.

2) Analysis of the Pessimistic Decision Criterion: We present the closed-form expressions to compute \( P_{DG} \) and \( P_{AG} \) for a system of \( n \) processes executing the 1-of-n selection algorithm with the pessimistic decision criterion when the probability of a message loss is \( q \). According to Alg. 4, and the specified predicates in Table I, if \( C_1 \) and \( C_2 \) are true for a process, it can decide to select a value.

Finding closed-form expressions for \( P_{DG} \): To derive closed-form expressions to calculate the probability of disagreement for the case of pessimistic decision criterion, we define Lemma 4 which is proved in the Appendix.

Lemma 4. In order to have disagreement among processes, it is necessary that all processes have complete views at the end of round \( R - 1 \).

There are two possible ways that the views of all the processes can be complete at the end of round \( r \), where \( 1 \leq r \leq R - 1 \):

Case I When all processes have incomplete views at the end of round \( r - 1 \) but they all have complete views after round \( r \).

Case II When exactly \( n - 1 \) processes have incomplete views at the end of round \( r - 1 \) and all processes have complete views after round \( r \).

Combining the probabilities for Case I and Case II, the probability of disagreement for the pessimistic decision criterion is computed using Eq. 7.

\[
P_{DG} = P_{DG(CaseI')} + P_{DG(CaseII')} = (1 - q^n) \cdot (1 - q^{R-1})^n \cdot \left( \sum_{r=2}^{R-1} \sum_{i=2}^{n} \left( \frac{n}{i} \cdot (1 - q^{r-1})^{n-i} \cdot q^{(r-1)i} \cdot (1 - q)^i \cdot n \cdot q^{(R-r)} \cdot (1 - q^{R-r})^{n-1} \right) \right)
\]

Finding closed-form expressions for \( P_{AG} \): We have agreement on a value if all processes decide to select a value. According to Alg. 4, a process \( p_i \) decides to select a value if it satisfies two conditions; First, if its view is complete at the end of round \( R - 1 \) and second if it receives messages from all other processes at some point during \( R \) rounds of execution indicating the completeness of the view of the corresponding sending process. The crucial observation is that having complete views by all of the processes at the end of round \( R - 1 \) is a necessary condition for agreement. There are two possible ways that the views of all the processes can be complete at the end of round \( r \), where \( 1 \leq r \leq R - 1 \):

Case I' When all processes have incomplete views at the end of round \( r - 1 \) and all processes have complete views by the end of round \( r \).

Case II' When exactly \( n - 1 \) processes have incomplete views at the end of round \( r - 1 \), then at the end of round \( r \) all processes have complete views.

We can show that other than these two cases, there is no other execution that agreement on a value may occur. Given that the views of all processes are complete at round \( r \), to compute the probability of agreement on a value, we have to consider that each process receives complete messages from all other processes. We derive closed-form expressions to calculate the probability of agreement on a value, \( P_{AG} \), for the pessimistic decision criterion with analysing the two cases, Case I' and Case II'. Combining the expressions for calculating the probability of agreement on a value among processes when they meet the conditions for Case I' and Case II', the probability of agreement on a value among the processes is computed from Eq. 8.

\[
P_{AG} = P_{AG(CaseI') + P_{AG(CaseII')} = (1 - q^n) \cdot (1 - q^{R-1})^n + \sum_{r=2}^{R-1} \sum_{i=2}^{n} \left( \frac{n}{i} \cdot (1 - q^{r-1})^{n-i} \cdot q^{(r-1)i} \cdot (1 - q)^i \cdot (1 - q^{(R-r)})^n + \sum_{r=2}^{R-1} \left( n \cdot q^{r-1} \cdot (1 - q^{r-1})^{n-1} \cdot (1 - q) \cdot (1 - q^{R-r})^{n-1} \right) \right)
\]

Finally, the probability of abort is \( P_{AB} = 1 - P_{DG} - P_{AG} \) where \( P_{DG} \) and \( P_{AG} \) are computed in Eq. 7 and Eq. 8, respectively.

