Probabilistic Analysis of Disagreement in Synchronous Consensus Protocols (Preliminary draft)

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Abstract

This report presents a probabilistic analysis of a family of simple synchronous round-based consensus algorithms aimed at solving the *1-of-n* selection problem. In this problem, a set of n nodes are to select one common value among a set of n proposed values. There are two possible outcomes of each node's selection process: it can decide either to *select a value*, or to *abort*. Agreement implies that all nodes select the same value, or all nodes decide to abort. We analyse this problem under the assumption of massive communication failures considering symmetric and asymmetric message losses. Previous research has shown that it is impossible to guarantee agreement among the nodes in a synchronous system subjected to an unbounded number of message losses. Our aim is to find algorithms for which the probability of disagreement is as low as possible. To this end, we study how the probability of disagreement varies for three different decision criteria, the *optimistic*, *pessimistic* and the *moderately pessimistic*. Our results show that that the probability of disagreement varies significantly with the number of nodes, the number of rounds, and the probability of message loss. In general, the optimistic decision criterion performs better (has a lower probability of disagreement) than the pessimistic one when the probability of message loss is less than 30% to 70%. On the other hand, the optimistic decision criterion has in general a higher maximum probability of disagreement compared to the pessimistic decision criterion. Moreover we show that the outcome of the moderately pessimistic decision criterion generally lies in between the two other decision criteria.

Keywords

distributed algorithms; consensus; reliability analysis; symmetric and asymmetric communication failures.

I. Introduction

The problem of reaching consensus on a value among a set of cooperating distributed computing units has been studied extensively over the last thirty years. Despite this, we still lack definite answers to how consensus is best solved in distributed systems that rely on wireless communication.

A main challenge in solving the consensus problem in wireless distributed systems is that the communication channel can be subjected to disturbances of varying duration and magnitude. Consequently, we cannot make any assumptions about the number of messages that can be lost during the execution of a distributed consensus algorithm in such systems. We know from previous research [1], [2] that it is impossible to construct an algorithm that guarantees consensus in the face of unrestricted communication failures.

Design of consensus algorithms that minimize the probability of disagreement in the presence of unrestricted communication failures is an important emerging problem in the area of cooperative systems. Examples of applications that demand fast and reliable real-time consensus include autonomous and semi-autonomous cooperative systems for improving traffic safety and fuel-efficiency of road vehicles, such as vehicle platooning, virtual traffic lights[1] and coordinated lane change [3]. Similar demands also emerge in applications for safe and fuel-efficient autonomous manoeuvring of air vehicles.

We are interested in investigating the possibility of using simple deterministic synchronous algorithms for fast consensus in safety-critical cooperative systems. To this end, we investigate in this paper a family of synchronous round-based consensus algorithms to solve the *1-of-n* selection problem in the presence of symmetric and asymmetric communication failures.

Since we know that we cannot construct an algorithm that solves this problem perfectly, our aim is to find algorithms for which the probability of disagreement is as low as possible.

In *1-of-n* selection, each node (in a system of n nodes) proposes a value and then all nodes must either select the same value, which has to be one of the proposed values, or decide to abort. Disagreement occurs if some nodes decide to select

[1]In a virtual traffic light, road vehicles approaching an intersection interact via wireless communication to form a virtual, or imaginary, traffic light.

a value, while the remaining nodes decide to abort. (The algorithms we study in this paper is safe in the sense that they guarantee that the nodes will never decide on different values.) As we will show later, the probability of disagreement for a given algorithm depends in general on three parameters: i) the number of nodes in the system, ii) the number of rounds of message exchange, and iii) the probability of message loss. In addition, it also depends on the decision criterion that determines whether a node should decide to abort or to select a value.

The main contributions of this report are as follows. We investigate three practical decision criteria called the *optimistic*, *pessimistic* and *moderately pessimistic* decision criterion. For these, we present closed-form expressions for calculating the probability of disagreement in the presence of symmetric communication failures. For asymmetric communication failures, we use a probabilistic model checking tool to calculate the probability of disagreement. We use this tool and the closed-form expressions to illustrate how the probability of disagreement varies for different system configurations.

The remainder of the paper is organized as follows. Section II describes the system model and our failure assumptions. In addition we present the description of the *1-of-n* selection algorithm and the three decision criteria. To provide an intuitive understanding of the impact of the decision criteria, we present in Section III an analysis of a simple system consisting of two processes executing a two-round consensus algorithm. We present closed-form expressions for calculating the probability of disagreement for symmetrical communication failures in Section IV. In Section V we briefly describe PRISM [4] the probabilistic model checking tool which we used to model the *1-of-n* selection algorithm. Section VI presents a number of diagrams that illustrates how the probability of disagreement varies for different protocol configurations. We discuss related work in Section VII, and finally in Section VIII we conclude and outline some directions for future research.

II. Protocol Description

A. System Model and Failure Assumptions

Our *1-of-n* selection algorithm is based on the classical round-based computational model used by many researchers, such as in [5], [6] and [7]. We consider a synchronous system consisting of a set of n processes. The processes are indexed respectively with their identifiers as: $\Pi = \{p_1, \ldots, p_n\}$. We assume that processes are fully connected to one another via wireless broadcast links and that they execute a deterministic protocol in R rounds of message exchange. Each round consists of three phases: *send, receive* and *compute*.

Failures occur as message losses and can occur on any communication link at any time during the execution of the algorithm. In other words, there are no restrictions on the number, timing or pattern of the lost messages.

We consider two different scenarios for a lost message: (i) when all the intended receivers of the message fail to receive the message (*symmetric message loss*), and (ii) when only a subset of the intended receivers fail to receive the message (*asymmetric message loss*).

For simplicity, we assume that the nodes are fault-free. Note, however, that a send omission failure of a node is equivalent to a symmetric message loss, and that a receive omission failure is equivalent to asymmetric message loss where only one node fails to receive a message.

B. The Consensus Algorithm

Alg. 1 shows the pseudocode of the *1-of-n* selection algorithm executed by each process $p_i \in \Pi$. We assume that all processes initially constructs a message, denoted as msg_i, that contains a proposed value ($proposed_i$) and a bit-vector (v_i) of length n that represents the view of process p_i. Initially, $v_i[j] = 0$ for all j (i.e., at this point p_i has not received any message from other processes in the system). We define v_i as *complete* if all elements of this vector are set to 1. Similarly, we define v_i as *incomplete* if at least one element of v_i is 0.

Algorithm 1 Generic algorithm for *1-of-n* selection (p_i)

```
msg_i ← {v_i, proposed_i};
for r = 1 to R do
    begin_round
    send_to_all(msg_i);
    receive_from_all();
    compute(msg_i);
    end_round
end for
execute_decision_algorithm();
```

After initialization, each process iterates the send, receive and compute phases in each of the R rounds. These phases work as follows:

Send: Process $p_i \in \Pi$ broadcasts msg_i to all other processes.

Receive: Process p_i receives messages from the other $n-1$ processes. If p_i does not receive a message from process p_j within a bounded time, it assumes that message to be lost.

Compute: Each process p_i performs the computations specified for each decision algorithm and updates its local state including msg_i, if necessary.

After a process finishes the send, receive and compute phases for all R rounds, it executes the decision algorithm. At the end of the execution of the algorithm each process either decides to *select a value* or decides to *abort*.

We consider three different decision criteria for the consensus algorithm given in Alg. 1; namely the *optimistic* decision criterion, *pessimistic* decision criterion and the *moderately pessimistic* decision criterion. Alg. 2 shows the description of the *optimistic* decision criterion. Executing the optimistic decision criterion, if the view of a process p_i is complete at the end of the R^{th} round it selects its $proposed_i$ as the highest value. Indeed a process with complete view optimistically assumes that all the other processes have also complete views and select a value. A process with incomplete view at the end of the R^{th} round decides to abort.

Algorithm 2 Optimistic decision criterion (p_i)

```
if v_i is complete then
    p_i selects proposed_i;
else
    abort;
end if
```

Alg. 3 shows the compute phase for a process p_i executing the optimistic criterion. p_i updates $proposed_i$ to $proposed_j$ if $proposed_j > proposed_i$.Process p_i also updates its v_i vector at the end of each round as follows: For all the elements of v_j that are set to 1, process p_i sets the corresponding elements in v_i to 1 also (If it is not already 1).

Algorithm 3 Compute phase: optimistic decision criterion (p_i)

```
∀p_j ∈ Π − {p_i}
if p_i received msg_j then
    if proposed_i < proposed_j then
        proposed_i = proposed_j;
    end if                                          ▷ update proposed_i

    ∀p_k ∈ Π − {p_i, p_j}
    if (v_j[k] = 1 and v_i[k] = 0) then
        v_i[k] = 1;
    end if
    v_i[j] = 1
                                                    ▷ update v_i
end if
```

Alg. 4 shows the description of the pessimistic decision criterion. A process p_i with *incomplete* v_i at the end of the execution of the algorithm decides to abort while a process p_i with a *complete* view pessimistically assumes that other processes do not have complete views unless they confirmed this at some point during R rounds of execution. If process p_i does not receive such confirmations from all processes it decides to abort. We define a vector C_i of size n for each process p_i in order to keep a record of the processes who have sent a confirmation to p_i (to confirm that their view is complete). Initially, $C_i[j] = 0$ for all j. When $(C_i)_j$ is set to 1, it means that p_i has received a message from p_j showing that v_j is complete. C_i is complete if all of its elements are set to 1. Alg. 5 shows the compute phase for the pessimistic decision criterion. At the end of all rounds except for the last round, process p_i updates its proposed value and its view vector in the same way as in the compute phase of the optimistic criterion. In addition at the end of all rounds p_i updates its C_i, its confirmation vector, by definition.

Alg. 6 shows the description of the compute phase defined for the moderately pessimistic decision criterion. When a process as p_i receives a message from a process as p_j it updates $proposed_i$ to $proposed_j$ if $proposed_j > proposed_i$.

Algorithm 4 Pessimistic decision criterion (p_i)

if v_i is *complete* **then**
  if C_i is *complete* **then**
    p_i selects $proposed_i$;
  else
    abort;
  end if
else
  abort;
end if

Algorithm 5 Compute phase: pessimistic decision criterion (p_i)

$\forall p_j \in \Pi - \{p_i\}$
if p_i received msg_j **then**
  if $r \neq R$ **then**
    if $proposed_i < proposed_j$ **then**
      $proposed_i = proposed_j$;
    end if ▷ update $proposed_i$
    $\forall p_k \in \Pi - \{p_i, p_j\}$
    if $(v_j[k] = 1$ and $v_i[k] = 0)$ **then**
      $v_i[k] = 1$;
    end if
    $v_i[j] = 1$ ▷ update v_i
  end if
  if v_j is *complete* **then**
    $C_i[j] = 1$;
  end if
end if

Algorithm 6 Compute phase: Moderately pessimistic decision criterion (p_i)

$\forall p_j \in \Pi - \{p_i\}$
if p_i received msg_j **then**
  if $r \neq R$ **then**
    if $proposed_i < proposed_j$ **then**
      $proposed_i = proposed_j$;
    end if
    $\forall p_k \in \Pi \setminus \{p_i, p_j\}$
    if $(v_j[k] = 1$ and $v_i[k] = 0)$ **then**
      $v_i[k] = 1$;
    end if
  end if
end if

Algorithm 7 Moderately pessimistic decision criterion (p_i)

if v_i is *complete* **then**
  if receives some incomplete view in round R **then**
    abort;
  else
    p_i selects *the highest value*;
  end if
else
  abort;
end if

Table I
POSSIBLE EXECUTIONS FOR A $n = 2$, $R = 2$ SYSTEM
msg_i = { v_i, $proposed_i$ }

Case	*round* 1				*round* 2			
	T_1	T_2	msg_1	msg_2	T_1	T_2	msg_1	msg_2
1	OK	OK	[{0},11]	[{0},12]	OK	OK	[{1},12]	[{1},12]
2	OK	OK	[{0},11]	[{0},12]	OK	Fail	[{1},12]	[–]
3	OK	OK	[{0},11]	[{0},12]	Fail	OK	[–]	[{1},12]
4	OK	OK	[{0},11]	[{0},12]	Fail	Fail	[–]	[–]
5	OK	Fail	[{0},11]	[–]	OK	OK	[{0},11]	[{1},12]
6	OK	Fail	[{0},11]	[–]	OK	Fail	[{0},11]	[–]
7	OK	Fail	[{0},11]	[–]	Fail	OK	[–]	[{1},12]
8	OK	Fail	[{0},11]	[–]	Fail	Fail	[–]	[–]
9	Fail	OK	[–]	[{0},12]	OK	OK	[{1},12]	[{0},12]
10	Fail	OK	[–]	[{0},12]	OK	Fail	[{1},12]	[–]
11	Fail	OK	[–]	[{0},12]	Fail	OK	[–]	[{0},12]
12	Fail	OK	[–]	[{0},12]	Fail	Fail	[–]	[–]
13	Fail	Fail	[–]	[–]	OK	OK	[{0},11]	[{0},12]
14	Fail	Fail	[–]	[–]	OK	Fail	[{0},11]	[–]
15	Fail	Fail	[–]	[–]	Fail	OK	[–]	[{0},12]
16	Fail	Fail	[–]	[–]	Fail	Fail	[–]	[–]

Process p_i also updates its v_i vector at the end of each round as follows: for all the elements of v_j that are set to 1, p_i sets the corresponding elements in v_i to 1 (if it is not already 1). The update of its proposed value and its view vector occurs at the end of all round except for the last round (i.e., round R).

Alg. 7 shows the description of the moderately pessimistic decision criterion. A process p_i executing the moderately pessimistic decision criterion decides to abort if its view, v_i is incomplete. Otherwise it checks the second **if** statement given at line 2 (See Alg. 7). If p_i at round R, receives a message from a process p_j indicating that v_j is incomplete, process p_i must abort, otherwise it selects its $proposed_i$ as the highest value. Process p_i disregards the lost messages in the last round and optimistically assume a complete view for the senders of lost messages.

III. ANALYSIS OF A SIMPLE SYSTEM

In order to provide an intuitive understanding of how the decision criterion influences the probability of the three possible outcomes of the decision process, i.e., *agreement on a value*, *disagreement*, *agreement on abort*, we will in this section compare the outcomes of the optimistic and the pessimistic decision criteria for a *1-of-2* selection algorithm using two rounds of message exchange (i.e., a *1-of-n* selection algorithm with $n = 2$ and $R = 2$). We assume that the two processes (called, p_1 and p_2) respectively propose *11* and *12* as their initial (ranking) values. The objective of the algorithm is to reach agreement on the highest value proposed by any of the two processes, i.e., the value *12* in our example.

In this algorithm, each of the two processes sends two messages, one in the the first round and one in second round. Thus, in total four messages are exchanged during the execution of the algorithm. Since we assume that any number of messages can be lost due to communication failures, there are $2^4 = 16$ permutations of lost and successful messages, see Table I.

The columns denoted T_1 refer to transmissions from p_1 to p_2, while columns denoted T_2 refers to transmissions from p_2 to p_1. A successful message transmission is marked as 'OK' while a transmission failure is marked as 'Fail'.

The columns denoted msg_1 and msg_2 state the contents of the messages received by each process in the corresponding round. msg_1 (resp. msg_2) is the message received by p_2 (resp. p_1) from p_1 (resp. p_2). As explained before, msg_i consists of v_i, the view vector of process p_i, and $proposed_i$, the value proposed by p_i. The content of a message is given within square brackets. The view vector is given with curly brackets. As an example, [{0},11] indicates that the view vector[2] is $\{0\}$ (this shows that the process has not yet received a message from the other process) while the proposed value is 11. '[–]' denotes the loss of a message due to a transmission failure. As a further example consider Case 1 in Table I: msg_1

[2]Note that the view vector actually consists of two bits. However, one of the bits represents the process's view of its own value and this bit is always set to 1. For simplicity, we have omitted this bit in the example.

Table II
RESULTS FOR *1-of-2* SELECTION ALGORITHM
AG:AGREEMENT DG: DISAGREEMENT

Case	*Optimistic decision criterion*	*Pessimistic decision criterion*
1	AG(12)	AG(12)
2	AG(12)	DG
3	AG(12)	DG
4	AG(12)	AG(abort)
5	AG(12)	AG(abort)
6	DG	AG(abort)
7	AG(12)	AG(abort)
8	DG	AG(abort)
9	AG(12)	AG(abort)
10	AG(12)	AG(abort)
11	DG	AG(abort)
12	DG	AG(abort)
13	AG(12)	AG(abort)
14	DG	AG(abort)
15	DG	AG(abort)
16	AG(abort)	AG(abort)

Table III
COMPARISON OF DECISION CRITERIA, VIEW OF PROCESS p_1

Case	msg_1 sent by p_1 to p_2 in round 2	msg_2 received by p_1 from p_2 in round 2	State of msg_1 after *compute* phase in round 2	Optimistic decision by p_1	Pessimistic decision by p_1
A	[{1},12]	[{1},12]	[{1},12]	12	12
B	[{1},12]	[{0},12]	[{1},12]	12	abort
C	[{1},12]	[–]	[{1},12]	12	abort
D	[{0},11]	[{1},12]	[{1},12]	12	abort
E	[{0},11]	[{0},12]	[{1},12]	12	abort
F	[{0},11]	[–]	[{0},11]	abort	abort

in *round 2* represents the message received by p_2 and consists of $[\{1\}, 12]$. 12 is the value that p_1 proposes in this round. (p_1 receives 12 from p_2 in round one, and since 12 is greater than its own value 11, p_1 proposes 12 in the second round.) $\{1\}$ is the view vector of p_1 and is set to 1 because p_1 received a message from p_2 in the first round. Table II shows the outcomes of the decision process for the 16 cases shown in Table I. As we can see in Table II there are 9 cases of agreement on a value, 6 cases of disagreement, one case of agreement on abort for the optimistic decision criterion. For the pessimistic decision criterion there are one case of agreement on a value, two cases of disagreement, and 13 cases of agreement on abort.

We now explain the outcomes shown in Table II. To this end, we refer the reader to Table III and Table IV which describe the six possible variants of information that process p_1 and p_2 can have after the execution of round 2. These cases are denoted A to F for p_1 in Table III and A' to F' for p_2 in Table IV.