IV. ASYMMETRIC COMMUNICATION FAILURES

In order to calculate the probabilities of each outcome of the I-of-n selection algorithm with asymmetric communication failures, we modelled the protocol using a probabilistic model checking tool named as PRISM [14]. In the following, we present the results for the asymmetric failure model along with a brief explanation on our PRISM models (For more details on the PRISM models see [15]).

A. Observations for Asymmetric Failures

![Fig. 4. Probability of agreement, abort and disagreement denoted as \( AG \), \( AB \) and \( DG \) respectively, as a function of \( Q \) for 1-of-3 consensus protocol with asymmetric failures \( (R = 2) \), a comparison of optimistic and pessimistic decision criterion.](image)
for the pessimistic decision criterion as $Q$ approaches one. The probability of disagreement, $P_{DG}$, shows, as expected, a distinct peak in all of the curves. The maximum probability of disagreement is considerably higher for the optimistic criterion than for the pessimistic criterion. On the other hand, the peak occurs for much lower values of $Q$ for the pessimistic criterion. These results show that there are pros and cons to both decision criteria. If we compare Fig. 1 and Fig. 4, we see that the asymmetric failure model have higher peaks for $P_{DG}$ compared to the symmetric failure model. It is interesting to see that $P_{DG}$ is as high as 68% for the optimistic criterion for asymmetric failures.

Fig. 5 shows $P_{DG}$ for a 1-of-3 consensus algorithm as a function of $Q$ with various values of $R$. Similar to the symmetric failure model, the peak values of $P_{DG}$ for the optimistic criterion are considerably higher than those for the pessimistic criterion, while for the pessimistic criterion the $P_{DG}$ peaks at lower values of $Q$ compared to the optimistic criterion. Interestingly, we see that the peak values increase as the number of rounds increases for the pessimistic criterion, while the trend for the optimistic criterion is the reverse. Comparing the results in Fig. 5 with those in Fig. 2, as expected, we see higher probabilities of disagreement for the asymmetric failure model than for the symmetric failure model. Similar to the results for the symmetric failure model, the peak of the curves move to the right for both decision criteria when the number of rounds increases.

Fig. 6 and 7 show $P_{DG}$ as a function of $Q$ assuming $R = 2$ and $R = 3$ respectively, for systems of various values of $n$ processes. For $R = 3$, we see that with increasing $n$, we obtain higher peak values of $P_{DG}$ for both decision criteria. Also, the peaks move to the left w.r.t. the x-axis for the pessimistic decision criterion, while they move to the right w.r.t the x-axis for the optimistic decision criterion. From Fig. 7, we also observe that the peak values for the pessimistic decision criterion is higher than for the optimistic one, which is the opposite to what we observed for the symmetric failures in Fig. 2. We can see that for systems with many processes under asymmetric failures, when $R \geq 3$, the optimistic criterion performs better than the pessimistic criterion in the sense of having higher probabilities of agreement and lower disagreement. Indeed, from Fig. 6, we see that $R = 2$ is an exception case where the maximum of $P_{DG}$ for the pessimistic criterion, remain around the same value with increasing $n$.

### B. PRISM models

We model the behaviour of the protocol using a discrete time markov chain containing both deterministic and probabilistic transitions. For the probabilistic transitions, the choice of the next state is determined by a discrete probability distribution. We use probabilistic transitions in order to model the probability of losing of a message due to a communication failure. Fig. 8 illustrates the conceptual model of a process for the case of $n = 3$, with the following transitions for process $p_1$ ($T_i$ denotes transition $i$, $S_0$ denotes state 0):

- **T1**: $S_1 \rightarrow S_2$: $p_1$ sends its message;
- **T2**: $S_2 \rightarrow S_3$: $p_1$ receives (with prob. 1-$Q$) or loses (with prob. $Q$) the message from $p_2$;
T3: S3 → S4: p₁ receives (with prob. 1-Q) or loses (with prob. Q) the message from p₃;
T4: S4 → S1: Not last round: p₁ executes compute phase;
T5: S4 → S5: Last round: p₁ execute compute phase;
T6: S5 → S0: p₁ computes the decision: agree or abort.

For a more detailed description of the PRISM models, we refer the reader to [15].

C. Asymmetric failures assuming variance of Q

In this section, our aim is to study the sensibility of the asymmetric failure model that assumes the same probability of message loss for all the communication links, to a more generic communication failure model for which the probability of message loss among processes becomes higher with increasing the physical distance among them. We compare the probability of disagreement for systems of 3 and 4 processes executing the 1-of-n selection algorithm for two, three and four rounds, assuming different asymmetric failure models. The results given in section are calculated using PRISM.

![Fig. 9](image_url)  
(a) n = 3  
(b) n = 4

Fig. 9. Distribution of vehicles in a row with difference distances

![Fig. 10](image_url)  
(a) n = 3, R = 2  
(b) n = 3, R = 4  
(c) n = 4, R = 3

Fig. 10. Probability of disagreement, ΔQ = 0, 0.1 and 0.2.

remains approximately the same for the case of the optimistic decision criterion (See Fig. 10(b) and 10(c)).

For all figures, with higher ΔQ, the curves move slightly to the right side of the x-axis. One explanation is the fact that although we adopted \( Q_M \) as the median between \( \min(Q_{ij}) \) and \( \max(Q_{ij}) \), for the given case studies in Fig. 9, there are more links with \( Q_M - \Delta Q \) than with \( Q_M + \Delta Q \), which means that the average of all \( Q_{ij} \) is actually lower than \( Q_M \).

It is important to note that, considering the expressions given in Fig. 9, the probability of message loss, \( Q_M \), becomes negative when \( Q_M < \Delta Q/2 \) and becomes greater than 1 when \( Q_M > (1 - \Delta Q/2) \). So, the given expressions in Fig. 9 are only valid for the interval of \( \Delta Q/2 < Q_M < (1 - \Delta Q/2) \). Therefore, for the intervals of \( 0 < Q_M < \Delta Q/2 \) and \( (1 - \Delta Q/2) < Q_M < 1 \), we replace \( \Delta Q \) with a reduced value called \( \Delta Q_R \). For \( 0 < Q_M < \Delta Q/2 \), we adopt \( \Delta Q_R = 2 \times Q_M \).
For \((1 - \Delta Q/2) \leq Q_M < 1\), we consider \(\Delta Q_R = 2 \times (1 - Q_M)\). These expressions progressively reduce \(\Delta Q_R\) such that for \(Q_M = 0\) and \(Q_M = 1\), we assume \(\Delta Q_R = 0\). In other words, at the two extreme points of the x-axis, the curve of \(Q_M\) with a given \(\Delta Q > 0\) is approximated to curve of \(Q_M\) with \(\Delta Q = 0\).

V. RELATED WORK

The consensus problem has been investigated widely in the area of distributed computing such as in [16]–[18] and is proved to be solvable under different failure assumptions. However most of the previous research has been concentrated on associating communication failures to process failures rather than investigating them explicitly as an independent phenomenon (e.g., see [19], [20]). Such unrealistic assumptions may lead to incorrect characterizations of a system. For example loss of a single message due to a transient communication failure is inscribed to a faulty behaviour of the sending or receiving process.

We study the consensus problem for systems subject to an arbitrary number of communication failures. Our failures model is inspired by the failure model called as the transmission fault model or the dynamic omission fault model which was first introduced by Santoro and Widmayer in [1]. It is shown in [1] that under this model, even if one considers a perfectly synchronous system, if omission failures are unrestricted, it is impossible to guarantee consensus. There are a large number of methods suggested in literature to circumvent the impossibility result in synchronous consensus protocols with dynamic omission faults [21]–[23]. However, the suggested methods have mostly a preventive approach toward this problem, such as restricting the communication failure patterns or limiting the number of failures in a round. Nevertheless, it is possible to design protocols that have a low probability of failing to reach consensus, so as to meet specific requirements on reliability and availability. This intuition has been explored to build protocols that maximize the probability of correctness by accumulating more information over a larger duration of the execution [24]. Researchers have also focused on stochastic models of verifying the probability of transition into an incorrect state [25].