We will first look at the outcomes of disagreement for the optimistic decision criterion. We start with Case 6 in Table I

Table IV
COMPARISON OF DECISION CRITERIA, VIEW OF PROCESS p_2

Case	*msg*$_2$ sent by p_2 to p_1 in round 2	*msg*$_1$ received by p_2 from p_1 in round 2	State of *msg*$_2$ after *compute* phase in round 2	Optimistic decision by p_2	Pessimistic decision by p_2
A'	[{1},12]	[{1},12]	[{1},12]	12	12
B'	[{1},12]	[{0},11]	[{1},12]	12	abort
C'	[{1},12]	[–]	[{1},12]	12	abort
D'	[{0},12]	[{1},12]	[{1},12]	12	abort
E'	[{0},12]	[{0},11]	[{1},12]	12	abort
F'	[{0},12]	[–]	[{0},12]	abort	abort

and II. In this case, the two messages sent by p_2 are both lost, whereas the two messages sent by p_1 are received by p_2. Since p_1 has not received any information from p_2, p_1 must abort according to Case F in Table III. p_2 receives both messages from p_1 and decides to select the value 12 according to Case B' in Table IV. Hence, p_1 and p_2 decides differently, which leads to disagreement. The same the thing happens in Case 11, where the two messages sent by p_1 are both lost, whereas the two messages sent by p_2 are received by p_1.

In Case 8, p_1 decides to abort according to Case F in Table III, while p_2 decides on 12 according to Case C' in Table IV. Case 12 is the same as Case 8, with p_1 and p_2 swapped. In Case 14, p_1 decides to abort according to Case F in Table III, while p_2 decides on 12 according to Case E' in Table IV. In Case 15 p_2 decides to abort according to Case F', while p_1 decides on 12 according to Case E. These examples illustrates why the decision criterion is called optimistic. Consider Case 15, where p_1 decides on 12 although it knows that p_2 did not received the message it sent in round 1: p_1 optimistically assumes that the message it sent in round 2 is successfully received by p_2.

We now consider the two cases of disagreement for the pessimistic decision criterion shown in Table II. Using the pessimistic decision criterion a process p_i will not decide on a value unless it has received confirmation that the other process p_j has received the value p_i sent in round 1. (In the general case of *1-of-n* selection, this is ensured by the second *if* statement in Alg. 3, which states that a process must have received complete views for all processes in order to decide on a value.) Thus, it is only for case A in Table III and case A' in Table IV that the pessimistic decision criterion decides on a value. This implies that there is disagreement if one of the messages sent in the last round is lost and all other messages are successful. Clearly, there are two such cases, namely Case 2 and 3 in Table II.

IV. CLOSED-FORM EXPRESSIONS FOR SYMMETRIC COMMUNICATION FAILURES

In this section, we derive closed-form expressions for calculating the probability of disagreement, P_{DG}, for the optimistic, pessimistic and moderately pessimistic decision criteria in the presence of symmetric communication failures. The derivation of closed-form expressions for the case of asymmetric communication failures is left for future work.

A. Sketch of Analysis and Useful Propositions

Safety-critical system that requires consensus needs to be predictable in the sense that appropriate level of assurance regarding any possible outcome of the consensus algorithm is guaranteed before the system is put in mission. To provide such assurance, the objective in this section is to efficiently count the number of different possibilities (out of total $2^{n \cdot R}$ possibilities) such that all the n processes agree/disagree under the assumption of symmetric communication failure. In addition, given the probability of a message loss, denoted by q (i.e., the probability that a send operation is unsuccessful due to symmetric communication failure), the probability of agreement/abort/disagreement among all the n processes is also computed. Such probabilistic analysis is useful, for example, to verify that the system's probability of disagreement is not greater than some tolerable limit.

In this section, the detailed analysis of both the optimistic and pessimistic decision criteria for the *1-of-n* selection algorithm is presented in order to efficiently compute the number of possible ways all the processes select the same value (*agreement on a value*) or all the processes decide to abort (*agreement on abort*) or all the processes can neither select the same value nor can decide to abort (*disagreement*). In particular, we compute `AG`, `AB`, and `DG` for some given n and R, where

- `AG` is the total number of ways all processes select the same value (i.e., agreement on a value)

- $\texttt{AB}$ is the total number of ways all processes decide to abort (i.e., agreement on abort)
- $\texttt{DG}$ is the total number of ways some processes select a value while others decide to abort (i.e., disagreement).

In addition, by considering the probability of a message loss (q), we also compute

- $P_{\texttt{AG}}$: the probability of agreement on value;
- $P_{\texttt{AB}}$: the probability of agreement to abort; and
- $P_{\texttt{DG}}$: the probability of disagreement

of the system where $(P_{\texttt{AG}} + P_{\texttt{AB}} + P_{\texttt{DG}}) = 1$.

According to the *1-of-n* selection algorithm given in Algorithm 1, each of the n processes executes the send operation in each of the R rounds. Since each send operation can either be successful (message reaches to every other process) or unsuccessful (message reaches to no process), there are $2^{n \cdot R}$ possible combinations of all the views of the n processes at the end of R^{th} round. Finding $\texttt{AG}$, $\texttt{AB}$ and $\texttt{DG}$ by considering the exponential number of possible views of all the processes (a trivial but exhaustive approach) can be computationally prohibitive when $n \cdot R$ is very large (e.g., $n = 20$ cars execute the consensus algorithm for $R = 5$ rounds in a road intersection). Finding an efficient way to compute $\texttt{AG}$, $\texttt{AB}$, and $\texttt{DG}$ is non-trivial and challenging problem. Our endeavour in Section IV-B and Section IV-C is to address this challenge to efficiently compute $\texttt{AG}$, $\texttt{AB}$, and $\texttt{DG}$ for optimistic and pessimistic criteria, respectively. It will be evident that $\texttt{AG}$, $\texttt{AB}$ and $\texttt{DG}$ can be computed in linear time by conducting elegant analysis of each decision making criterion for the *1-of-n* selection algorithm. The following propositions will be useful in Section IV-B and Section IV-C.

Proposition 1. *Two or more processes fail to send during all the $1 \ldots K$ rounds if and only if all the n processes have incomplete view at the end of K^{th} round.*

Proof: **(only if part)** Assume a contradiction that there is at least one process, say process p_x, that has complete view at the end of K^{th} round. If process p_x has complete view at the end of K^{th} round, then each of the processes in set $\{p_1, \ldots, p_{x-1}, p_{x+1}, \ldots p_n\}$ successfully sends during one or more of the K rounds. This contradicts the fact that two or more processes fail to send in all K rounds since $|\{p_1, \ldots, p_{x-1}, p_{x+1}, \ldots p_n\}| = (n-1)$.

(if part) Assume a contradiction that zero or one process fails to send in all K rounds. If zero, i.e., no process fails to send in all K rounds, then the view of each of the n processes is complete at the end of K^{th} round. And, if exactly one process fails to send in all the K rounds, then each of the remaining $(n-1)$ processes successfully sends during at least one of the K rounds. This implies that the only process that fails to send during all the K rounds has the complete view. Therefore, if zero or one process fails to send during all K rounds, then at least one process has complete view at the end of K^{th} round (contradiction!). ■

Proposition 2. *Each process successfully sends message during at least one of the K rounds if and only if each of the n processes has complete view at the end of K^{th} round.*

Proof: **(only if part)** Assume a contradiction that there is at least one process that has incomplete view at the end of K^{th} round. This can happen only if there is at least one process fails to send in all the K rounds (contradiction!).

(if part) Assume a contradiction that there is at least one process that fails to send during all the K rounds. This implies that the view of all the processes cannot be complete at the end of K^{th} round (contradiction!). ■

Proposition 3. *Exactly one process fails to send in all K rounds if and only if one process has complete view and the remaining $(n-1)$ nodes have incomplete view at the end of K^{th} round.*

Proof: It is evident from the proof of Proposition 1 that at most one process can fail to send in all the K rounds if and only if at least one process has complete view at the end of K^{th} round. And, according to Proposition 2, no process fails to send in all the K rounds if and only if all processes have complete view. Combining these two observations, it is not difficult to see that exactly one process fails in all K rounds if and only if the view of this process is complete while the views of the remaining $(n-1)$ nodes are incomplete. ■

Proposition 1—3 will be useful to find $\texttt{AG}$, $\texttt{AB}$ and $\texttt{DG}$ for the optimistic decision criterion of the *1-of-n* selection algorithm. Each process select a value or decided to abort based on the different **IF** conditions of Algorithm 2 and Algorithm 3 respectively for the optimistic and pessimistic criteria. For ease of presentation, we denote C0, C1, C2 the various **IF** conditions present in Algorithm 2 and Algorithm 3. The semantics for the conditions C0, C1 and C2 being true or false for some process p are given below:

- C0 is true: The view of the process is complete.
- C0 is false: The view of process is incomplete.

The analysis of the optimistic decision criterion to find $\texttt{AG}$, $\texttt{AB}$ and $\texttt{DG}$ is now presented in Section IV-B.

B. Analysis of Optimistic Decision Criterion

The optimistic decision criterion (given in Algorithm 2) for each process is simple: if the view of a process is complete (i.e., C0 is true) at the end of the R^{th} round, then the process selects a value; otherwise, it decides to abort. We determine $\texttt{AG}$, $\texttt{AB}$ and $\texttt{DG}$ for optimistic decision criterion in next three subsections.

1) Finding P_{AG} for Optimistic Criterion: We have to determine the number of ways the view of each of the n processes can be complete at the end of R^{th} round (i.e., the condition C0 is true for all processes). According to Proposition 2, the views of all the n processes are complete if and only if each process successfully sends at least once during R rounds. Since each send operation can either be successful or unsuccessful, there are $\sum_{i=1}^{R}\binom{R}{i}$ possible combinations for each of which at least one of the R send operations by some process could be successful. Consequently, there are n processes, we have

$$\texttt{AG} = \left(\sum_{i=1}^{R}\binom{R}{i}\right)^n = (2^R - 1)^n \tag{1}$$

Note that i send operations of each process are successful while $(R-i)$ send operations of are unsuccessful for each i in Eq. (1). If q is the probability of a message loss, then the probability that all the nodes selects the a value (i.e., agreement on value) is given as follows:

$$P_{\texttt{AG}} = \left(\sum_{i=1}^{R}\binom{R}{i}\cdot(1-q)^i\cdot q^{R-i}\right)^n = (1-q^R)^n \tag{2}$$

2) Finding P_{AB} for Optimistic Criterion: We have to find the number of ways the view of each of the n processes can be incomplete at the end of R^{th} round (i.e., the condition C0 is false for all processes). According to Proposition 1, the views of all the n processes are incomplete if and only if at least two processes fail to send during all R rounds. If there are i processes that fail to send in all the R rounds, where $2 \le i \le n$, then each of the remaining $(n-i)$ processes successfully sends in at least one of the R rounds. Given some i, $2 \le i \le n$, there are $(\sum_{j=1}^{R}\binom{R}{j})^{n-i} = (2^R-1)^{n-i}$ possibilities for each of which each of the $(n-i)$ processes successfully sends during at least one of the R rounds for some given i. And, the i processes from n processes can be selected in $\binom{n}{i}$ ways, where $2 \le i \le n$. Consequently, the number of possibilities all the processes decide to abort is given as follows:

$$\texttt{AB} = \sum_{i=2}^{n}\binom{n}{i}\cdot(2^R-1)^{n-i} \tag{3}$$

Given that the probability of a message loss is q, the probability that exactly i processes fails to send in all R rounds is $(q^R)^i$ while the probability that each of the $(n-i)$ processes successfully sends during at least one of the R rounds is $[\sum_{j=1}^{R}\binom{R}{j}\cdot(1-q)^j\cdot q^{R-j}]^{n-i} = (1-q^R)^{n-i}$, where $2 \le i \le n$. Consequently, the probability that all the processes agree to abort is given as follows:

$$P_{\texttt{AB}} = \sum_{i=2}^{n}\binom{n}{i}\cdot(q^R)^i\cdot(1-q^R)^{n-i} \tag{4}$$

3) Finding P_{DG} for Optimistic Criterion: Since $(\texttt{AG} + \texttt{AB} + \texttt{DG}) = 2^{n\cdot R}$, the value of $\texttt{DG}$ is computed as follows:

$$\texttt{DG} = 2^{n\cdot R} - \texttt{AG} - \texttt{AB} \tag{5}$$

where $\texttt{AG}$ and $\texttt{AB}$ are computed in Eq. (1) and Eq. (10), respectively. Similarly, the probability of disagreement is given as follows:

$$P_{\texttt{DG}} = 1 - P_{\texttt{AG}} - P_{\texttt{AB}} \tag{6}$$

where $P_{\texttt{AG}}$ and $P_{\texttt{AB}}$ are computed in Eq. (2) and Eq. (11), respectively. This completes the analysis of the optimistic decision criteria. Next sections present the analysis for pessimistic criterion.

C. Analysis of Pessimistic Decision Criterion

In this subsection, the closed-form expressions to compute P_{DG} and P_{AG} are presented by analyzing the pessimistic decision criterion (Algorithm 4) where the probability of a message loss is q.

1) Finding P_{DG} for Pessimistic Decision Criterion: Disagreement occurs if some processes decide to abort while others decide on a value. According to the pessimistic decision criterion (given in Algorithm 4), if the view of some process, say process p_i, is not complete by round $(R-1)$, then p_i decides to abort. Notice that if process p_i does not have complete view by round $(R-1)$, it is also guaranteed that none of the other processes can receive a confirmation from process p_i in any round. Consequently, all other processes (regardless whether their views are complete or not) also decide to abort and there is no disagreement. The crucial observation is that having complete view by each of the processes at the end of round $(R-1)$ is a necessary condition for disagreement. There are two possible ways the views of all the processes can be complete at the end of round r, where $1 \leq r \leq (R-1)$:

- **Case (i):** all the processes have incomplete views by the end of round $(r-1)$ and all the processes have complete views in round r, and
- **Case (ii)** exactly $(n-1)$ processes have incomplete views by the end of round $(r-1)$ and all the processes have complete views in round r.

Notice that other than these two cases, there is no other case in round r for which disagreement may occur. Given that the views of all the processes are complete at round r, to compute the probability of disagreement, we have to consider that *exactly* one process, say process p_x, receives confirmation from *all* other processes while others do not receive all confirmations[3]. The probability of disagreement for pessimistic decision criterion is computed by analyzing each of the above two cases.

Analysis of Case (i): For this case, all n processes have incomplete views during rounds $(r-1)$ and all n processes have complete views in round r, where $1 \leq r \leq R-1$. We consider two different subcases for this case:

- **Subcase (i)** all the processors have complete views at round $r = 1$,
- **Subcase (ii)** all the processors have incomplete views at round $(r-1)$ and have complete views at round r, where $2 \leq r \leq (R-1)$.

Subcase (i): The probability that all the processes have complete views at the end of round 1 is $(1-q)^n$. Disagreement can occur if during rounds $2 \ldots R$, there is exactly one process, say p_x, that does not send confirmation in any round while the other $(n-1)$ processes send confirmation in at least one of the rounds $2 \ldots R$. The probability of not sending any confirmation by p_x in any round $2 \ldots R$ is $q^{(R-1)}$ and the probability of sending confirmation by each of the processes in $\Pi - \{p_x\}$ during at least one of the rounds $2 \ldots R$ is $(1-q^{R-1})^{n-1}$. Since the process p_x can be selected in n possible ways, the probability of disagreement when all processes have complete views at the end of 1^{st} round is given as follows:

$$(1-q)^n \cdot q^{(R-1)} \cdot (1-q^{R-1})^{n-1} \cdot n$$

Subcase (ii) If the views of all the processes are incomplete at the end of round $(r-1)$, where $2 \leq r \leq (R-1)$, then at least two or more processes have failed to send in all $1, \ldots (r-1)$ rounds. For a given round r, the probability that all the processes has incomplete view at the end of $(r-1)$ rounds is $\sum_{i=2}^{n} \left(\binom{n}{i} (1-q^{r-1})^{n-i} \cdot q^{(r-1)\cdot i} \right)$ where i out of n processes fail to send in all $(r-1)$ rounds and $(n-i)$ processes successfully send in at least one of the $(r-1)$ rounds, where $2 \leq i \leq n$. Because the view of all the processes are complete at the end of round r for this case, all these i processes must successfully send during round r and this has the probability $(1-q)^i$. Therefore, for some given r, the probability that all the processes have incomplete views at the end of round $(r-1)$ and have complete views at the end of round r is $\sum_{i=2}^{n} \left(\binom{n}{i} (1-q^{r-1})^{n-i} \cdot q^{(r-1)\cdot i} \cdot (1-q)^i \right)$.

After the view of all the processes are complete at round r, disagreement occurs if exactly one process, say p_x, does not send confirmation in any of the remaining $(R-r)$ rounds (has probability $q^{(R-r)}$) while each of the other $(n-1)$ processes send confirmation in at least one of the remaining $(R-r)$ rounds (has probability $(1-q^{R-r})^{n-1}$). The process p_x can be selected in n possible ways. For a given r, the probability of disagreement for this subcase is

$$\sum_{i=2}^{n} \left(\binom{n}{i} (1-q^{r-1})^{n-i} \cdot q^{(r-1)\cdot i} \cdot (1-q)^i \cdot n \cdot q^{(R-r)} \cdot (1-q^{R-r})^{n-1} \right)$$

Since r ranges from 2 to $R-1$, the probability that disagreement occurs when all the processes have complete view at the end of any round $2 \ldots R-1$ is given as follows:

[3] Note that more than one process receive confirmation from all other processes if and only if each process receives confirmation from all other processes (i.e., all processes decide on a value).

$$\sum_{r=2}^{R-1}\sum_{i=2}^{n}\left(\binom{n}{i}(1-q^{r-1})^{n-i}\cdot q^{(r-1)\cdot i}\cdot(1-q)^{i}\cdot\right.$$
$$\left. n\cdot q^{(R-r)}\cdot(1-q^{R-r})^{n-1}\right)$$

Combining the probabilities for subcase (i) and subcase (ii), we have

$$\left((1-q)^{n}\cdot q^{(R-1)}\cdot(1-q^{R-1})^{n-1}\cdot n\right) \quad +$$
$$\sum_{r=2}^{R-1}\sum_{i=2}^{n}\left(\binom{n}{i}(1-q^{r-1})^{n-i}\cdot q^{(r-1)\cdot i}\cdot(1-q)^{i}\cdot\right.$$
$$\left. n\cdot q^{(R-r)}\cdot(1-q^{R-r})^{n-1}\right)$$

Analysis of Case (ii): For this case, exactly $(n-1)$ processes have incomplete views during rounds $(r-1)$ and all the processes have complete views in round r, where $2 \leq r \leq R-1$. This can happen if exactly one process, say p_x, fails to send in all the $(r-1)$ rounds and other $(n-1)$ processes send at least in one of the $(r-1)$ rounds. Given a particular round r, $2 \leq r \leq (R-1)$, the probability that any one of the n processes fails to send in all the $(r-1)$ rounds is $n \cdot q^{(r-1)}$ and the probability that each of the other $(n-1)$ processes sends in at least one of the $(r-1)$ rounds is $(1-q^{r-1})^{n-1}$. The process p_x must send successfully at round r (has probability $(1-q)$) because the views of all the processes are complete at the end of round r. The probability that the view of each of the processes is complete at the end of round r is $\left(n\cdot q^{(r-1)}\cdot(1-q^{r-1})^{n-1}\cdot(1-q)\right)$. Since the view of process p_x is complete at the end of round $r-1$, the send operation by process p_x in round r is also the confirmation of p_x to all other processes.