This paper continues our previous works [7] and [8] in design and analysis of decision making algorithms to be run on top of a simple consensus protocol with the main purpose of minimizing the probability of failing to reach consensus. We evaluate and compare the effectiveness of different decision algorithms by means of using probabilistic model checking tools as well as deriving closed-form expressions to calculate the probability of disagreement among processes. Our results may be applied also for on-line verification and adaptation to cope with variable probabilities of communication failures.

Recently, the consensus algorithms have been applied in cooperating automotive and aeronautics systems under a wireless environment, such as in the control strategy of a platoon formation of highway vehicles [9] or in protocols for electing the leader of a virtual traffic light in a road intersection [7] which motivates our work to evaluate the sensibility of our failure assumptions in such applications. For this, we compare our results for a number of scenarios to the results we obtain for a new asymmetric failure model where the probability of losing a message depends on different factors such as the distance among the vehicles [10].

VI. CONCLUSION AND FUTURE WORK

We have presented closed-form expressions for calculating the probability of disagreement in the presence of unbounded number of symmetric failures for a family of simple synchronous consensus protocols. Our work is motivated by the need to develop fast and reliable consensus algorithm for distributed cooperative systems for the transportation sector. Since it is impossible to construct an algorithm that solves the consensus problem for a system that uses wireless, and thereby, unreliable communication, we are interested in exploring the design of adaptive consensus protocols that are equipped with an on-line mechanism which can temporarily shut down the protocol in situations where the likelihood for disagreement becomes unacceptably high. We are therefore interested in finding computational effective ways of calculating the probability of disagreement on-line. Additionally, we performed a study of the sensibility of our asymmetric failure model for a simple scenario of a system of three and four vehicles, to a more realistic failure model in which the probability of message loss, \(Q\), varies depending on the physical distance among the cars. Our initial results show that for low values of \(n\) and \(R\), the probability of disagreement is not significantly different for the two asymmetric failure models, which motivates one of our future works to find closed-form expressions for calculating the probability of disagreement under the assumption of asymmetric message losses.

REFERENCES

Proof of Proposition 1

The given proposition is a theorem of the form "A If And Only If B". The A part is true, when two or more processes fail to send their messages in all the 1...K rounds and the B part is true when all n processes have incomplete views at the end of Kth round. So, in order to prove the given proposition, we break the "If And Only If" proposition into two lemmas and prove them separately. First, we prove If A Then B, then we prove If B Then A.

Lemma 5. If two or more processes fail to broadcast their message in all the 1...K rounds then all processes will have incomplete views at the end of Kth round.

We prove Lemma 5 by contradiction. We show that Assump. 6 implies a contradiction.

Assumption 6. If there are less than two processes (i.e., zero or one process) that fails to broadcast its message for all K rounds, then all processes will have incomplete views at the end of Kth round.

First, it is clear that if no process (i.e., zero process) fails to broadcast its message in all K rounds, then all processes receive messages from all other processes in the system and as a result, the views of all processes are complete at the end of Kth round. On the other hand, if exactly one process as px fails to broadcast its message for K rounds, while each of the remaining n − 1 processes successfully broadcast their message in at least one of the K rounds, then only process px will have the complete view, and all other processes have incomplete views of the system. So we can conclude that there is no execution of the algorithm for which less than two processes fail to broadcast their message for K rounds and all processes have complete views at the end of Kth round (i.e., at least one process must have complete view). This contradicts our initial assumption and consequently shows the validity of Lemma 5.

Lemma 7. If all processes have incomplete views at the end of Kth round, then two or more processes have failed to broadcast their messages in all the 1...K rounds.

To prove Lemma 7 we use proof by contradiction by making Assum. 8:

Assumption 8. There is at least one process, say process px, that has complete view at the end of Kth round, and two or more processes have failed to broadcast their messages in all the 1...K rounds.

A process as px with a complete view at the end of Kth round must have received messages from each of the n − 1 processes at least once. This means that each of the processes in set \{p1, . . . , px−1, px+1, . . . , pn\} must have successfully broadcast their message at least once during K rounds of executions. This contradicts the initial assumption that two or more processes fail to send in all K rounds since |{p1, . . . , px−1, px+1, . . . , pn}| = n − 1. So, Lemma 7 is valid.

Proof of Proposition 2

To prove Proposition 2, we divide the proposition into two lemmas. Lemma 9 and Lemma 11.

Lemma 9. If all processes have complete views at the end of the Kth round, then each process must have successfully broadcast its message in at least one of the K rounds.

We use proof by contradiction to prove Lemma 9. First we define the following assumption:

Assumption 10. All processes have complete views at the end of the Kth round, however some processes failed to broadcast their message in all of the K rounds.

We show that the Assum. 10 implies a contradiction. If all processes have complete views at the end of the Kth round, it means that each of the processes has received a message from the other processes at least in one of the K rounds of execution. This contradicts the assumption that some processes failed to broadcast their message in all of the K rounds.
To prove the validity of Lemma 11 we define the following
contradictory assumption:

**Assumption 12.** All processes successfully broadcast their
message in at least one of the \( K \) rounds, and there are some
processes that have incomplete views at the end of \( K \)th round.
If all processes have successfully broadcast their message
at least once in \( K \) rounds of execution, then all processes
should have received the message from all other processes
and therefore have a complete view of the system at the end
of the \( K \)th round. This contradicts the assumption that there
are some processes that have incomplete views at the end of
\( K \)th round. Given that Lemma 9 and Lemma 11 are proved
using contradiction, we conclude that Proposition 2 is proved
to be valid.

**Proof of Proposition 3**

It is evident from the proof of Proposition 1 that at most one
process can fail to send in all the \( K \) rounds if and only if at
least one process has complete view at the end of \( K \)th round.
And, according to Proposition 2, no process fails to send in
all the \( K \) rounds if and only if all processes have complete view.
Combining these two observations, it is not difficult to
see that exactly one process fails in all \( K \) rounds if and only
if the view of this process is complete while the views of the
remaining \( n - 1 \) nodes are incomplete.

**Finding \( P_{DG} \) for Pessimistic Decision Criterion**

**Lemma 4.** In order to have disagreement among processes,
it is necessary that all processes have complete views at the
end of round \( R - 1 \).

We prove the validity of Lemma 4 by contradiction. We
consider that disagreement occurs if some processes decide to abort
while other processes decide to select a value. Based on
Algorithm 4, a process as \( p_x \) decides to abort if its view is
not complete at the end of round \( R - 1 \) (i.e., \( C_1 \) is false for
\( p_x \)). On the other hand, if \( p_x \) does not have a complete view at
the end of round \( R - 1 \), none of the other processes receive a
complete view from process \( p_x \) in any round including round \( R \)
(i.e., \( C_2 \) is false for all other processes). So, all other processes
also decide to abort and as a result, there is no disagreement
but agreement to abort.

**Analysis of Case I.** We divide this case into two different
subcases as follows:

- Case I. All processors have complete views at the end
  of round \( r = 1 \).
- Case II. All processors have incomplete views at the end
  of round \( r = 1 \) and have complete views at round
  \( r \), where \( 2 \leq r \leq R - 1 \).

**Analysis of Case II.** The probability that all processes
have complete views at the end of the first round is \( (1 - q)^n \).
Disagreement occurs if during rounds \( 2 \ldots R \), there is exactly
one process, say \( p_x \), that does not broadcast a complete view in
any round while the other \( (n - 1) \) processes broadcast complete
views in at least one of the rounds \( 2 \ldots R \). The probability that
a process \( p_x \) does not broadcast its complete view in any round
\( 2 \ldots R \) is \( q^{R-r-1} \) and the probability that all other processes in
\( P \setminus \{ p_x \} \) broadcast their complete view in at least one of the
rounds \( 2 \ldots R \) is \( (1 - q^{R-r-1})^{n-1} \). Since the process \( p_x \) can
be selected in \( n \) possible ways, the probability of disagreement
when all processes have complete views at the end of first
round is given as below:

\[
P_{DG \text{ Case I.I} \_1} = (1 - q)^n \cdot q^{(R-r-1)} \cdot (1 - q^{R-r-1})^{n-1} \cdot n \quad (9)
\]