After the view of all the processes are complete in round r, disagreement occurs if exactly one process, say p_y, where $p_x \neq p_y$, does not send confirmation in any of the remaining $(R-r)$ rounds (has probability $q^{(R-r)}$) while each of the other $(n-2)$ processes in $\Pi - \{p_x, p_y\}$ send confirmation in at least one of the remaining $(R-r)$ rounds (has probability $(1-q^{R-r})^{n-2}$). The process p_y can be selected in $(n-1)$ possible ways from set $\Pi - \{p_x\}$ and the probability of disagreement, given than all processes have complete views at round r, is equal to

$$n\cdot q^{r-1}\cdot(1-q^{r-1})^{n-1}\cdot(1-q)\cdot(n-1)\cdot q^{(R-r)}\cdot(1-q^{R-r})^{n-2}$$

Since r ranges from 2 to $R-1$, the probability of disagreement, given that all the processes have complete views in any of the $2 \dots R-1$ rounds, is given as follows:

$$\sum_{r=2}^{R-1}\left(n\cdot q^{r-1}\cdot(1-q^{r-1})^{n-1}\cdot(1-q)\cdot\right.$$
$$\left.(n-1)\cdot q^{(R-r)}\cdot(1-q^{R-r})^{n-2}\right)$$

Combining the probabilities for case (i) and case (ii), the probability of disagreement for the pessimistic decision criterion is computed as follows:

$$
\begin{aligned}
P_{DG} = & \Big((1-q)^n \cdot q^{(R-1)} \cdot (1-q^{R-1})^{n-1} \cdot n \Big) \\
& + \sum_{r=2}^{R-1} \sum_{i=2}^{n} \Bigg(\binom{n}{i} (1-q^{r-1})^{n-i} \cdot q^{(r-1)\cdot i} \cdot (1-q)^i \cdot \\
& \qquad n \cdot q^{(R-r)} \cdot (1-q^{R-r})^{n-1} \Bigg) \\
& + \sum_{r=2}^{R-1} \Big(n \cdot q^{r-1} \cdot (1-q^{r-1})^{n-1} \cdot (1-q) \cdot \\
& \qquad (n-1) \cdot q^{(R-r)} \cdot (1-q^{R-r})^{n-2} \Big)
\end{aligned} \tag{7}
$$

2) Finding P_{AG} for Pessimistic Decision Criterion: Agreement occurs if all processes decide to select a value. According to the pessimistic decision criterion (given in Algorithm 4), a process p_i decides to select a value if its view is complete by round $(R-1)$. In addition if p_i receives confirmation from all other nodes. The crucial observation is that having complete view by each of the processes at the end of round $(R-1)$ is a necessary condition for agreement. There are two possible ways the views of all the processes can be complete at the end of round r, where $1 \leq r \leq (R-1)$:

- **Case (i):** all the processes have incomplete views by the end of round $(r-1)$ and all the processes have complete views in round r, and
- **Case (ii)** exactly $(n-1)$ processes have incomplete views by the end of round $(r-1)$ and all the processes have complete views in round r.

Notice that other than these two cases, there is no other case in round r for which agreement may occur. Given that the views of all the processes are complete at round r, to compute the probability of agreement, we have to consider that each process received confirmation from all other processes. The probability of agreement P_{AG} for pessimistic decision criterion is computed by analyzing each of the above two cases.

Analysis of Case (i): For this case, all n processes have incomplete views during rounds $(r-1)$ and all n processes have complete views in round r, where $1 \leq r \leq R-1$. We consider two different subcases for this case:

- **Subcase (i)** all the processors have complete views at round $r = 1$,
- **Subcase (ii)** all the processors have incomplete views at round $(r-1)$ and have complete views at round r, where $2 \leq r \leq (R-1)$.

Subcase (i): The probability that all the processes have complete views at the end of round 1 is $(1-q)^n$. Agreement can occur if during rounds $2 \ldots R$, each of the n processes successfully sends (confirmation) in at least one of the $2 \ldots R$ rounds. The probability of sending any confirmation by one process in any round $2 \ldots R$ is $(1-q^{(R-1)})$ and the probability of sending confirmation by all the n processes in $(1-q^{(R-1)})^n$. The probability of agreement when all the processes have complete views at the end of 1^{st} round is given as follows:

$$(1-q)^n \cdot (1-q^{R-1})^n$$

Subcase (ii) If the views of all the processes are incomplete at the end of round $(r-1)$, where $2 \leq r \leq (R-1)$, then at least two or more processes have failed to send in all $1, \ldots (r-1)$ rounds. For a given round r, the probability that all the processes has incomplete view at the end of $(r-1)$ rounds is $\sum_{i=2}^{n} \Bigg(\binom{n}{i} (1-q^{r-1})^{n-i} \cdot q^{(r-1)\cdot i} \Bigg)$ where i out of n processes fail to send in all $(r-1)$ rounds and $(n-i)$ processes successfully send in at least one of the $(r-1)$ rounds, where $2 \leq i \leq n$. Because the view of all the processes are complete at the end of round r for this case, all these i processes must successfully send during round r and this has the probability $(1-q)^i$. Therefore, for some given r, the probability that all the processes have incomplete views at the end of round $(r-1)$ and have complete views at the end of round r is $\sum_{i=2}^{n} \Bigg(\binom{n}{i} (1-q^{r-1})^{n-i} \cdot q^{(r-1)\cdot i} \cdot (1-q)^i \Bigg)$.

After the view of all the processes are complete at round r, agreement occurs if during rounds $(r+1) \ldots R$, each of the n processes successfully sends (confirmation) in at least one of the $(r+1) \ldots R$ rounds. The probability of sending any confirmation by one process in any round $(r+1) \ldots R$ is $(1-q^{(R-r)})$ and the probability of sending confirmation by all the n processes in $(1-q^{(R-r)})^n$. The probability of agreement when all the processes have complete views at the end of r^{th} round is given as follows:

$$\sum_{i=2}^{n}\left(\binom{n}{i}(1-q^{r-1})^{n-i}\cdot q^{(r-1)\cdot i}\cdot(1-q)^{i}\cdot\right.$$
$$\left.(1-q^{(R-r)})^n\right)$$

Since r ranges from 2 to $R-1$, the probability that disagreement occurs when all the processes have complete view at the end of any round $2\ldots R-1$ is given as follows:

$$\sum_{r=2}^{R-1}\sum_{i=2}^{n}\left(\binom{n}{i}(1-q^{r-1})^{n-i}\cdot q^{(r-1)\cdot i}\cdot(1-q)^{i}\cdot\right.$$
$$\left.(1-q^{(R-r)})^n\right)$$

Combining the probabilities for subcase (i) and subcase (ii), we have

$$(1-q)^n\cdot(1-q^{R-1})^n \quad +$$
$$\sum_{r=2}^{R-1}\sum_{i=2}^{n}\left(\binom{n}{i}(1-q^{r-1})^{n-i}\cdot q^{(r-1)\cdot i}\cdot(1-q)^{i}\cdot\right.$$
$$\left.(1-q^{(R-r)})^n\right)$$

Analysis of Case (ii): For this case, exactly $(n-1)$ processes have incomplete views during rounds $(r-1)$ and all the processes have complete views in round r, where $2\le r\le R-1$. This can happen if exactly one process, say p_x, fails to send in all the $(r-1)$ rounds and other $(n-1)$ processes send at least in one of the $(r-1)$ rounds. Given a particular round r, $2\le r\le(R-1)$, the probability that any one of the n processes fails to send in all the $(r-1)$ rounds is $n\cdot q^{(r-1)}$ and the probability that each of the other $(n-1)$ processes sends in at least one of the $(r-1)$ rounds is $(1-q^{r-1})^{n-1}$. The process p_x must send successfully at round r (has probability $(1-q)$) because the views of all the processes are complete at the end of round r. The probability that the view of each of the processes is complete at the end of round r is $\left(n\cdot q^{(r-1)}\cdot(1-q^{r-1})^{n-1}\cdot(1-q)\right)$. Since the view of process p_x is complete at the end of round $r-1$, the send operation by process p_x in round r is also the confirmation of p_x to all other processes.

After the view of all the processes are complete in round r, agreement occurs if each of the $(n-1)$ other processes in $(\Pi-\{p_x\})$ successfully sends in at least one of the remaining $(R-r)$ rounds (has probability $(1-q^{R-r})^{n-1}$). The probability of agreement, given than all processes have complete views at round r, is equal to

$$n\cdot q^{r-1}\cdot(1-q^{r-1})^{n-1}\cdot(1-q)\cdot(1-q^{R-r})^{n-1}$$

Since r ranges from 2 to $R-1$, the probability of disagreement, given that all the processes have complete views in any of the $2\ldots R-1$ rounds, is given as follows:

$$\sum_{r=2}^{R-1}\left(n\cdot q^{r-1}\cdot(1-q^{r-1})^{n-1}\cdot(1-q)\cdot(1-q^{R-r})^{n-1}\right)$$

Combining the probabilities for case (i) and case (ii), the probability of disagreement for the pessimistic decision criterion is computed as follows:

$$P_{AG} = (1-q)^n \cdot (1-q^{R-1})^n \quad + \\ \sum_{r=2}^{R-1}\sum_{i=2}^{n}\left(\binom{n}{i}(1-q^{r-1})^{n-i}\cdot q^{(r-1)\cdot i}\cdot(1-q)^i\cdot \right. \\ \left.(1-q^{(R-r)})^n\right) \quad + \\ \sum_{r=2}^{R-1}\left(n\cdot q^{r-1}\cdot(1-q^{r-1})^{n-1}\cdot(1-q)\cdot(1-q^{R-r})^{n-1}\right) \tag{8}$$

It is not difficult to see that the above equation can be computed in polynomial time. It is not difficult to see that the above equation can be computed in polynomial time. The probability of abort is $P_{AB} = 1 - P_{DG} - P_{AG}$ where P_{DG} and P_{AG} are computed in Eq. (7) and Eq. (8), respectively.

D. Analysis of Moderately Pessimistic Decision Criterion

In this subsection, we present the closed-form expressions to compute P_{DG} and P_{AB} by analyzing the moderately pessimistic decision criterion (See Alg 7) assuming symmetric message losses with the probability of q.

1) Finding P_{DG} for Moderately Pessimistic Decision Criterion: Same as for other decision criteria, disagreement occurs if some processes select a value while some other decide to abort. We assume there is a set of processes Π_x which decide to select a value, while all other processes as p_k in $\Pi - \Pi_x$ decide to abort. There are two conditions for a process as p_x to decide on selecting a value. First p_x should have complete view by the end of round $R-1$. Second, process p_x should not receive any message from any process indicating that their view is incomplete at round R.

We show that the set Π_x consists of exactly one process (we call this process as p_x). We prove this using contradiction. Our assumptions are as follows:

Assumption 1

There are two processes, p_x and $p_{x'}$, in Π_x which decide to select a value.

Assumption 2

All processes in $\Pi - \Pi_x$ decide to abort.

Based on **Assumption 1** process p_x must have complete view at the end of round $R-1$ (See Alg. 7). This means that all messages sent from the $n-2$ processes in $\Pi - \Pi_x$ and $p_{x'}$ are successfully delivered in at least one of the $R-1$ rounds. Also from **Assumption 1** we know that process $p_{x'}$ must have complete view at the end of round $R-1$ and this shows that all messages sent from the $n-2$ processes in $\Pi - \Pi_x$ and p_x are successfully delivered in at least one of the $R-1$ rounds. Therefore we can conclude that according to **Assumption 1**, messages sent from all processes in $(\Pi - \Pi_x) \cup \{p_x\} \cup \{p_{x'}\}$ are successfully delivered in at least one of the $R-1$ rounds. Obviously $(\Pi - \Pi_x) \cup \{p_x\} \cup \{p_{x'}\}$ indicates the set of all processes (i.e., Π). In other words from **Assumption 1** and Proposition 2 we can conclude that all processes have successfully delivered their messages in at least one of the $R-1$ rounds, which results in complete views for all n processes by the end of round $R-1$. This contradicts **Assumption 2**.

So using proof with contradiction we showed that there is exactly one process in Π_x, as p_x which has a complete view at the end of round $R-1$ while all other processes have incomplete views at this point and in order for p_x to decide to select a value it must not receive any message from other $n-1$ processes in round R. Eq. (9) shows the closed form solution to calculate the probability of disagreement for moderately pessimistic decision criterion.

$$P_{DG} = n \cdot q^{R-1} \cdot (1-q^{R-1})^{n-1} \cdot q^{n-1} \tag{9}$$

We explain Eq. (9) as follows. As exactly one process as p_x has a complete view at round $R-1$, none of the other $n-1$ processes have received a message from p_x during $R-1$ rounds with the probability of $q^{R-1} \cdot (1-q^{R-1})^{n-1}$. On the other hand q^{n-1} refers to the probability that all messages sent from the $n-1$ processes in $\Pi - \Pi_x$ are lost in round R. Finally the process p_x can be selected in n possible ways from n processes (See Eq. (9)).

2) Finding P_{AB} for Moderately Pessimistic Decision Criterion: In order to derive the closed form solution to calculate the probability of abort we consider two cases.

Case (i): All processes have incomplete views by the end of round $R-1$.

Case (ii): Some processes in the set of Π_x have complete views by the end of round $R-1$ but in round R they receive incomplete views from all or some of the processes in $\Pi - \Pi_x$. As a result they decide to abort.

First we explain how we calculate the probability for Case (i). We have to find the number of ways the view of each of the n processes can be incomplete at the end of round $R-1$. This case is similar to calculate the probability of abort in Section. IV-B2 for the optimistic approach when the total number of rounds are $R-1$. So according to Proposition 1, the views of all the n processes are incomplete if and only if at least two processes fail to send during all $R-1$ rounds. If there are i processes that fail to send in all the $R-1$ rounds, where $2 \leq i \leq n$, then each of the remaining $(n-i)$ processes successfully sends in at least one of the $R-1$ rounds. Given some i, $2 \leq i \leq n$, there are $(\sum_{j=1}^{R} \binom{R}{j})^{n-i} = (2^R - 1)^{n-i}$ possibilities for each of which each of the $(n-i)$ processes successfully sends during at least one of the $R-1$ rounds for some given i. And, the i processes from n processes can be selected in $\binom{n}{i}$ ways, where $2 \leq i \leq n$. Consequently, the number of possibilities that all the processes have incomplete views by round $R-1$ is given as follows:

$$Case(i) = \sum_{i=2}^{n} \binom{n}{i} \cdot (2^R - 1)^{n-i} \tag{10}$$

Given that the probability of a message loss is q, the probability that exactly i processes fail to send in all $R-1$ rounds is $(q^{R-1})^i$ while the probability that each of the $(n-i)$ processes successfully send during at least one of the $R-1$ rounds is $[\sum_{j=1}^{R-1} \binom{R-1}{j} \cdot (1-q)^j \cdot q^{R-1-j}]^{n-i} = (1-q^{R-1})^{n-i}$, where $2 \leq i \leq n$. Consequently, the probability that all the processes given in Case(i) agree to abort is given as follows:

$$P_{Case(i)} = \sum_{i=2}^{n} \binom{n}{i} \cdot (q^{R-1})^i \cdot (1-q^{R-1})^{n-i} \tag{11}$$

Now we explain how we calculate the probability of Case(ii). First we prove that the set of Π_x consists of exactly one process as p_x. We show the proof by contradiction. We consider the following assumptions:

Assumption 1

Process p_x and $p_{x'}$ in Π_x decide to abort. They both have complete views by the end of round $R-1$ but in round R they receive incomplete from some or all processes in $\Pi - \Pi_x$.

Assumption 2

All processes in $\Pi - \Pi_x$ have incomplete views by round $R-1$ and as a result they all decide to abort.

Based on **Assumption 1** process p_x must have complete view at the end of round $R-1$ (See Alg. 7). This means that all messages sent from the $n-2$ processes in $\Pi - \Pi_x$ and $p_{x'}$ are successfully delivered in at least one of the $R-1$ rounds. From **Assumption 1** we know that process $p_{x'}$ has also complete view at the end of round $R-1$ and this shows that all messages sent from the $n-2$ processes in $\Pi - \Pi_x$ and p_x are successfully delivered in at least one of the $R-1$ rounds. Therefore we can conclude that according to **Assumption 1**, messages sent from all processes in $(\Pi - \Pi_x) \cup \{p_x\} \cup \{p_{x'}\}$ are successfully delivered in at least one of the $R-1$ rounds. Obviously $(\Pi - \Pi_x) \cup \{p_x\} \cup \{p_{x'}\}$ indicates the set of all processes (i.e., Π). In other words from **Assumption 1** and Proposition 2 we can conclude that all processes have successfully delivered their messages in at least one of the $R-1$ rounds, which results in complete views for all n processes by the end of round $R-1$. This contradicts **Assumption 2**.

So Π_x consists of exactly one process as p_x which fails to send its message in in all $R-1$ rounds. This happens with the probability of q^{R-1}. As a result $n-1$ processes have incomplete views by the end of round $R-1$. All other $n-1$ processes successfully deliver their message at least once during $R-1$ rounds of execution, so that the view of p_x is complete by round $R-1$. In round R, p_x receives incomplete views from at least one of $n-1$ processes with the probability of $(1-q^{n-1})$. As a result p_x decides to abort.