**Analysis of Case III.** According to Proposition 1, if the
views of all processes are incomplete at the end of round \( r - 1 \),
where \( 2 \leq r \leq R - 1 \), then at least two or more processes
have failed to send in all \( 1 \ldots r - 1 \) rounds. For a given round \( r \),
the probability that all processes have incomplete views at the
end of round \( r - 1 \) is calculated from

\[
\sum_{i=2}^{n} \binom{n}{i} (1 - q^{R-r-1})^{n-i} \cdot q^{(r-r-1) \cdot i} \cdot q^{(r-1)^{\cdot i}}
\]

where the messages sent from \( i \) processes are lost in all \( r - 1 \)
rounds, but the other \( n - i \) processes successfully broadcast
their message in at least one of the \( r - 1 \) rounds, where \( 2 \leq i \leq n \).
According to Case I.II, the view of all processes are
complete at the end of round \( r \). So, during round \( r \), all of the \( i \)
processes successfully broadcast their message which happens
with the probability \( (1 - q)^i \). Therefore, for a given round \( r \), the
probability that all processes have incomplete views at the end
of round \( r - 1 \) and have complete views at the end of round \( r \) is
calculated from \( \sum_{i=2}^{n} \binom{n}{i} (1 - q^{R-r-1})^{n-i} \cdot q^{(r-r-1) \cdot i} \cdot q^{(r-1)^{\cdot i}} \cdot (1 - q)^i \).

Assuming that all processes have complete views at the end
of round \( r \), disagreement occurs if exactly one process, say
\( p_x \), does not successfully broadcast its complete view in any
of the remaining \( R - r \) rounds with the probability of \( q^{R-r} \),
while all remaining \( n - 1 \) processes successfully broadcast
their complete views in at least one of the remaining \( R - r \)
rounds, with the probability of \( (1 - q^{R-r})^{n-1} \). Process \( p_x \) can
be selected in \( n \) possible ways. For a given \( r \), the probability
of disagreement for Case I.II is:

\[
P_{DG \text{ Case I.II}} = \sum_{i=2}^{n} \binom{n}{i} (1 - q^{R-r-1})^{n-i} \cdot q^{(r-r-1) \cdot i} \cdot (1 - q)^i \cdot q^{(R-r) \cdot i} \cdot (1 - q^{R-r})^{n-1} \quad (10)
\]

Since \( r \) ranges from 2 to \( R - 1 \), the probability that
disagreement occurs when all processes have complete views
at the end of any round \( 2 \ldots R - 1 \) is given as follows:

\[
P_{DG \text{ Case I.II}} = \sum_{r=2}^{R-1} \sum_{i=2}^{n} \binom{n}{i} (1 - q^{R-r-1})^{n-i} \cdot q^{(r-r-1) \cdot i} \cdot (1 - q)^i \cdot q^{(R-r) \cdot i} \cdot (1 - q^{R-r})^{n-1} \quad (11)
\]

Combining the probabilities for Case I and Case I.II, we
have the probability of disagreement for Case I given in 12.

\[
P_{DG \text{ Case I} = (1 - q)^n \cdot q^{(R-r)} \cdot (1 - q^{R-r})^{n-1} \cdot n + \\
\sum_{r=2}^{R-1} \sum_{i=2}^{n} \binom{n}{i} (1 - q^{R-r-1})^{n-i} \cdot q^{(r-r-1) \cdot i} \cdot (1 - q)^i \cdot q^{(R-r) \cdot i} \cdot (1 - q^{R-r})^{n-1} \quad (12)
\]

**Analysis of Case II.** In this case, exactly \( n - 1 \) processes
have incomplete views during rounds \( r - 1 \) and all the processes
have complete views in round \( r \), where \( 2 \leq r \leq R - 1 \). This
can happen if exactly one process, say \( p_x \), fails to send in all
the \( r - 1 \) rounds and other \( n - 1 \) processes send at least in one
of the \( r - 1 \) rounds. Given a particular round \( r \), \( 2 \leq r \leq R - 1 \),
the probability that any of the \( n - 1 \) processes fail to send in all
\( r - 1 \) rounds is \( n \cdot q^{(r-1)} \) and the probability that each of the
other \( n - 1 \) processes successfully broadcasts its message in at least one of the \( r - 1 \) rounds is \((1 - q^{r-1})^{n-1}\).