Eq. 12 shows the probability of case(ii). Then in Eq. 13 the probability of reaching to an agreement on abort is given, which is the sum of the probabilities of two cases, case (i) and (ii).

$$P_{Case(ii)} = n \cdot (1-q^{R-1})^{n-1} \cdot q^{R-1} \cdot (1-q^{n-1}) \tag{12}$$

$$P_{(AB)} = n \cdot (1-q^{R-1})^{n-1} \cdot q^{R-1} \cdot (1-q^{n-1}) + \sum_{i=2}^{n} \binom{n}{i} (1-q^{R-1})^{n-i} \cdot (q^{R-1})^i \tag{13}$$

Eq. 14 shows how we can calculate the probability of agreement on a value, considering that we have the given closed form solutions to calculate the probability of disagreement and abort.

$$P_{AG} = 1 - P_{DG} - P_{AB} \tag{14}$$

V. Probabilistic Model Checking of the *1-of-n* Selection Algorithm

We model the given consensus algorithm in Section II-B using a probabilistic model checking tool, PRISM [4]. We assume the given system model and failure assumptions in Section II-A.

Model checking is the problem of automatically checking whether a model of a system satisfies it specifications or not. A model checker receives as an input a state transition model and a specification, and it verifies whether the given model satisfies the specification. In the case of probabilistic model checking, the state transition model contains stochastic behaviour, such as the probabilistic choice among enabled transitions. The probabilistic model checker performs the reachability analysis of the transition system and, in addition, it calculates the likelihoods of reaching the states using numerical methods.

PRISM supports four different classes of models: discrete time Markov chain (`dtmc`); Markov decision process (`mdp`), continuous time Markov chain (`ctmc`), and probabilistic timed automata (`pa`). Among these models, we do not use `ctmc` and `pa` since the consensus algorithm does not require the modelling of time intervals. Both `dmtc` and `mdp` allow the specification of the deterministic and probabilistic transitions. For probabilistic transitions, the choice of the next state is determined by a discrete probability distribution. The difference between `mdp` and `dmtc` is that `mdp` also allows the specification of non-deterministic transitions (not associated with any probability distribution), while `dmtc` does not. As the consensus algorithm does not require the use of non-deterministic transition, we define the models of the consensus algorithm as `dmtc`.

Following we describe in detail the PRISM models which we designed in order to verify the correctness of the *1-of-n* selection algorithm under the given failure assumptions in a probabilistic manner. Using PRISM we calculate the probability of reaching to an agreement on a value, the probability of all processes deciding to abort and finally the probability of having disagreement among the processes.

We start with explaining the PRISM model we designed for a system consisting of three processes with the assumption of having symmetric and asymmetric communication failures only. Then in the next sections, we show how the PRISM models can be modified for systems with more processes.

A. PRISM Model For Three Processes With Symmetric Failure Model

1) Model Overview: Our PRISM model is composed of three parts: *declarations*, *modules* and *expressions*. *Declarations* contain the list of constant values and global variables. The *modules* describe the behaviour of the processes; Each process is defined as a module. The message exchange among processes is modelled using global variables that are written/read by the modules. *expressions* are the expressions that can be used to avoid repetition of code in the module definition. Synchronization among processes is achieved using a global variable (called `token`) and the decision criteria are embedded in expressions.

Before explaining each part in detail we look at an overview of a process module. Modules in general are divided in two parts: *declaration of local variables* and *description of transitions*. Fig. 1 illustrates the module of a process. Table V describes the transitions defined for a process. There are four states assumed for a process, `S0`, `S1`, `S2` and `S3`. The probabilistic choice occurs between states `S1` and `S2` and corresponds to the cases where a message is successfully transmitted with the probability of $(1-q)$ or is lost with the probability of q. `S3` refers to the state in which the process is in its last round of execution. `S0` is the final state and a process in this state should decide either to *select a value* or to *abort*.

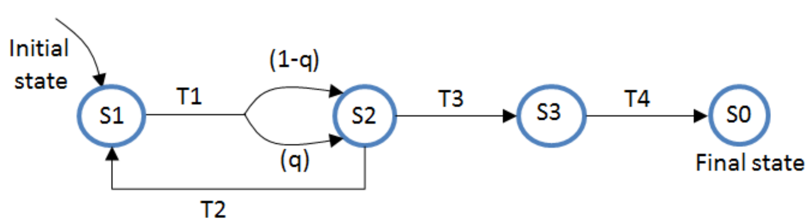


Figure 1. Process model for symmetric failures

Table V
TRANSITIONS FOR THE SYMMETRIC FAILURE MODEL

Transition	From	To	Probability	Description
T1	S1	S2	1-q	Process succeed in sending its message
T1	S1	S2	q	Process fails in sending its message
T2	S2	S1	1	Not last round: process receives the messages and computes the round (not last round)
T3	S2	S3	1	Last round: The process receives the messages and computes the round (last round)
T4	S3	S0	1	Make a decision (agree or abort)

2) Global Declarations: Constants: The first line of a PRISM model declares its class which in our case is (`dtmc`). Then we insert the list of constant values. The given constant values in Listing 2 define the system settings under which the

```
dtmc
```

Listing 1. Declaring the class of the PRISM Model

1-of-n selection algorithm is run: the number of processes (`N`), the number of rounds of execution of the algorithm (`RN`), the probability of losing a message (`q`) and finally the decision criterion (`DC`). Currently, the model supports two different decision criteria: optimistic `(DC=3)` and pessimistic `(DC=1)`. Constants `N1`, `N2` and `N3` are the processes' identifiers and

```
const N=3;      // Number of processes in the network (N cannot be modified)
const RN=2;         // Number of rounds in the protocol (RN>=2)
const double q;     // Probability of losing a message (0<=q<=1)
const DC=3;         // Decision criterion: OP=1; PS=3;
```

Listing 2. Declaring the constant variables

are used to define which process has the `token` and shall fire the next transition (see expression `next` in Section V-B7). Constant `v_max` defines the highest value among the processes' initial values and is used to limit the range of the variables that stores processes' values. Constants `v1_ini`, `v2_ini` and `v3_ini` store the initial value for each process. These values can be chosen randomly between `1` and `v_max`. We know that the choice of the initial values does not affect the results of the algorithm. Regardless of the number of failures, if a process decides to select a value it selects the correct value. Therefore we leave the choice of defining the initial values to the user instead of implementing them as a probabilistic choice in the model, which unnecessarily increases the number of states and transitions.

```
const N1=1;          // Identity number of Process 1
const N2=2;          // Identity number of Process 2
const N3=3;          // Identity number of Process 3
```

Listing 3. Declaring the processes' identities

```
const v_max=2;       // Maximum value of a process
const v1_ini=1;      // Initial value of Process 1
const v2_ini=2;      // Initial value of Process 2
const v3_ini=1;      // Initial value of Process 3
```

Listing 4. Declaring the processes' initial values

Finally, constants `not_last` and `last` are used to define which process receives the `token` after the current process (see expression `next` in Section V-A4).

```
const not_last=1;    // Auxiliary constant to define the next process
const last=0;        // Auxiliary constant to define the next process
```

Listing 5. Declaring auxiliary constants

3) Global Declarations: Global Variables: We use global variables to model the message exchange among processes. At each round, each process writes its current value and view in the global variables and reads the values and views of the other processes.

Variable `vi_ext` contains the value of `Process` i and must be defined within the range of `[0..v_max]`. As the range of acceptable values is between `1` and `v_max`, value `0` is used to indicate the loss of a message sent by a process as `Process` i.

```
global v1_ext : [0..v_max] init 0;  // Message value of Process 1
global v2_ext : [0..v_max] init 0;  // Message value of Process 2
global v3_ext : [0..v_max] init 0;  // Message value of Process 3
```

Listing 6. Declaring auxiliary constants

Variable `wi_vj_ext` contains the `Process` i 's view of `Process` j. This variable is *boolean* and indicates whether or not `Process` i has received the value of `Process` j during the previous rounds.

```
global w1_v2_ext : bool init false; // Process 1 view of Process 2
global w1_v3_ext : bool init false; // Process 1 view of Process 3

global w2_v1_ext : bool init false; // Process 2 view of Process 1
global w2_v3_ext : bool init false; // Process 2 view of Process 3

global w3_v1_ext : bool init false; // Process 3 view of Process 1
global w3_v2_ext : bool init false; // Process 3 view of Process 2
```

Listing 7. Declaring auxiliary constants

The global variable `token` is used to coordinate the processes and is within the range of `[1..N]`. `token`'s value indicates the identifier of a process that must perform the next transition. When `token=2`, only a transition of `Process 2` can be enabled. When `Process 2` performs the enabled transition it passes the token to the next process (`Process 3`) by assigning a new value to it `token=3`, then only a transition of `Process 3` can be enabled.

The variable `m_lost` stores the number of lost messages during an execution of the algorithm. This variable is used for verification purposes (e.g. to determine the minimum number of lost messages which results in having disagreement among processes).

```
global token : [1..N] init 1;        // Token used to coordinate the processes
global m_lost : [0..(RN*N)] init 0; // Number of lost messages
```

Listing 8. Declaring auxiliary constants

Table VI
VALUES OF THE VARIABLES FOR THE NEXT EXPRESSION.

Process	**N1**	**not_last**	**next**
1	1	1	2
2	N2=2	1	3
3	N2=3	last=0	1

Table VII
REPLACEMENT OF VARIABLES OF V1_NEW EXPRESSION.

Process1	v1	v2_ext	v3_ext
Process2	v2	v3_ext	v1_ext
Process3	v3	v1_ext	v2_ext

4) Global Declarations: expressions: The given expressions in the following are defined from the perspective of `Process 1`, i.e., with the appropriate variables for `Process 1`. Then, when we define the modules for other processes, we must indicate how the global and local variables are replaced, so that the same expression can be used by other processes. The

```
formula next = N1*not_last+1;   // Define the next process to receive the token
```

Listing 9. Definition of `next` expression

first expression (`next`) determines the next value of the `token`. Table VI shows how the variables defined for the `next` expression vary for each process. In the case of `Process 1`, the variables are not replaced, as the expression is defined from the perspective of this process. For `Process 2`, `N1` is replaced by `N2` and `not_last` is not replaced, resulting in `next=3`. Finally, for `Process 3`, `N1` is replaced by `N3` and `not_last` is replaced by the constant `last` (defined as `0`), which results in `next=1`.

The expression `v1_new` computes the new value of a process after a round by comparing the current value (`v1`, which is an internal variable of the module `Process 1`) with the values received from other processes (`v2_ext` and `v3_ext`). Table VII presents how the variables of this expression are replaced when it is used by `Process 2` and `Process 3`. Additional explanations about how to redefine variables for other processes are given in Section V-B7.

```
formula v1_new = max(v1,v2_ext,v3_ext); // Process 1 compute new value
```

Listing 10. Definition of expression `v1_new`

The *boolean* expressions `w1_v2_new` and `w1_v3_new` compute the view that `Process 1` has of `Process 2` and `Process 3`, respectively. For the case of `w1_v2_new` the result is *true* if at least one of the following conditions is satisfied:

1) It was already *true* in the previous round (`w1_v2` is *true*, `w1_v2` is an internal variable of the module `Process 1`);
2) `Process 1` received the message of `Process 2` in the current round (`v2_ext!=0`); or
3) `Process 1` received the message of `Process 3` and `Process 3` has received the message from `Process 2` in a previous round (`w3_v2_ext` is *true*).

We observe that, in the case of symmetric failure, the last condition is equal to the first one (when `Process 3` receives the message, `Process 1` also receives it). However, we leave this condition on the expression in order to be consistent with the given algorithm. Table VIII and IX present how the variables of these expressions are replaced when they are used

```
formula w1_v2_new = w1_v2 | (v2_ext!=0) | w3_v2_ext; // Process 1 update its view of Process 2
formula w1_v3_new = w1_v3 | (v3_ext!=0) | w2_v3_ext; // Process 1 update its view of Process 3
```

Listing 11. Processes update their views `v1_new`

by `Process 2` and `Process 3`. We observe that in the case of `Process 2`, the expression `w1_v2_new` determines

Table VIII

REPLACEMENT OF VARIABLES OF `w1_v2_NEW` EXPRESSION.

Process1	w1_v2	v2_ext	w3_v2_ext
Process2	w2_v3	v3_ext	w1_v3_ext
Process3	w3_v1	v1_ext	w2_v1_ext

Table IX

REPLACEMENT OF VARIABLES OF `w1_v3_NEW` EXPRESSION.

Process1	w1_v3	v3_ext	w2_v3_ext
Process2	w2_v1	v1_ext	w3_v1_ext
Process3	w3_v3	v2_ext	w1_v2_ext

the view that `Process 2` has of `Process 3`, and in the case of `Process 3`, it determines the view that `Process 3` has of `Process 1`.

expression `w1_c2_new` determines whether or not `Process 1` has received confirmation from `Process 2` indicating that its view is complete. The result is a *boolean* value that is *true* if at least one of the following conditions is satisfied:

1) It was already *true* in the previous round (`w1_c2` is *true*, `w1_c2` is an internal variable of the module `Process 1`);
2) The message received from `Process 2` in the current round shows that `Process 2` has a complete view (`w2_v1_ext` and `w2_v3_ext` are *true*).

Table X presents how the variables of this expression are replaced when it is used by `Process 2` and `Process 3`.

```
formula w1_c2_new = w1_c2 | (w2_v1_ext & w2_v3_ext); // Process 1 knows that Process 2 view is complete
formula w1_c3_new = w1_c3 | (w3_v1_ext & w3_v2_ext); // Process 1 knows that Process 3 view is complete
```

Listing 12. expressions to verify the completeness of views

The next expressions verify the decision criteria and provide a *boolean* outcome: *true* means to decide on a value and *false* means decide to abort. For these expressions, the replacement of variables when the expression is called by `Process 2` and `Process 3` is the same as indicated in the previous tables (see also Section V-B7). The expression `decision_OP` verifies whether or not `Process 1` has a complete view, i.e, has the view of `Process 2` (`w1_v2` is *true*) and `Process 3` (`w1_v3`).

```
// Optimistic Decision
formula decision_OP = w1_v2 & w1_v3; // Process 1 has complete view at RN
```

Listing 13. Optimistic decicion criterion

It returns *true* if the view of `Process 1` is complete and the received messages are also complete. It is important to observe that the requirement that the view of `Process 1` must be complete at `RN-1` is not explicitly embedded in the expression. This requirement is satisfied by not updating the view of `Process 1` in the last round (see Section V-A6).

The expression `decision_MP` verifies whether or not the view of `Process 1` is complete (`w1_v2` and `w1_v3` are *true*) and `Process 1` has received a confirmation that `Process 2` and `Process 3` also have complete view (`w1_c2` and `w1_c3` are *true*). Finally, the expression `decision` combines the three decision criteria in a single expression using the value of the constant `DC`.

5) Module Description: Internal Variables: In PRISM, the definition of a module starts with the word module, followed by its name (`Process 1`). The internal variables of `Process 1` are: its current stage in the execution of the algorithm

Table X

REPLACEMENT OF VARIABLES OF **W1-V3-NEW** EXPRESSION.

Process1	w1_c2	w2_v1_ext	w2_v3_ext
Process2	w2_c3	w3_v2_ext	w3_v1_ext
Process3	w3_c1	w1_v3_ext	w1_v2_ext

```
formula decision_PS = w1_v2 & w1_v3 & w1_c2 & w1_c3; // Process 1 has complete view at (RN-1) and has received complete view from all processes at RN
```

Listing 14. Pessimistic decicion criterion

```
formula decision = ((DC=1) & decision_OP) | ((DC=3) & decision_PS); // Combine all decision criterea in a single formula
```

Listing 15. General decicion expression

(S1), its current round (RN1), its decision (d1), its current value (v1, initiated as v1_ini), its current views of other processes' values (w1_v2 and w1_v3), and its current confirmations that other processes view are complete (w1_c2 and w1_c3).

```
s1 : [0..3] init 1;  // Process 1 current state
RN1 : [0..RN] init 0; // Current round
d1 : bool init false; // Process 1 decision
v1 : [0..v_max] init v1_ini; // Process 1 value

// Process 1 view of other processes
w1_v2 : bool init false; // Process 1 has the view of Process 2
w1_v3 : bool init false; // Process 1 has the view of Process 3

// Process 1 has confirmation that other processes have complete view
w1_c2 : bool init false; // Process 1 has confirmation from Process 2
w1_c3 : bool init false; // Process 1 has confirmation from Process 3
```

Listing 16. Process 1 module

The variable S1 assumes its value as described in Table V and Fig. 1 (S1=0 is equivalent to S0, and so on). It is important to observe that S1 is defined in order to help the organization and understanding of the module. The actual state of the module results from the combination of the value of all its variables, not only S1.

6) Module Description: Transitions: The definition of a transition starts with square brackets. They are used to specify the synchronisation of transitions between modules. As the models we describe here have no synchronisation of transitions, all the transitions described here starts with empty brackets [].

A transition is composed of a guard and an action separated by an arrow: ([] guard $\rightarrow$ action). The guard is a *boolean* expression that specifies the condition under which the transition can be executed. The action specifies how internal and global variables are updated when the transition is performed. The update of each variable must be included in parentheses. The parentheses are combined with an & operator. Example: [] guard $\rightarrow (update_1)$ & $(update_2)$ & $(update_3)$.

In the case of probabilistic transitions, the action is composed of a set of possible actions with the corresponding probabilities, and separated by plus signals. Example: [] guard $\rightarrow p1$: action 1 $+p2$: action 2 $+p3$: action 3. As for the case of deterministic transitions, each action may be composed of one or more updates. We observe that here we broke the text of a transition in many lines to have a better understanding. In PRISM a transition must be specified in a single line.

The first transition of the module performs the message sending. The guard of this transition specifies that the module must be in the initial state (S1=1), have the token (RN1=1), and must not have complete the last round (RN1<RN). The condition m_lost<(RN*N) is added to avoid compilation errors, it assures that the update of m_lost ($\texttt{m\_lost}' = \texttt{m\_lost} + 1$) will not violate the variable range ([0..(RN*N)]). With probability of $(1-q)$ the message is sent successfully: the global variables associated with Process 1's value (v1_ext) and view (w1_v2_ext and w1_v3_ext) are updated with the current values of the internal variables. With probability of q the message is lost: the global variable v1_ext receives 0, w1_v2_ext and w1_v3_ext are set to *false*, and the number of lost messages (m_lost) is incremented.

In both cases, the current round of Process 1 is incremented (RN1'=RN1+1), the token is passed to the next process (token'=next), and Process 1 moves to state S1=2.

```
[] s1=1 & token=N1 & RN1<RN & m_lost<(RN*N) -> (1-q):(s1'=2) & (token'=next) & (v1_ext'=v1) & (w1_v2_ext'=w1_v2) & (w1_v3_ext'=w1_v3) & (RN1'=RN1+1) + q:(s1'=2) &
      (token'=next) & (v1_ext'=0) & (w1_v2_ext'=false) & (w1_v3_ext'=false) & (RN1'=RN1+1) & (m_lost'=m_lost+1) ;
```

Listing 17. The message sent from Process 1 is either lost or delivered successfully

The following transition describes the computation of the round when Process 1 is not in the last round (RN1<RN). When this transition is enabled, all processes have already sent their messages and the token has returned to Process 1 (token=N1). The round computation consists of updating the internal variables of Process 1 (value:v1; views: w1_v2

and w1_v3; and confirmations: w1_c2 and w1_c3), using the corresponding expressions. Then, the token is passed to the next process and Process 1 returns to S1=1. The following transition describes the computation of the last round

```
[] s1=2 & token=N1 & RN1<RN -> 1: (s1'=1) & (v1'=v1_new) & (w1_v2'=w1_v2_new) & (w1_v3'=w1_v3_new) & (w1_c2'=w1_c2_new) & (w1_c3'=w1_c3_new) & (token'=next);
```

Listing 18. The message sent from Process 1 is either lost or delivered successfully

(RN1=RN). The main difference between this and the previous transition is that the value and views of Process 1 are updated if and only if the decision criterion is the optimistic one (DC=1). The expression $x' = (c)?a : b$ means that if c is true $x = a$, otherwise $x = b$. Also differently from the previous transition, in this case Process 1 goes to S1=3 and does not pass the token. After computing the last round, Process 1 is at S1=3 and makes a decision using the

```
[] s1=2 & token=N1 & RN1=RN -> 1: (s1'=3) & (w1_c2'=w1_c2_new) & (w1_c3'=w1_c3_new) & (v1'=(DC=1)?v1_new:v1) & (w1_v2'=(DC=1)?w1_v2_new:w1_v2) &
       (w1_v3'=(DC=1)?w1_v3_new:w1_v3);
```

Listing 19. The message sent from Process 1 is either lost or delivered successfully

corresponding expression. It then goes to the final state S1=0 and passes the token to the next process. The module is closed with end_module.

```
[] s1=3 & token=N1 -> 1: (s1'=0) & (token'=next) & (d1'= decision);
endmodule
```

Listing 20. The process makes the decision

7) Modules of other processes: Process 2 and 3 are defined as a copy of Process 1. In this case, PRISM imposes that all the internal variables must be renamed. The external variables may be replaced or not by other external variables that have already been defined in the appropriate section.

The basic idea is that, in the definition of Process 1, a reference to Process 2 (which is the next process after Process 1), shall be replaced by a reference to Process 3 for the case of Process 2 (Process 3 is the next process after Process 2). Likewise for the case of Process 3 it should be referenced by Process 1 (which is the next process after Process 3). Similarly, in the definition of Process 1, a reference to Process 3 (second next process after Process 1), shall be replaced, in the case of Process 2, by a reference to Process 1 (second next process after Process 2) and, in the case of Process 3 by a reference to Process 2 (second next process after Process 3).

The general rule adopted in this work for the definition of a new Process i $(1 < i <= N)$ as a copy of Process 1 is:

For each internal variable, global variable or constant used by Process 1 and named X$_j$ or X$_j$_Y or X$_j$ _Yw, where X and Y are the variable names and j and w are references to other processes in the interval [1..N]:

- If $(j + i - 1 <= n)$ and/or $(w + i - 1 <= n)$ than replace it with $(j + i - 1)$ and/or $(w + i - 1)$.
- If $(j + i - 1 > n)$ and/or $(w + i - 1 > n)$ than replace it with $(j + i - 1 - n)$ and/or $(w + i - 1 - n)$.

Only for the definition of Process i, add the replacement not_last=last. The application of the general rule results in the following definition of Process 2 and 3:

```
module Process_3=Process_1 [N1=N3, s1=s3, v1=v3, d1=d3, RN1=RN3, w1_v2=w3_v1, w1_v3=w3_v2, w1_c2=w3_c1, w1_c3=w3_c2, v1_ext=v3_ext, v2_ext=v1_ext, v3_ext=v2_ext,
       w1_v2_ext=w3_v1_ext, w1_v3_ext=w3_v2_ext, w2_v1_ext=w1_v3_ext, w2_v3_ext=w1_v2_ext, w3_v1_ext=w2_v3_ext, w3_v2_ext=w2_v1_ext, v1_ini=v3_ini, not_last=last]
       endmodule
```

Listing 21. The process makes the decision

8) Verification of properties: We specify the verification properties using an extension of probabilistic temporal logic, which combines temporal relationships between events with probabilistic quantifiers. The specification language used by PRISM is named the Probabilistic Computation Tree Logic $(PCTL)$, derived from the well-known Computation Tree Logic (CTL). We used $PCTL$ to calculate the probability that all processes reach the final state in a given condition (agreement, abort or disagreement). For this purpose, the following properties are specified: To better understand the given properties we explain Property (1) given in Listing 22. This property refers to the probability (P=?) that eventually (F) the system

```
P=? [ F (s1=0)&(s2=0)&(s3=0)&((d1!=d2)|(d2!=d3))]
```

Listing 22. Property (1), Probability of disagreement

```
P= ? [ F(s1=0)&(s2=0)&(s3=0)&(d1=false)&(d2=false)&(d3=false)]
```

Listing 23. Property (2), Probability of abort

```
P= ? [ F(s1=0)&(s2=0)&(s3=0)&(d1=true)&(d2=true)&(d3=true)]
```

Listing 24. Property (3), Probability of agreement

```
P= ? [ F(s1=0)&(s2=0)&(s3=0)&((d1!=d2)|(d2!=d3))&(m-lost=1)]
```

Listing 25. Property (4), Probability of disagreement with a specific number of lost messages

reaches a state where all the processes are in the final state `((s1=1)&(s2=1)&(s3=1)&(s4=1))` and have reached different decisions `((d1!=d2)|(d2!=d3))` (disagreement). One limitation of probabilistic model checking is that due to the problem of state explosion, we are only able to calculate probabilities for a network with 3 processes. For a larger number of processes, PRISM is can estimate the value of property using simulation, with a given tolerance and interval of confidence, or with a fixed number of runs.

B. PRISM Model For Three Processes With Asymmetric Failure Model

1) Model overview: In the case of asymmetric failure, each process may receive or lose a message independently from the other processes. As a consequence, the number of states and transitions of the process's module depends on the number of processes in the network. Generally, the process's module is composed of `N+3` transitions and `N+3` states. Fig. 2 illustrates the module of a process for the case of `N=3`, while Table XI describes its transitions. For larger number of processes, the probabilistic transition shall be repeated `N-1` times.

2) Global Declarations: Constants : The only modification introduced in the constants' declaration is the replacement of constant `q` with constant `Q`.

```
const double Q=0.5;     // Probability of losing a message (0<=q<=1)
```

Listing 26. Probability of losing a message

3) Global Declarations: Global Variables : The only modification introduced in the declaration of global variables is the range of `m_lost`, which is enlarged to `[0.. RN*N*(N-1)]`.

```
global m_lost : [0..(RN*N*(N-1))] init 0;   // Number of lost messages
```

Listing 27. Number of lost messages

4) Global Declarations: expressions: We assume that in the case of having asymmetric failures, all messages are always sent, which means that their values and views are always copied the global variables. In order to register the occurrence of a failure, each `Process` *i* has a set of internal variables, named `ni_nfj`, that indicates whether `Process` *i* has received (`ni_nfj` = true) or not (`ni_nfj= true`) the message of `Process` *j* in the last round. The expressions are redefined in order to consider other processes' values and views only when they have been received by `Process 1`. The expression `v1_new` for computing the new value of `Process 1` uses the condition operator ? to replace the value of `v2_ext` and `v3_ext` with 0 when the corresponding message has not been received by `Process 1`.

```
formula v1_new = max(v1,(n1_nf2?v2_ext:0),(n1_nf3?v3_ext:0));  // Process 1 compute new value
```

Listing 28. `Process 1` computes new value

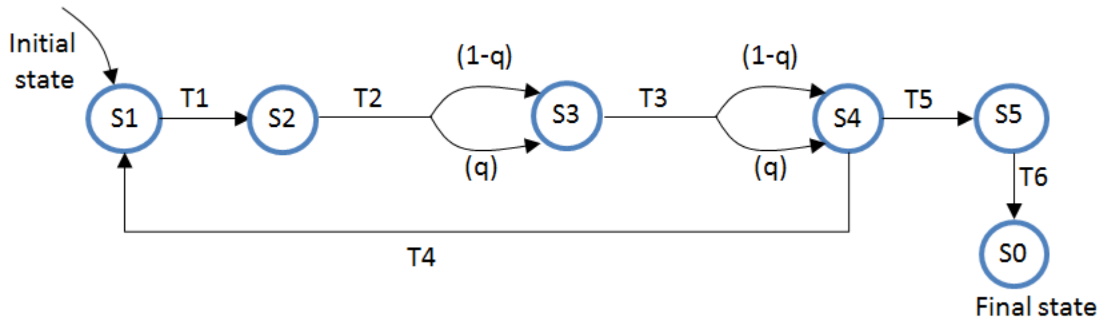


Figure 2. Process model for asymmetric failures

The expressions that compute the views of `Process 1` (`w1_v2_new` and `w1_v3_new`) are also modified. Taking as an example `w1_v2_new`, instead of using `v2_next` to check if the message of `Process 2` has been received, `n1_nf2` is used. Also, the view of `Process 2` can only be acquired through `Process 3` (`w3_v2_ext`), when the message from `Process 3` has been received (`n1_nf3` is true).

```
formula w1_v2_new = w1_v2 | n1_nf2 | (n1_nf3 & w3_v2_ext); // Process 1 update its view of Process 2
formula w1_v3_new = w1_v3 | n1_nf3 | (n1_nf2 & w2_v3_ext); // Process 1 update its view of Process 3
```

Listing 29. `Process 1` updates its view

Similarly, the expressions that compute the confirmations of `Process 1` (`w1_c2_new` and `w1_c3_new`) consider whether or not the messages have been received by checking `n1_nf2` and `n1_nf3`. The expressions associated with the deci-

```
formula w1_c2_new = w1_c2 | (n1_nf2 & (w2_v1_ext & w2_v3_ext)); // Process 1 knows that Process 2 view is complete
formula w1_c3_new = w1_c3 | (n1_nf3 & (w3_v1_ext & w3_v2_ext)); // Process 1 knows that Process 3 view is complete
```

Listing 30. `Process 1` updates its view of completeness of other processes's view

sion criteria are not modified as they use only internal variables. The only exception is for the expression `received_message_complet` where the condition (`vi_ext=0`) is replaced by `n1_nfi`.

5) Module Description: Internal Variables: The range of variable `s1` is modified to `[0..N+2]`. Variables `n1_nf2` and

```
s1 : [0..N+2] init 1;  // Process 1 current state
```

Listing 31. `Process 1` current state

`n1_nf3` are added to the list of internal variables.

```
n1_nf3 : bool init true; // Process 1 has not received the message of Process 3
```

Listing 32. Status of the message from the other processes

6) Module Description: Transitions: The first transition is modified so that `Process 1` always sends its message, i.e., copies its value and views to the global variables. A set of `N−1` probabilistic transitions are added in order to define, at

```
// Process 1 sends its message;
[] s1=1 & token=N1 & RN1<RN -> 1:(s1'=2) & (token'=next) & (v1_ext'=v1) & (w1_v2_ext'=w1_v2) & (w1_v3_ext'=w1_v3) & (RN1'=RN1+1);
```

Listing 33. `Process 1` sends its message

Table XI
TRANSITIONS FOR THE ASYMMETRIC FAILURE MODEL N=3

Transition	From	To	Probability	Description
T1	S1	S2	1	Process 1 sends its message
T2	S2	S3	1-q	Process succeed in receiving the message of Process 2
T2	S2	S3	q	Process fails in receiving the message of Process 2
T3	S3	S4	1-q	Process succeed in receiving the message of Process 3
T3	S3	S4	q	Process fails in receiving the message of Process 3
T4	S4	S1	1	Not last round: process receives the messages and computes the round (not last round)
T5	S4	S5	1	Last round: process receives the messages and computes the round (last round)
T6	S5	S0	1	Make a decision (agree or abort)

each round, whether or not the message of `Process` *i* `(1<i<=N)` has been received by `Process 1`. When the message is lost, `n1_nfi` is set to false and `m_lost` is incremented. Finally, for the next transitions, the initial and final values of

```
// Process 1 receives or loses the message of Process 2
[] s1=2 & token=N1 & RN1<=RN & (m_lost<(RN*N*(N-1))) -> (1-Q): (s1'=3) & (n1_nf2'=true) + Q: (s1'=3) & (n1_nf2'=false) & (m_lost'=m_lost+1);
// Process 1 receives or loses the message of Process 3
[] s1=3 & token=N1 & RN1<=RN & (m_lost<(RN*N*(N-1))) -> (1-Q): (s1'=4) & (n1_nf3'=true) + Q: (s1'=4) & (n1_nf3'=false) & (m_lost'=m_lost+1);
```

Listing 34. `Process 1` either receives or loses the message sent by other processes

variable `s1` are set according to Fig. 2.

7) Modules of other Processes: The variables `n1_nfi` must be renamed in the definition of `Process 2` and `3`.

8) Verification of Properties: No modification is introduced in the list of properties.

```
// Not last round, Process 1 computes the messages of other processes: updates its value, views and confirmations;
[] s1=N+1 & token=N1 & RN1<RN -> 1: (s1'=1) & (v1'=v1_new) & (w1_v2'=w1_v2_new) & (w1_v3'=w1_v3_new) & (w1_c2'=w1_c2_new) & (w1_c3'=w1_c3_new) & (token'=next);

// Last round, Process 1 computes the messages of other processes: updates its confirmations and, only for OP, updates value and views;
[] s1=N+1 & token=N1 & RN1=RN -> 1: (s1'=N+2) & (w1_c2'=w1_c2_new) & (w1_c3'=w1_c3_new) & (v1'=(DC=1)?v1_new:v1) & (w1_v2'=(DC=1)?w1_v2_new:w1_v2) &
      (w1_v3'=(DC=1)?w1_v3_new:w1_v3);

// Process 1 decides -> agree or abort
[] s1=N+2 & token=N1 -> 1: (s1'=0) & (token'=next) & (d1'= decision);
endmodule
```

Listing 35. `Process 1` state transitions

```
module Process_2=Process_1 [N1=N2, s1=s2, v1=v2, d1=d2, RN1=RN2, n1_nf2=n2_nf3, n1_nf3=n2_nf1, w1_v2=w2_v3, w1_v3=w2_v1, w1_c2=w2_c3, w1_c3=w2_c1, v1_ext=v2_ext,
       v2_ext=v3_ext, v3_ext=v1_ext, w1_v2_ext=w2_v3_ext, w1_v3_ext=w2_v1_ext, w2_v1_ext=w3_v2_ext, w2_v3_ext=w3_v1_ext, w3_v1_ext=w1_v2_ext, w3_v2_ext=w1_v3_ext,
       v1_ini=v2_ini] endmodule
module Process_3=Process_1 [N1=N3, s1=s3, v1=v3, d1=d3, RN1=RN3, n1_nf2=n3_nf1, n1_nf3=n3_nf2, w1_v2=w3_v1, w1_v3=w3_v2, w1_c2=w3_c1, w1_c3=w3_c2, v1_ext=v3_ext,
       v2_ext=v1_ext, v3_ext=v2_ext, w1_v2_ext=w3_v1_ext, w1_v3_ext=w3_v2_ext, w2_v1_ext=w1_v3_ext, w2_v3_ext=w1_v2_ext, w3_v1_ext=w2_v3_ext, w3_v2_ext=w2_v1_ext,
       v1_ini=v3_ini, not_last=last] endmodule
```

Listing 36. `Process 2` and `3`

C. PRISM Model For More Than Three Processes With Symmetric Failure Model

This section describes how to expand the consensus algorithm model described in Section V-A for the case of 4 processes. The same procedure should be repeated to obtain models with more than 4 processes.

1) Model overview: In the case of symmetric failure no additional transition or state is added to the model of a process.