Process \( p_x \) at round \( r \) with the probability of \( 1 - q \), successfully broadcasts its message and as a result, the views, of all the processes are complete at the end of round \( r \). We know that the probability that the views of all processes are complete at the end of round \( r \) is \( n \cdot q^{(r-1)} \cdot (1 - q^{r-1})^{n-1} \cdot (1 - q) \). Since the view of process \( p_x \) is complete at the end of round \( r - 1 \), the send operation by process \( p_x \) in round \( r \) also confirms the completeness of the view of \( p_x \) to all other processes.

Assuming that all processes have a complete view of the system at the round \( r \), disagreement occurs if exactly one process, say \( p_y \), where \( p_x \neq p_y \), does not send its complete view in any of the remaining \( R - r \) rounds with the probability of \( q^{R-r} \), while each of the other \( n - 2 \) processes in \( \Pi - \{p_x, p_y\} \) broadcast their complete view in at least one of the remaining \( R - r \) rounds with the probability of \( (1 - q^{R-r})^{n-2} \). Process \( p_y \) can be selected in \( n - 1 \) possible ways from set \( \Pi - \{p_x\} \). So, for a given \( r \), the probability of disagreement assuming Case II, given that all processes have complete views at the end of round \( r \), can be calculated using \( n \cdot q^{r-1} \cdot (1 - q^{r-1})^{n-1} \cdot (1 - q) \cdot (n - 1) \cdot q^{R-r} \cdot (1 - q^{R-r})^{n-2} \).

Since \( r \) ranges from 2 to \( R - 1 \), the probability of disagreement, given that all the processes have complete views in any of the \( 2 \ldots R - 1 \) rounds, is given in 13:

\[
P_{\text{Case II}} = \sum_{r=2}^{R-1} \sum_{i=2}^{n} n \cdot q^{r-1} \cdot (1 - q^{r-1})^{n-1} \cdot (1 - q) \cdot (n - 1) \cdot q^{R-r} \cdot (1 - q^{R-r})^{n-2}
\]  

Finding \( P_{\text{Ag}} \) for Pessimistic Decision Criterion

**Analysis of Case I’**: We divide this case into two subcases as follows:

Case I.1 All processes have complete views at the end of round \( r = 1 \).

Case I.2 All processes have incomplete views at the end of round \( r - 1 \) and have complete views at the end of round \( r \), where \( 2 \leq r \leq R - 1 \).

**Analysis of Case I.1**: The probability that all processes have complete views at the end of round 1 is \((1 - q)^n\).

Agreement can occur if during rounds 2 \ldots R, each of the \( n \) processes successfully sends its complete view in at least one of the 2 \ldots R rounds. The probability of a successful message broadcast by a process in any round 2 \ldots R is \((1 - q^{R-1})^n\), and the probability of successful complete message broadcast by all of the \( n \) processes is \((1 - q^{R-1})^n\). The probability of agreement when all processes have complete views at the end of 1st round is given as follows:

\[
P_{\text{AgCase I.1}} = (1 - q)^n \cdot (1 - q^{R-1})^n
\]  