2) Global Declarations: Constants: The following constants are added or modified: number of processes, identity number of `Process 4`, initial value of `Process 4`.

```
const N=4;      // Number of processes in the network (N cannot be modified)
const RN=2;         // Number of rounds in the protocol (RN>=2)
const double q;     // Probability of losing a message (0<=q<=1)
const DC=3;         // Decision criterion: OP=1; PS=3;

const N1=1;         // Identity number of Process 1
const N2=2;         // Identity number of Process 2
const N3=3;         // Identity number of Process 3
const N4=4;         // Identity number of Process 4

const v_max=2;      // Maximum value of a process
const v1_ini=1;     // Initial value of Process 1
const v2_ini=2;     // Initial value of Process 2
const v3_ini=1;     // Initial value of Process 3
const v4_ini=1;     // Initial value of Process 4
```

Listing 37. Global declarations

3) Global Declarations: Global Variables: The following variables must be added to the list of global variables: the value of `Process 4`, the view each other process has of `Process 4`, and the view `Process 4` has of each other process. The expression for computing the value, views and confirmations of `Process 1` are updated to consider the value and views of `Process 4`. Two new expressions are created for computing `Process 1` view and confirmation of `Process 4`. The decision criteria expressions are also updated to include the view and confirmation of `Process 4`. In addition,

```
global v1_ext : [0..v_max] init 0;  // Message value of Process 1
global v2_ext : [0..v_max] init 0;  // Message value of Process 2
global v3_ext : [0..v_max] init 0;  // Message value of Process 3
global v4_ext : [0..v_max] init 0;  // Message value of Process 4

global w1_v2_ext : bool init false;     // Process 1 view of Process 2
global w1_v3_ext : bool init false;     // Process 1 view of Process 3
global w1_v4_ext : bool init false;     // Process 1 view of Process 4

global w2_v1_ext : bool init false;     // Process 2 view of Process 1
global w2_v3_ext : bool init false;     // Process 2 view of Process 3
global w2_v4_ext : bool init false;     // Process 2 view of Process 4

global w3_v1_ext : bool init false;     // Process 3 view of Process 1
global w3_v2_ext : bool init false;     // Process 3 view of Process 2
global w3_v4_ext : bool init false;     // Process 3 view of Process 4

global w4_v1_ext : bool init false;     // Process 4 view of Process 1
global w4_v2_ext : bool init false;     // Process 4 view of Process 2
global w4_v3_ext : bool init false;     // Process 4 view of Process 3
```

Listing 38. Declaring more global variables

```
formula v1_new = max(v1,v2_ext,v3_ext,v4_ext);  // Process 1 compute new value

formula w1_v2_new = w1_v2 | (v2_ext!=0) | w3_v2_ext | w4_v2_ext; // Process 1 update its view of Process 2
formula w1_v3_new = w1_v3 | (v3_ext!=0) | w2_v3_ext | w4_v3_ext; // Process 1 update its view of Process 3
formula w1_v4_new = w1_v4 | (v4_ext!=0) | w2_v4_ext | w3_v4_ext; // Process 1 update its view of Process 4

formula w1_c2_new = w1_c2 | (w2_v1_ext & w2_v3_ext & w2_v4_ext); // Process 1 knows that Process 2 view is complete
formula w1_c3_new = w1_c3 | (w3_v1_ext & w3_v2_ext & w3_v4_ext); // Process 1 knows that Process 3 view is complete
formula w1_c4_new = w1_c4 | (w4_v1_ext & w4_v2_ext & w4_v3_ext); // Process 1 knows that Process 4 view is complete
```

Listing 39. Updating expressions

the expression `received_message_complete` must check if all other processes have the view of `Process 4` and if `Process 4` has the view of all other processes.

```
// Optimistic Decision
formula decision_OP = w1_v2 & w1_v3 & w1_v4; // Process 1 has complete view at RN





// Pessimistic Decision
formula decision_PS = w1_v2 & w1_v3 & w1_v4 & w1_c2 & w1_c3 & w1_c4; // Process 1 has complete view at (RN-1) and has received complete view from all processes at RN
```

Listing 40. Updating expressions for decision criteria

4) Module Description: Internal Variables: `Process 1` view and confirmation of `Process 4` are added to the list of internal variables.

```
// Process 1 view of other Process
w1_v2 : bool init false; // Process 1 has the view of Process 2
w1_v3 : bool init false; // Process 1 has the view of Process 3
w1_v4 : bool init false; // Process 1 has the view of Process 4

// Process 1 has confirmation that other processes have complete view
w1_c2 : bool init false; // Process 1 has confirmation from Process 2
w1_c3 : bool init false; // Process 1 has confirmation from Process 3
w1_c4 : bool init false; // Process 1 has confirmation from Process 4
```

Listing 41. `Process 1` update the confirmation views

5) Module Description: Transitions: The first transition is modified so that `Process 1` also sends or fails to send its view of `Process 4`. Furthermore, at the compute phase, `Process 1` view and confirmation of `Process 4` must be updated.

```
// Process 1 sends or loses its message;
[] s1=1 & token=N1 & RN1<RN & m_lost<(RN*N) -> (1-q):(s1'=2) & (token'=next) & (v1_ext'=v1) & (w1_v2_ext'=w1_v2) & (w1_v3_ext'=w1_v3) & (w1_v4_ext'=w1_v4) &
      (RN1'=RN1+1) + q:(s1'=2) & (token'=next) & (v1_ext'=0) & (w1_v2_ext'=false) & (w1_v3_ext'=false) & (w1_v4_ext'=false) & (RN1'=RN1+1) & (m_lost'=m_lost+1) ;

// Not last round, Process 1 computes the messages of other processes: updates its value, views and confirmations;
[] s1=2 & token=N1 & RN1<RN -> 1: (s1'=1) & (v1'=v1_new) & (w1_v2'=w1_v2_new) & (w1_v3'=w1_v3_new) & (w1_v4'=w1_v4_new) & (w1_c2'=w1_c2_new) & (w1_c3'=w1_c3_new) &
      (w1_c4'=w1_c4_new) & (token'=next);

// Last round, Process 1 computes the messages of other processes: updates its confirmations and, only for OP, updates value and views;
[] s1=2 & token=N1 & RN1=RN -> 1: (s1'=3) & (w1_c2'=w1_c2_new) & (w1_c3'=w1_c3_new) & (w1_c4'=w1_c4_new) & (v1'=(DC=1)?v1_new:v1) & (w1_v2'=(DC=1)?w1_v2_new:w1_v2) &
      (w1_v3'=(DC=1)?w1_v3_new:w1_v3) & (w1_v4'=(DC=1)?w1_v4_new:w1_v4);
```

Listing 42. `Process 1` sends or loses its message

6) Modules of other Processes: In the definition of `Process 2` and `3`, the new internal and global variables related to `Process 4` are introduced in the list of variables that are renamed or replaced. Moreover, all the renaming and replacing previously defined for the case of `N=3` must be revised in other to follow the general rule defined in Section V-B7. One example is the variable `w1_v3` in the definition of `Process 3`: in the case of `N=3` it is renamed as `w2_v1`, while for `N=4` it is renamed as `w2_v4`. Finally, the definition of `Process 4` as a copy of `Process 1` is added to the model.

7) Verification of properties: The properties are updated to include the final state and the decision of `Process 4`:

D. PRISM Model For More Than Three Processes With Asymmetric Failure Model

The extension of the asymmetric model for the case of `N=4` follows the same steps of the symmetric model. Additionally, a transition is added to `Process 1` in order to include the reception of the message from `Process 4` (See Appendix A)

```
module Process_2=Process_1 [N1=N2, s1=s2, v1=v2, d1=d2, RN1=RN2, w1_v2=w2_v3, w1_v3=w2_v4, w1_v4=w2_v1, w1_c2=w2_c3, w1_c3=w2_c4, w1_c4=w2_c1, v1_ext=v2_ext,
        v2_ext=v3_ext, v3_ext=v4_ext, v4_ext=v1_ext, w1_v2_ext=w2_v3_ext, w1_v3_ext=w2_v4_ext, w1_v4_ext=w2_v1_ext, w2_v1_ext=w3_v2_ext, w2_v3_ext=w3_v4_ext,
        w2_v4_ext=w3_v1_ext, w3_v1_ext=w4_v2_ext, w3_v2_ext=w4_v3_ext, w3_v4_ext=w4_v1_ext, w4_v1_ext=w1_v2_ext, w4_v2_ext=w1_v3_ext, w4_v3_ext=w1_v4_ext,
        v1_ini=v2_ini] endmodule
module Process_3=Process_1 [N1=N3, s1=s3, v1=v3, d1=d3, RN1=RN3, w1_v2=w3_v4, w1_v3=w3_v1, w1_v4=w3_v2, w1_c2=w3_c4, w1_c3=w3_c1, w1_c4=w3_c2, v1_ext=v3_ext,
        v2_ext=v4_ext, v3_ext=v1_ext, v4_ext=v2_ext, w1_v2_ext=w3_v4_ext, w1_v3_ext=w3_v1_ext, w1_v4_ext=w3_v2_ext, w2_v1_ext=w4_v3_ext, w2_v3_ext=w4_v1_ext,
        w2_v4_ext=w4_v2_ext, w3_v1_ext=w1_v3_ext, w3_v2_ext=w1_v4_ext, w3_v4_ext=w1_v2_ext, w4_v1_ext=w2_v3_ext, w4_v2_ext=w2_v4_ext, w4_v3_ext=w2_v1_ext,
        v1_ini=v3_ini] endmodule
module Process_4=Process_1 [N1=N4, s1=s4, v1=v4, d1=d4, RN1=RN4, w1_v2=w4_v1, w1_v3=w4_v2, w1_v4=w4_v3, w1_c2=w4_c1, w1_c3=w4_c2, w1_c4=w4_c3, v1_ext=v4_ext,
        v2_ext=v1_ext, v3_ext=v2_ext, v4_ext=v3_ext, w1_v2_ext=w4_v1_ext, w1_v3_ext=w4_v2_ext, w1_v4_ext=w4_v3_ext, w2_v1_ext=w1_v4_ext, w2_v3_ext=w1_v2_ext,
        w2_v4_ext=w1_v3_ext, w3_v1_ext=w2_v4_ext, w3_v2_ext=w2_v1_ext, w3_v4_ext=w2_v3_ext, w4_v1_ext=w3_v4_ext, w4_v2_ext=w3_v1_ext, w4_v3_ext=w3_v2_ext,
        v1_ini=v4_ini, not_last=last] endmodule
```

Listing 43. `Processes'` modules

```
$P = ? [F(s1=0)\&(s2=0)&(s3=0)&(s4=0)&((d1!=d2)|(d2!=d3)|(d3!=d4))]$
```

Listing 44. Property (1), Probability of disagreement

```
$ P = ?[F(s1=0)&(s2=0)&(s3=0)&(s4=0)&(d1=false)&(d2=false)&(d3=false)&(d4=false)]$
```

Listing 45. Property (2), Probability of abort

```
$ P = ? [F(s1=0)&(s2=0)&(s3=0)&(s4=0)&(d1=true)&(d2=true)&(d3=true)&(d4=true)]$
```

Listing 46. Property (3), Probability of agreement

```
$ P = ? [F(s1=0)&(s2=0)&(s3=0)&(s4=0)&((d1!=d2)|(d2!=d3)|(d3!=d4))&(m-lost=1)]$
```

Listing 47. Property (4), Probability of disagreement with a specific number of lost messages

VI. Comparison of the Optimistic and the Pessimistic Decision Criterion

In this section, we present several graphs showing how the probability of disagreement,P_{DG}, varies with respect to the main system parameters: n the number of nodes involved in the decision process, and R the number of rounds of message exchange.

For symmetric failures, we assume a fixed probability of message loss, q, for all messages. For asymmetric failures we assume that all receivers have a fixed probability, Q, of not receiving a message. The results for the symmetric failure model are obtained using the closed-form expressions derived in Section IV, while the results for the asymmetric failure model have been calculated by means of the PRISM models described in Section **??**.

A. Symmetric versus asymmetric failures

Fig. 3 shows, for symmetric failures, the probability of agreement, disagreement and abort, as a function of q, for a *1-of-3* selection algorithm with two rounds of message exchange ($R = 2$). Fig. 4 shows the corresponding results for asymmetric failures, as a function of Q. We see that the probability of agreement on a value drops much more rapidly towards 0 for the pessimistic decision criterion compared to the optimistic decision criterion as q and Q approach one. Similarly, and as expected, we see that the probability of agreement to abort increase more rapidly for the pessimistic decision criterion (hence the name pessimistic!) as q and Q approach one. The probability of disagreement, P_{DG}, shows, as expected, a distinct peak in all the curves. For both failure models, the maximum probability of disagreement is considerably higher for the optimistic criterion than for the pessimistic criterion. On the other hand, the peak occurs for much lower values of q and Q for the pessimistic criterion. These results show that there are pros and cons to both decision criteria. If we compare Fig. 3 and Fig. 4, we see that the asymmetric failure model have higher peaks for P_{DG} compared to the symmetric failure model. It is interesting to see that P_{DG} is as high as 68% for the optimistic criterion for asymmetric failures.

B. Observations for Symmetric Failures

In this section, we investigate variations in the probability of disagreement in the presence of symmetric failures only.

Fig. 5 shows P_{DG} for a *1-of-3* consensus algorithm as a function of q. The solid curves show the results for the optimistic decision criterion with $R = 2, 3, 4$ and 6. The dotted curves show the corresponding results for the pessimistic criterion. We see that the peak values of P_{DG} for the optimistic criterion are considerably higher than those for the pessimistic criterion. We also see, as expected from the results shown in Fig. 3, that the P_{DG} for the optimistic criterion peaks at higher values of

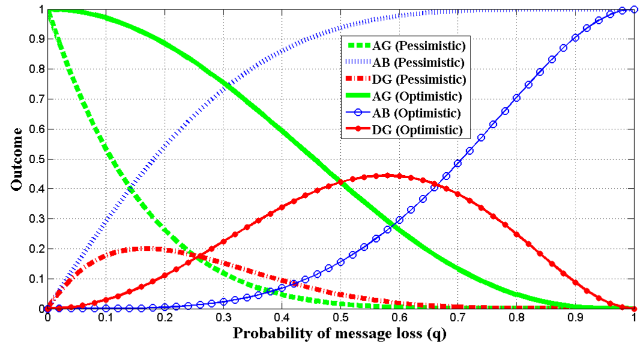


Figure 3. Probability of agreement (AG), abort (AB) and disagreement (DG) as a function of q for *1-of-3* selection algorithm with *symmetric* failures ($R = 2$), a comparison of optimistic and pessimistic criteria.

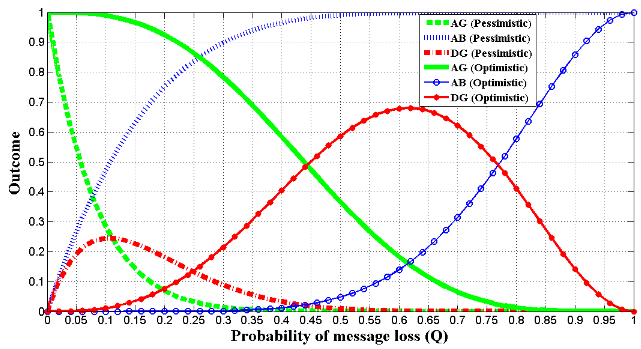


Figure 4. Probability of agreement (AG), abort (AB) and disagreement (DG) as a function of Q for *1-of-3* selection algorithm with *asymmetric* failures ($R = 2$), a comparison of optimistic and pessimistic criteria.

q compare to the pessimistic approach. For both decision criteria, the peaks of the curves move to the right if we increase the number of rounds. This is expected because when the number rounds increases so does the probability that all processes have a complete view at the last round.

Fig. 6 shows the probability of disagreement for a *1-of-n* selection algorithm with R= 3 rounds. The different curves represent the results for systems with different number of processes ($n = 2, 3, 4$ and 6). For both decision criteria, with increasing n, the peak of the curves move to the left. Hence, if the number of processes increase in a system with a fixed number of rounds, then the peak of P_{DG} moves towards points with lower probabilities of message loss. This implies that

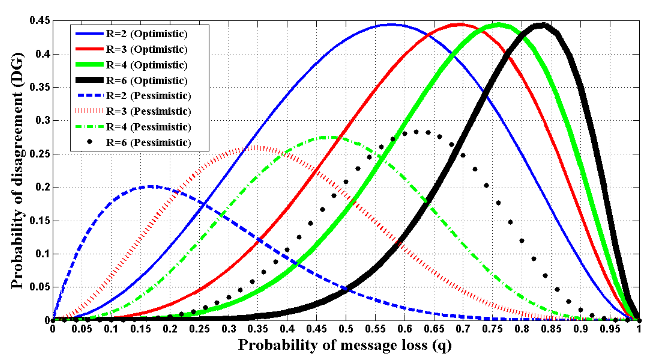


Figure 5. Probability of disagreement (P_{DG}) for a *1-of-3* selection algorithm with ($n = 3$, $R = 2, 3, 4$ and 6) for *symmetric* failures. Comparison of the optimistic decision criteria (solid curves) and the pessimistic decision criteria (dotted curves).

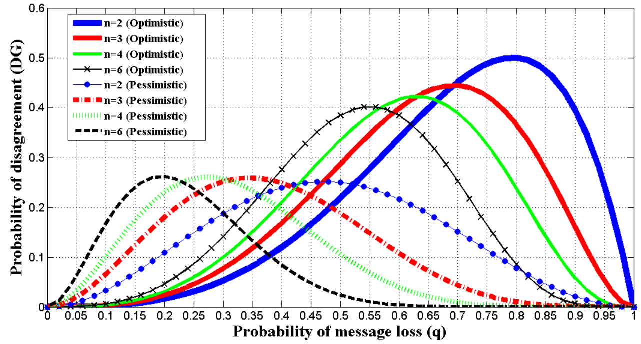


Figure 6. Probability of disagreement (P_{DG}) for *1-of-n* selection algorithm with ($R = 3$, $n = 2, 3, 4$ and 6) for *symmetric* failures. Comparison of the optimistic decision criteria (solid curves) and the pessimistic decision criteria (dotted curves).

the probability of agreement to abort increases when the number of processes increases.

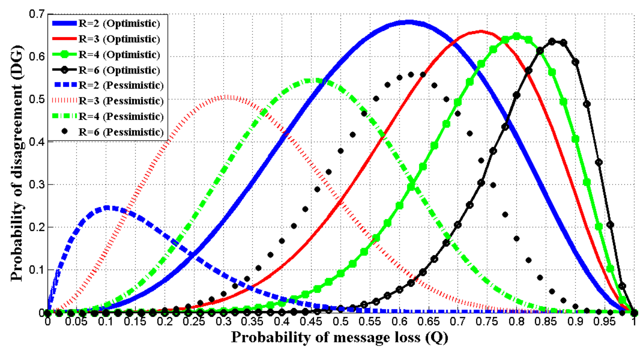


Figure 7. Probability of disagreement (P_{DG}) for a *1-of-3* selection algorithm with ($n = 3$, $R = 2, 3, 4$ and 6) for *asymmetric* failures. Comparison of the optimistic decision criteria (solid curves) and the decision pessimistic criteria (dotted curves).

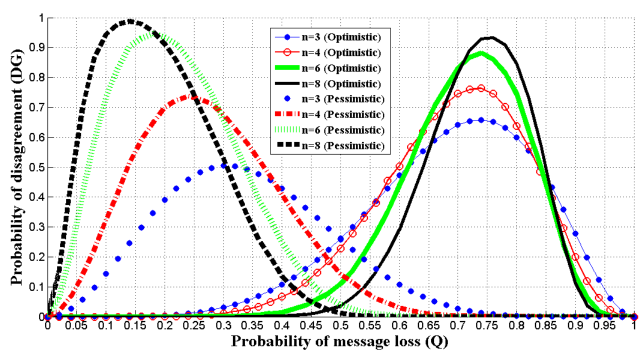


Figure 8. Probability of disagreement (P_{DG}) for *1-of-n* selection algorithm with ($R = 3$, $n = 3, 4, 6$ and 8) for *asymmetric failures*. Comparison of the optimistic decision criteria (solid curves) and the pessimistic criteria (dotted curves).

1) An analysis and comparison of moderately pessimistic decision criterion: Fig. 9 shows the probabilities of the three outcomes of the given decision criteria for a *1-of-3* selection algorithm with 2 rounds of execution as a function of the probability of message loss (Q). As we see in Fig. 9 for any value of Q, the optimistic decision criterion has the highest probability of agreement while the pessimistic one has the lowest. On the other hand, the pessimistic decision criterion has the highest probability of abort compare to the other two decision criteria.

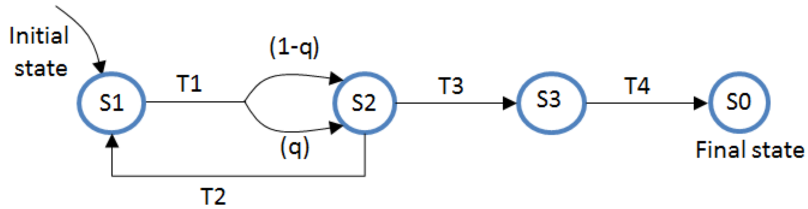
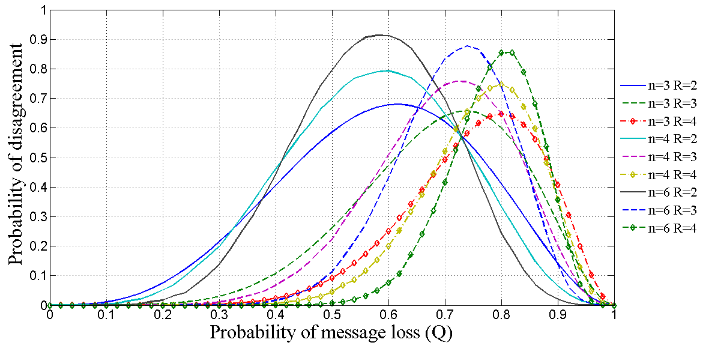


Figure 9. Probability of agreement, disagreement and abort for *1-of-3* selection algorithm for different decision criteria and $R = 2$ under asymmetric failures.

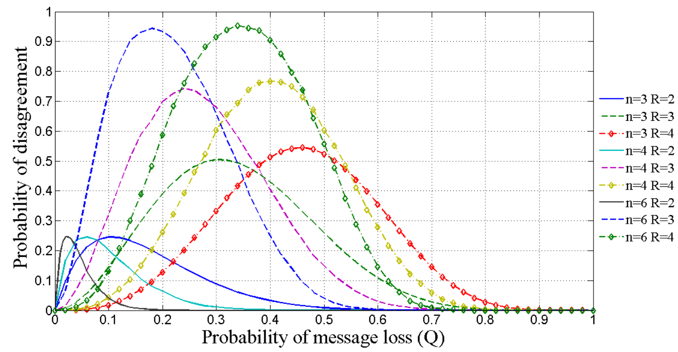
For all decision criteria the probability of disagreement shows a distinct peak. The maximum probability of disagreement varies significantly for different decision criteria. The optimistic decision criterion has the highest maximum probability of disagreement which is which is on the most right side of the x-axis (a large value of Q). The peak of disagreement for the pessimistic decision criterion on the other hand is on the most left side of x-axis (small values of Q) and is the smallest maximum probability of disagreement of all decision criteria, although not much larger than the maximum probability of disagreement for the moderately pessimistic decision criterion. Fig. 10 illustrates how the probability of disagreement is affected by varying the number of processes n and rounds R. In this figure the probability of disagreement for three different system configurations ($n = 3, 4, 6$) each running the *1-of-n* selection algorithm in two, three and four rounds of execution. The three given sub graphs 10(a), 10(b) and 10(c) correspond to the three decision criteria, optimistic, pessimistic and moderately pessimistic respectively.

Considering a fixed number of processes, a larger number of rounds means that a process has more chance to complete its view, therefore the probability of agreement increases for all the decision criteria. Consequently the peak of the probability of disagreement moves to the right side of the x-axis and is achieved for larger values of Q. For the optimistic decision criterion, the maximum value of disagreement decreases slightly with increasing R, but for the pessimistic and moderately pessimistic decision criteria, it increases significantly. An immediate conclusion is that, for all decision criteria, increasing the number of rounds does not guarantee lower probabilities of disagreement if the probability of message loss Q cannot be limited. Varying n affects the probability of disagreement in a different way. In the case of the optimistic decision criterion (Fig. 10(a)), when we increase the number of processes, the maximum probability of disagreement increases significantly (For $n = 6$ the probability of disagreement becomes larger than 80%). However, with increasing n the peak of disagreement does not move to the left or right side of the x-axis unlike the results we get with increasing R.

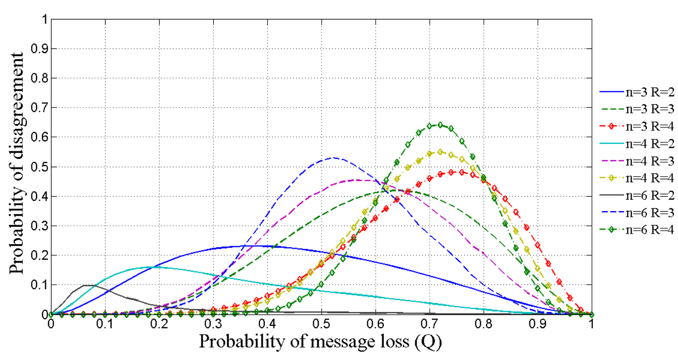
In the case of the pessimistic and moderately pessimistic decision criteria (Fig. 10(b) and Fig. 10(c)), we distinguish two different behaviours according to the number of rounds. For $R = 2$, in order to have an outcome of agreement or disagreement, all the messages should be successfully transmitted in the first round. When we increase the number of processes, the agreement region is reduced because it is less likely that a process completes its view in the first round. As a consequence, with increasing n the curve of disagreement moves to the left w.r.t. the x-axis when we have $R = 2$. In the case of the pessimistic decision criterion, when we increase n, the maximum probability of disagreement remains around the same value, that is 0.25. However, for the moderately pessimistic decision criterion, it declines significantly (not larger than 10% for $n = 6$. For $R > 2$, with increasing n the maximum probability of disagreement increases significantly for both the pessimistic and moderately pessimistic decision criteria. Fig. 11 shows a comparison of the three decision criteria for system configurations of three, four and six processes executing the *1-of-n* selection algorithm in $R = 2$ rounds. For any number of processes, the optimistic decision criterion have the highest peak of the probability of disagreement. For a given application with fixed n, if Q is unknown, disagreement can be minimized by adopting the moderately pessimistic decision criterion with $R = 2$. If the range of Q can be estimated, then we have a threshold and for lower values of Q the minimum disagreement is achieved by the optimistic decision criterion, while for higher values of Q the pessimistic decision criterion has the lowest probabilities of disagreement. For example the case of $n = 3$, this threshold is around $Q = 0.24$, while for $n = 6$ it is around $Q = 0.15$. Fig. 12 illustrates a comparison of probability of disagreement (P_{DG}) for three decision criteria, for a system executing the *1-of-n* selection algorithm with $n = 3, 4$ and $R = 2, 3$ assuming symmetric failures. As we see in this figure, in most cases the maximum value of P_{DG} is significantly lower when compared with the corresponding results for asymmetric failure. For the moderately pessimistic decision criterion, when we increase the number of processes with $R = 2$, the probability of disagreement is reduced. Furthermore, for all decision criteria, when we increase the number of rounds, the peak of P_{DG} moves to the right w.r.t. the x-axis but does not decrease, similar to the case of asymmetric failure.



(a) Optimistic decision criterion



(b) Pessimistic decision criterion



(c) Moderately pessimistic decision criterion

Figure 10. Probability of disagreement for *1-of-n* selection algorithm for $(n = 3, 4, 6)$ with $R = 2, 3, 4$ under asymmetric failures.

C. Observations for Asymmetric Failures

In this section, we study how the probability of disagreement varies for different values of Q, n and R in the presence of asymmetric failures. Fig. 7 shows P_{DG} for a *1-of-3* consensus algorithm as a function of Q, the probability of message loss in the asymmetric failure model.

As for the symmetric failures, we see that the peak values of P_{DG} for the optimistic criterion are considerably higher than those for the pessimistic criterion, while for the pessimistic criterion the P_{DG} peaks occur at lower values of Q compared to the P_{DG} peaks for the optimistic criterion. Interestingly, we see that the peak values increase as the number of rounds increase for pessimistic criterion, while trend for the optimistic criterion is the reverse.

Comparing the results in Fig. 7 with those in Fig. 5, as expected, we see higher probabilities of disagreement for the asymmetric failure model than for the symmetric failure model. Similar to the results for the symmetric failure model, the

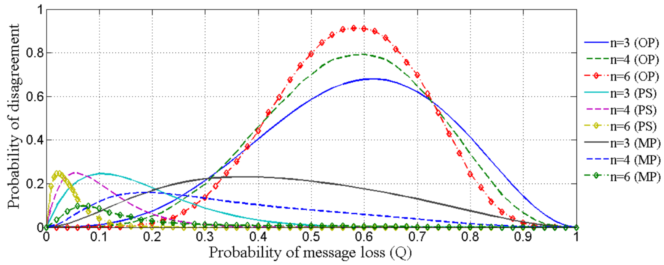


Figure 11. Probability disagreement for *1-of-n* selection algorithm for ($n = 3, 4, 6$) for different decision criteria and $R = 2$ under asymmetric failures.

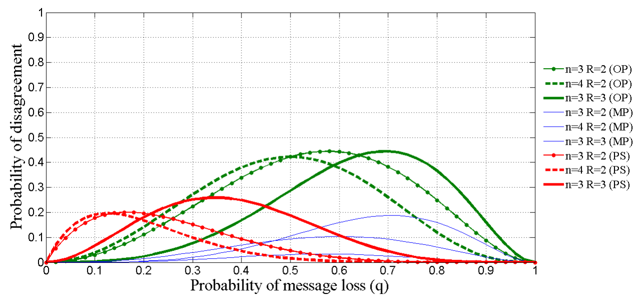


Figure 12. Probability of disagreement *1-of-n* selection algorithm ($n = 3, 4$ and $R = 2, 3$) for different decision criteria under symmetric failures.

peak of the curves move to the right for both decision criteria when the number of rounds increases.

Fig. 8 shows P_{DG} as a function of Q with $R = 3$ and $n = 3, 4, 6$ and 8. We see that with increasing n, we obtain higher peak values of P_{DG} for both decision criteria. The peak of the curves move to the left with increasing n for the pessimistic criterion, while they move to the right with increasing n for the optimistic criterion. We also see that the peak values for the pessimistic criterion is higher than for the optimistic criterion, which is the opposite to what we observed for the symmetric failures in Fig. 5 . Hence, for systems with many processes, the pessimistic criterion is worse than the optimistic criterion for asymmetric failures.

VII. Related Work

The problem of reaching agreement among the processes of a fault-tolerant distributed system has been investigated widely since thirty years ago [8]–[10]. The consensus problem has been proved to be solvable under different failure assumptions such as in [8], [9], [11]. Most of previous research were based on different classes of process failures only, with assuming reliable communication links among processes. Some were simply associating communication failures to process failures rather than investigating them explicitly as an independent phenomenon (e.g., see [12], [13]) which may lead to incorrect characterization of systems.[4] There are also perception-based hybrid failure models proposed in literature such as in [2] in which the sender-caused link faults are considered as process faults and the term link fault is used to denote the receiver-caused failures. Such failure models also may lead to undesirable conclusions for a system. Perfect communications abstraction with retransmission schemes is used in data-link layer protocols [14], but due to the high execution time needed in these solutions they are only useful for asynchronous systems when real-time properties are not so important.

On the other hand, considering highly unpredictable wireless environments, it is important to consider the communication failures explicitly in order to assure dependability and safety of critical distributed systems. Therefore our focus is to study the synchronous consensus problem for systems subject to transient and dynamic communication failures. In such a model, failures may occur on any communication link at any time when there are no limitations on the number or pattern of the lost messages. We define our failure model based on the model introduced by Santoro and Widmayer in [1] denoted as the *transmission fault model*. We are mainly considering dynamic and transient omission faults. We know from the results given in [1] and [15] that any non-trivial form of agreement is impossible to solve if $n1$ or more messages can be lost per communication round in a system with n processes. The impossibility result given by Santoro and Widmayer is indeed a

[4]For example attributing the transmission faults to the sending or receiving processes one may reach to incorrect conclusions such as assuming the entire processes to be faulty due to only a single failure of a message broadcast.

generalization of the given results by Akkoyulunu et al. in [16] and later in 1978 by Gray [17] in which they show that there is no deterministic solution to the consensus problem between two processes with unreliable communication links.

Santoro and Widmayer in [15] define a faulty transmission resulting in omission, corruption or addition of a message and then provide an extensive map of possible and impossible computations in the presence of transmission faults. Later in [18] they define bounds on the number of dynamic faults with expressing the connectivity requirements to achieve any non-trivial agreement.

In [14], Afek et al. employed randomization techniques to solve the k-consensus problem in presence of communication failures. In a k-consensus problem, at least k processes among n processes decide on the same value such that $k > n/2$. They show that the safety properties of consensus (i.e., validity and agreement) are ensured in presence of even unrestricted communication failures, however in order to satisfy the liveness property (i.e., termination) of the consensus algorithm the number of faults in a round should be restricted.

There are a large number of methods suggested in literature to circumvent the given impossibility result in synchronous consensus systems with dynamic omission faults such as [18]–[20]. However, most of the suggested methods take a preventive approach toward this problem, such as restricting the communication failure patterns or limiting the number of failures in a round.

Nevertheless, it is possible to design protocols that have a low probability of failing to reach consensus, so as to meet specific requirements on reliability and availability. This intuition has been explored to build protocols that maximize the probability of correctness by accumulating more information over a larger duration of the execution [21]. Researchers have also focused on stochastic models of verifying the probability of transition into an incorrect state [22].

Our goal in this paper is to design decision making algorithms to run on top of a simple consensus protocol with the main purpose of minimizing the probability of failing to reach consensus. We evaluate and compare the effectiveness of different decision algorithms by means of using probabilistic model checking tools as well as deriving closed form expressions to calculate the probability of disagreement among processes. Our results may be applied also for on-line verification and adaptation to cope with variable probabilities of communication failures. Our work focuses on probabilistic analysis of round-based consensus protocols in which processes communicate in rounds of message exchange in order to decide on a consistent output [9]. Our system model is inspired by a general computational model named the *heard-of* model introduced by Schiper and Charron-Bost [23] and is used to specify systems with any type of benign failures.

VIII. Conclusion And Future Work

We have presented closed-form expressions for calculating the probability of disagreement in the presence of symmetric message losses for a family of simple synchronous consensus algorithms. Our work is motivated by the need to develop fast and reliable consensus algorithm for distributed cooperative systems for the transportation sector. Since it is impossible to construct an algorithm that solves the consensus problem for a system that uses wireless, and thereby, unreliable communication, we are interested in exploring the design of adaptive consensus algorithms that are equipped with an on-line mechanism which can temporarily shut down the algorithm in situations where the likelihood for disagreement becomes unacceptably high. We are therefore interested in finding computational effective ways of calculating the probability of disagreement on-line. The next step in our pursuit for such algorithms will be to find closed-form expressions for calculating the probability of disagreement for simple algorithms under the assumption of asymmetric message losses.

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APPENDIX

```
dtmc

const N=3;      // Number of processes in the network (N cannot be modified)
const RN=2;         // Number of rounds in the protocol (RN>=2)
const double q;     // Probability of losing a message (0<=q<=1)
const DC=3;         // Decision criterion: OP=1; PS=3;

const N1=1;         // Identity number of Process 1
const N2=2;         // Identity number of Process 2
const N3=3;         // Identity number of Process 3

const v_max=2;      // Maximum value of a process
const v1_ini=1;     // Initial value of Process 1
const v2_ini=2;     // Initial value of Process 2
const v3_ini=1;     // Initial value of Process 3

const not_last=1;   // Auxiliary constant to define the next process
const last=0;       // Auxiliary constant to define the next process

global v1_ext : [0..v_max] init 0;  // Message value of Process 1
global v2_ext : [0..v_max] init 0;  // Message value of Process 2
global v3_ext : [0..v_max] init 0;  // Message value of Process 3

global w1_v2_ext : bool init false; // Process 1 view of Process 2
global w1_v3_ext : bool init false; // Process 1 view of Process 3

global w2_v1_ext : bool init false; // Process 2 view of Process 1
global w2_v3_ext : bool init false; // Process 2 view of Process 3

global w3_v1_ext : bool init false; // Process 3 view of Process 1
global w3_v2_ext : bool init false; // Process 3 view of Process 2

global token : [1..N] init 1;       // Token used to coordinate the processes
global m_lost : [0..(RN*N)] init 0; // Number of lost messages

formula next = N1*not_last+1;   // Define the next process to receive the token

formula v1_new = max(v1,v2_ext,v3_ext); // Process 1 compute new value

formula w1_v2_new = w1_v2 | (v2_ext!=0) | w3_v2_ext; // Process 1 update its view of Process 2
formula w1_v3_new = w1_v3 | (v3_ext!=0) | w2_v3_ext; // Process 1 update its view of Process 3

formula w1_c2_new = w1_c2 | (w2_v1_ext & w2_v3_ext); // Process 1 knows that Process 2 view is complete
formula w1_c3_new = w1_c3 | (w3_v1_ext & w3_v2_ext); // Process 1 knows that Process 3 view is complete

// Optimistic Decision
formula decision_OP = w1_v2 & w1_v3; // Process 1 has complete view at RN




// Pessimistic Decision
formula decision_PS = w1_v2 & w1_v3 & w1_c2 & w1_c3; // Process 1 has complete view at (RN-1) and has received complete view from all processes at RN

// General Decision Formula
formula decision = ((DC=1) & decision_OP) | ((DC=3) & decision_PS); // Combine all decision criterea in a single formula

module Process_1
s1 : [0..3] init 1;  // Process 1 current state
RN1 : [0..RN] init 0; // Current round
d1 : bool init false; // Process 1 decision
v1 : [0..v_max] init v1_ini; // Process 1 value

// Process 1 view of other processes
w1_v2 : bool init false; // Process 1 has the view of Process 2
w1_v3 : bool init false; // Process 1 has the view of Process 3

// Process 1 has confirmation that other processes have complete view
w1_c2 : bool init false; // Process 1 has confirmation from Process 2
w1_c3 : bool init false; // Process 1 has confirmation from Process 3

// Process 1 sends or loses its message;
[] s1=1 & token=N1 & RN1<RN & m_lost<(RN*N) -> (1-q):(s1'=2) & (token'=next) & (v1_ext'=v1) & (w1_v2_ext'=w1_v2) & (w1_v3_ext'=w1_v3) & (RN1'=RN1+1) + q:(s1'=2) &
       (token'=next) & (v1_ext'=0) & (w1_v2_ext'=false) & (w1_v3_ext'=false) & (RN1'=RN1+1) & (m_lost'=m_lost+1) ;

// Not last round, Process 1 computes the messages of other processes: updates its value, views and confirmations;
[] s1=2 & token=N1 & RN1<RN -> 1: (s1'=1) & (v1'=v1_new) & (w1_v2'=w1_v2_new) & (w1_v3'=w1_v3_new) & (w1_c2'=w1_c2_new) & (w1_c3'=w1_c3_new) & (token'=next);

// Last round, Process 1 computes the messages of other processes: updates its confirmations and, only for OP, updates value and views;
[] s1=2 & token=N1 & RN1=RN -> 1: (s1'=3) & (w1_c2'=w1_c2_new) & (w1_c3'=w1_c3_new) & (v1'=(DC=1)?v1_new:v1) & (w1_v2'=(DC=1)?w1_v2_new:w1_v2) &
       (w1_v3'=(DC=1)?w1_v3_new:w1_v3);

// Process 1 decides -> agree or abort
[] s1=3 & token=N1 -> 1: (s1'=0) & (token'=next) & (d1'= decision);
endmodule

module Process_2=Process_1 [N1=N2, s1=s2, v1=v2, d1=d2, RN1=RN2, w1_v2=w2_v3, w1_v3=w2_v1, w1_c2=w2_c3, w1_c3=w2_c1, v1_ext=v2_ext, v2_ext=v3_ext, v3_ext=v1_ext,
       w1_v2_ext=w2_v3_ext, w1_v3_ext=w2_v1_ext, w2_v1_ext=w3_v2_ext, w2_v3_ext=w3_v1_ext, w3_v1_ext=w1_v2_ext, w3_v2_ext=w1_v3_ext, v1_ini=v2_ini] endmodule
```