**Analysis of Case I.2**: If the views of all processes are incomplete at the end of round \( r - 1 \), where \( 2 \leq r \leq R - 1 \), then the messages sent from at least two or more processes have been lost in all \( 1 \ldots r - 1 \) rounds. For a given round \( r \), the probability that all processes have incomplete views at the end of round \( r - 1 \) is \( \sum_{i=2}^{N} \binom{n}{i} (1 - q^{r-1})^{n-i} \cdot q^{(r-1)i} \), where message broadcasts from \( i \) out of \( n \) processes are failed in all \( r - 1 \) rounds and \( n - i \) processes successfully broadcast their message in at least one of the \( r - 1 \) rounds, where \( 2 \leq i \leq n \). Since the views of all processes are complete at the end of round \( r \) for this case, all \( i \) processes must successfully broadcast their message during round \( r \) which happens with the probability of \((1 - q)^i\). Therefore, for some given \( r \), the probability that all processes have incomplete views at the end of round \( r - 1 \) and have complete views at the end of round \( r \) can be calculated from \( \sum_{i=2}^{N} \binom{n}{i} (1 - q^{r-1})^{n-i} \cdot q^{(r-1)i} \cdot (1 - q)^i \).

If the views of all processes are complete at the end of round \( r \), agreement on a value occurs, if each of the \( n \) processes successfully broadcast their complete message in at least one of the rounds \((r + 1) \ldots R\). The probability of a successful message broadcast by one process in any round \((r + 1) \ldots R\) is \(1 - q^{(R-r)}\) and the probability of successful message broadcast by all \( n \) processes in \((1 - q^{(R-r)})^n\).

The probability of agreement when all processes have complete views at the end of \( r \)th round is:

\[
P_{\text{AgCase I.2}} = \sum_{i=2}^{N} \binom{n}{i} (1 - q^{r-1})^{n-i} \cdot q^{(r-1)i} \cdot (1 - q)^i \cdot (1 - q^{(R-r)})^n
\]  

Since \( r \) ranges from 2 to \( R - 1 \), the probability that disagreement occurs when all the processes have complete view at the end of any round 2 \ldots \( R - 1 \) is given as follows:

\[
P_{\text{AgCase I.2}} = (1 - q)^n \cdot (1 - q^{R-1})^n + \sum_{r=2}^{R-1} \sum_{i=2}^{N} \binom{n}{i} (1 - q^{r-1})^{n-i} \cdot q^{(r-1)i} \cdot (1 - q)^i \cdot (1 - q^{(R-r)})^n
\]  

**Analysis of Case II’**: This case occurs if exactly one process, say \( p_x \), fails to broadcast its message in all of the \( r - 1 \) rounds while other \( n - 1 \) processes successfully broadcast their message in at least one of the \( r - 1 \) rounds. Given a \( r \), 2 \leq r \leq R - 1, the probability that the message broadcast from any of the \( n \) processes fails in all the \( r - 1 \) rounds is \( n \cdot q^{r-1} \) and the probability that each of the other \( n - 1 \) processes successfully broadcasts their message in at least one of the \( r - 1 \) rounds is \((1 - q^{r-1})^{n-1}\). The process \( p_x \) successfully broadcasts its message with the probability of \( 1 - q \) at round \( r \) and as a result, the views of all other processes are complete at the end of round \( r \). The probability that the view of each of the processes is complete at the end of round \( r \) is equal to \( n \cdot q^{r-1} \cdot (1 - q^{r-1})^{n-1} \cdot (1 - q) \). Since the view of process \( p_x \) is complete at the end of round \( r - 1 \), the send operation by process \( p_x \) at round \( r \) also confirms that \( p_x \) has a complete view of the system. If the views of all processes are complete at the end of round \( r \), agreement on a value occurs if each of the \( n - 1 \) processes in \( \Pi - \{p_x\}\) successfully broadcast their message in at least one of the remaining \( R - r \) rounds. This occurs with the probability of \((1 - q^{R-r})^n\). The probability of agreement, given that all processes have complete views at round \( r \), is equal to \( n \cdot q^{r-1} \cdot (1 - q^{r-1})^{n-1} \cdot (1 - q) \cdot (1 - q^{R-r})^n\).

As \( r \) ranges from 2 to \( R - 1 \), the probability of disagreement, if all processes have complete views in any of the \( 2 \ldots R - 1 \) rounds, is as follows:

\[
P_{\text{AgCase II’}} = \sum_{r=2}^{R-1} n \cdot q^{r-1} \cdot (1 - q^{r-1})^{n-1} \cdot (1 - q) \cdot (1 - q^{R-r})^n
\]  

\( (18) \)