Listing 48.   N=3 with Symmetric Failures

```
dtmc

const N=4;      // Number of processes in the network (N cannot be modified)
const RN=2;          // Number of rounds in the protocol (RN>=2)
const double q;      // Probability of losing a message (0<=q<=1)
const DC=3;          // Decision criterion: OP=1; PS=3;

const N1=1;          // Identity number of Process 1
const N2=2;          // Identity number of Process 2
const N3=3;          // Identity number of Process 3
const N4=4;          // Identity number of Process 4

const v_max=2;       // Maximum value of a process
const v1_ini=1;      // Initial value of Process 1
const v2_ini=2;      // Initial value of Process 2
const v3_ini=1;      // Initial value of Process 3
const v4_ini=1;      // Initial value of Process 4

const not_last=1;    // Auxiliary constant to define the next Process
const last=0;        // Auxiliary constant to define the next Process

global v1_ext : [0..v_max] init 0;  // Message value of Process 1
global v2_ext : [0..v_max] init 0;  // Message value of Process 2
global v3_ext : [0..v_max] init 0;  // Message value of Process 3
global v4_ext : [0..v_max] init 0;  // Message value of Process 4

global w1_v2_ext : bool init false;     // Process 1 view of Process 2
global w1_v3_ext : bool init false;     // Process 1 view of Process 3
global w1_v4_ext : bool init false;     // Process 1 view of Process 4

global w2_v1_ext : bool init false;     // Process 2 view of Process 1
global w2_v3_ext : bool init false;     // Process 2 view of Process 3
global w2_v4_ext : bool init false;     // Process 2 view of Process 4

global w3_v1_ext : bool init false;     // Process 3 view of Process 1
global w3_v2_ext : bool init false;     // Process 3 view of Process 2
global w3_v4_ext : bool init false;     // Process 3 view of Process 4

global w4_v1_ext : bool init false;     // Process 4 view of Process 1
global w4_v2_ext : bool init false;     // Process 4 view of Process 2
global w4_v3_ext : bool init false;     // Process 4 view of Process 3

global token : [1..N] init 1;       // Token used to coordinate the processes
global m_lost: [0..(RN*N)] init 0;  // Number of lost messages

formula next = N1*not_last+1;       // Define the next process in the network

formula v1_new = max(v1,v2_ext,v3_ext,v4_ext);  // Process 1 compute new value

formula w1_v2_new = w1_v2 | (v2_ext!=0) | w3_v2_ext | w4_v2_ext; // Process 1 update its view of Process 2
formula w1_v3_new = w1_v3 | (v3_ext!=0) | w2_v3_ext | w4_v3_ext; // Process 1 update its view of Process 3
formula w1_v4_new = w1_v4 | (v4_ext!=0) | w2_v4_ext | w3_v4_ext; // Process 1 update its view of Process 4

formula w1_c2_new = w1_c2 | (w2_v1_ext & w2_v3_ext & w2_v4_ext); // Process 1 knows that Process 2 view is complete
formula w1_c3_new = w1_c3 | (w3_v1_ext & w3_v2_ext & w3_v4_ext); // Process 1 knows that Process 3 view is complete
formula w1_c4_new = w1_c4 | (w4_v1_ext & w4_v2_ext & w4_v3_ext); // Process 1 knows that Process 4 view is complete

// Optimistic Decision
formula decision_OP = w1_v2 & w1_v3 & w1_v4; // Process 1 has complete view at RN





// Pessimistic Decision
formula decision_PS = w1_v2 & w1_v3 & w1_v4 & w1_c2 & w1_c3 & w1_c4; // Process 1 has complete view at (RN-1) and has received complete view from all processes at RN

// General Decision Formula
formula decision = ((DC=1) & decision_OP) | ((DC=3) & decision_PS); // Combine all decision criterea in a single formula

module Process_1
s1 : [0..3] init 1;  // Process 1 current state
RN1 : [0..RN] init 0; // Current round
d1 : bool init false; // Process 1 decision
v1 : [0..v_max] init v1_ini; // Process 1 value

// Process 1 view of other Processs
w1_v2 : bool init false; // Process 1 has the view of Process 2
w1_v3 : bool init false; // Process 1 has the view of Process 3
w1_v4 : bool init false; // Process 1 has the view of Process 4

// Process 1 has confirmation that other processes have complete view
w1_c2 : bool init false; // Process 1 has confirmation from Process 2
w1_c3 : bool init false; // Process 1 has confirmation from Process 3
w1_c4 : bool init false; // Process 1 has confirmation from Process 4

// Process 1 sends or loses its message;
[] s1=1 & token=N1 & RN1<RN & m_lost<(RN*N) -> (1-q):(s1'=2) & (token'=next) & (v1_ext'=v1) & (w1_v2_ext'=w1_v2) & (w1_v3_ext'=w1_v3) & (w1_v4_ext'=w1_v4) &
       (RN1'=RN1+1) + q:(s1'=2) & (token'=next) & (v1_ext'=0) & (w1_v2_ext'=false) & (w1_v3_ext'=false) & (w1_v4_ext'=false) & (RN1'=RN1+1) & (m_lost'=m_lost+1) ;

// Not last round, Process 1 computes the messages of other processes: updates its value, views and confirmations;
[] s1=2 & token=N1 & RN1<RN -> 1: (s1'=1) & (v1'=v1_new) & (w1_v2'=w1_v2_new) & (w1_v3'=w1_v3_new) & (w1_v4'=w1_v4_new) & (w1_c2'=w1_c2_new) & (w1_c3'=w1_c3_new) &
       (w1_c4'=w1_c4_new) & (token'=next);

// Last round, Process 1 computes the messages of other processes: updates its confirmations and, only for OP, updates value and views;
[] s1=2 & token=N1 & RN1=RN -> 1: (s1'=3) & (w1_c2'=w1_c2_new) & (w1_c3'=w1_c3_new) & (w1_c4'=w1_c4_new) & (v1'=(DC=1)?v1_new:v1) & (w1_v2'=(DC=1)?w1_v2_new:w1_v2) &
       (w1_v3'=(DC=1)?w1_v3_new:w1_v3) & (w1_v4'=(DC=1)?w1_v4_new:w1_v4);

// Process 1 decides -> agree or abort
[] s1=3 & token=N1 -> 1: (s1'=0) & (token'=next) & (d1'= decision);
endmodule

module Process_2=Process_1 [N1=N2, s1=s2, v1=v2, d1=d2, RN1=RN2, w1_v2=w2_v3, w1_v3=w2_v4, w1_v4=w2_v1, w1_c2=w2_c3, w1_c3=w2_c4, w1_c4=w2_c1, v1_ext=v2_ext,
       v2_ext=v3_ext, v3_ext=v4_ext, v4_ext=v1_ext, w1_v2_ext=w2_v3_ext, w1_v3_ext=w2_v4_ext, w1_v4_ext=w2_v1_ext, w2_v1_ext=w3_v2_ext, w2_v3_ext=w3_v4_ext,
       w2_v4_ext=w3_v1_ext, w3_v1_ext=w4_v2_ext, w3_v2_ext=w4_v3_ext, w3_v4_ext=w4_v1_ext, w4_v1_ext=w1_v2_ext, w4_v2_ext=w1_v3_ext, w4_v3_ext=w1_v4_ext,
       v1_ini=v2_ini] endmodule
module Process_3=Process_1 [N1=N3, s1=s3, v1=v3, d1=d3, RN1=RN3, w1_v2=w3_v4, w1_v3=w3_v1, w1_v4=w3_v2, w1_c2=w3_c4, w1_c3=w3_c1, w1_c4=w3_c2, v1_ext=v3_ext,
       v2_ext=v4_ext, v3_ext=v1_ext, v4_ext=v2_ext, w1_v2_ext=w3_v4_ext, w1_v3_ext=w3_v1_ext, w1_v4_ext=w3_v2_ext, w2_v1_ext=w4_v3_ext, w2_v3_ext=w4_v1_ext,
       w2_v4_ext=w4_v2_ext, w3_v1_ext=w1_v3_ext, w3_v2_ext=w1_v4_ext, w3_v4_ext=w1_v2_ext, w4_v1_ext=w2_v3_ext, w4_v2_ext=w2_v4_ext, w4_v3_ext=w2_v1_ext,
       v1_ini=v3_ini] endmodule
module Process_4=Process_1 [N1=N4, s1=s4, v1=v4, d1=d4, RN1=RN4, w1_v2=w4_v1, w1_v3=w4_v2, w1_v4=w4_v3, w1_c2=w4_c1, w1_c3=w4_c2, w1_c4=w4_c3, v1_ext=v4_ext,
       v2_ext=v1_ext, v3_ext=v2_ext, v4_ext=v3_ext, w1_v2_ext=w4_v1_ext, w1_v3_ext=w4_v2_ext, w1_v4_ext=w4_v3_ext, w2_v1_ext=w1_v4_ext, w2_v3_ext=w1_v2_ext,
       w2_v4_ext=w1_v3_ext, w3_v1_ext=w2_v4_ext, w3_v2_ext=w2_v1_ext, w3_v4_ext=w2_v3_ext, w4_v1_ext=w3_v4_ext, w4_v2_ext=w3_v1_ext, w4_v3_ext=w3_v2_ext,
       v1_ini=v4_ini, not_last=last] endmodule
```

Listing 49. N>3 with Symmetric Failures

```
dtmc

const N=3;      // Number of processes in the network (N cannot be modified)
const RN=2;          // Number of rounds in the protocol (RN>=2)
const double Q=0.5;     // Probability of losing a message (0<=q<=1)
const DC=3;          // Decision criterion: OP=1; MP=2; PS=3;

const N1=1;          // Identity number of Process 1
const N2=2;          // Identity number of Process 2
const N3=3;          // Identity number of Process 3

const v_max=2;       // Maximum value of a process
const v1_ini=1;      // Initial value of Process 1
const v2_ini=2;      // Initial value of Process 2
const v3_ini=1;      // Initial value of Process 3

const not_last=1;    // Auxiliary constant to define the next process
const last=0;        // Auxiliary constant to define the next process

global v1_ext : [0..v_max] init 0;  // Message value of Process 1
global v2_ext : [0..v_max] init 0;  // Message value of Process 2
global v3_ext : [0..v_max] init 0;  // Message value of Process 3

global w1_v2_ext : bool init false;     // Process 1 view of Process 2
global w1_v3_ext : bool init false;     // Process 1 view of Process 3

global w2_v1_ext : bool init false;     // Process 2 view of Process 1
global w2_v3_ext : bool init false;     // Process 2 view of Process 3

global w3_v1_ext : bool init false;     // Process 3 view of Process 1
global w3_v2_ext : bool init false;     // Process 3 view of Process 2

global token : [1..N] init 1;           // Token used to coordinate the processes
global m_lost : [0..(RN*N*(N-1))] init 0;   // Number of lost messages

formula next = N1*not_last+1; // Define the next Process in the network

formula v1_new = max(v1,(n1_nf2?v2_ext:0),(n1_nf3?v3_ext:0));  // Process 1 compute new value

formula w1_v2_new = w1_v2 | n1_nf2 | (n1_nf3 & w3_v2_ext); // Process 1 update its view of Process 2
formula w1_v3_new = w1_v3 | n1_nf3 | (n1_nf2 & w2_v3_ext); // Process 1 update its view of Process 3

formula w1_c2_new = w1_c2 | (n1_nf2 & (w2_v1_ext & w2_v3_ext)); // Process 1 knows that Process 2 view is complete
formula w1_c3_new = w1_c3 | (n1_nf3 & (w3_v1_ext & w3_v2_ext)); // Process 1 knows that Process 3 view is complete

// Optimistic Decision
formula decision_OP = w1_v2 & w1_v3; // Process 1 has complete view at RN





// Pessimistic Decision
formula decision_PS = w1_v2 & w1_v3 & w1_c2 & w1_c3; // Process 1 has complete view at (RN-1) and has received complete view from all processes at RN

// General Decision Formula
formula decision = ((DC=1) & decision_OP) | ((DC=3) & decision_PS); // Combine all decision criterea in a single formula

module Process_1
s1 : [0..N+2] init 1;  // Process 1 current state
RN1 : [0..RN] init 0; // Current round
v1 : [0..v_max] init v1_ini; // Process 1 value
d1 : bool init false; // Process 1 decision

// Status of the message from the other processs
n1_nf2 : bool init true; // Process 1 has not received the message of Process 2
n1_nf3 : bool init true; // Process 1 has not received the message of Process 3

// Process 1 view of other processes
w1_v2 : bool init false; // Process 1 has the view of Process 2
w1_v3 : bool init false; // Process 1 has the view of Process 3

// Process 1 has confirmation that other processes have complete view
w1_c2 : bool init false; // Process 1 has confirmation from Process 2
w1_c3 : bool init false; // Process 1 has confirmation from Process 3

// Process 1 sends its message;
[] s1=1 & token=N1 & RN1<RN -> 1:(s1'=2) & (token'=next) & (v1_ext'=v1) & (w1_v2_ext'=w1_v2) & (w1_v3_ext'=w1_v3) & (RN1'=RN1+1);

// Process 1 receives or loses the message of Process 2
[] s1=2 & token=N1 & RN1<=RN & (m_lost<(RN*N*(N-1))) -> (1-Q): (s1'=3) & (n1_nf2'=true) + Q: (s1'=3) & (n1_nf2'=false) & (m_lost'=m_lost+1);
// Process 1 receives or loses the message of Process 3
[] s1=3 & token=N1 & RN1<=RN & (m_lost<(RN*N*(N-1))) -> (1-Q): (s1'=4) & (n1_nf3'=true) + Q: (s1'=4) & (n1_nf3'=false) & (m_lost'=m_lost+1);

// Not last round, Process 1 computes the messages of other processes: updates its value, views and confirmations;
[] s1=N+1 & token=N1 & RN1<RN -> 1: (s1'=1) & (v1'=v1_new) & (w1_v2'=w1_v2_new) & (w1_v3'=w1_v3_new) & (w1_c2'=w1_c2_new) & (w1_c3'=w1_c3_new) & (token'=next);

// Last round, Process 1 computes the messages of other processes: updates its confirmations and, only for OP, updates value and views;
[] s1=N+1 & token=N1 & RN1=RN -> 1: (s1'=N+2) & (w1_c2'=w1_c2_new) & (w1_c3'=w1_c3_new) & (v1'=(DC=1)?v1_new:v1) & (w1_v2'=(DC=1)?w1_v2_new:w1_v2) &
       (w1_v3'=(DC=1)?w1_v3_new:w1_v3);

// Process 1 decides -> agree or abort
[] s1=N+2 & token=N1 -> 1: (s1'=0) & (token'=next) & (d1'= decision);
endmodule

module Process_2=Process_1 [N1=N2, s1=s2, v1=v2, d1=d2, RN1=RN2, n1_nf2=n2_nf3, n1_nf3=n2_nf1, w1_v2=w2_v3, w1_v3=w2_v1, w1_c2=w2_c3, w1_c3=w2_c1, v1_ext=v2_ext,
       v2_ext=v3_ext, v3_ext=v1_ext, w1_v2_ext=w2_v3_ext, w1_v3_ext=w2_v1_ext, w2_v1_ext=w3_v2_ext, w2_v3_ext=w3_v1_ext, w3_v1_ext=w1_v2_ext, w3_v2_ext=w1_v3_ext,
       v1_ini=v2_ini] endmodule
module Process_3=Process_1 [N1=N3, s1=s3, v1=v3, d1=d3, RN1=RN3, n1_nf2=n3_nf1, n1_nf3=n3_nf2, w1_v2=w3_v1, w1_v3=w3_v2, w1_c2=w3_c1, w1_c3=w3_c2, v1_ext=v3_ext,
       v2_ext=v1_ext, v3_ext=v2_ext, w1_v2_ext=w3_v1_ext, w1_v3_ext=w3_v2_ext, w2_v1_ext=w1_v3_ext, w2_v3_ext=w1_v2_ext, w3_v1_ext=w2_v3_ext, w3_v2_ext=w2_v1_ext,
       v1_ini=v3_ini, not_last=last] endmodule
```

Listing 50. N=3 with Asymmetric Failures

```
dtmc

const N=4;      // Number of processes in the network (N cannot be modified)
const RN=2;          // Number of rounds in the protocol (RN>=2)
const double Q=0.5;      // Probability of losing a message (0<=q<=1)
const DC=3;          // Decision criterion: OP=1; PS=3;

const N1=1;          // Identity number of Process 1
const N2=2;          // Identity number of Process 2
const N3=3;          // Identity number of Process 3
const N4=4;          // Identity number of Process 4

const v_max=2;       // Maximum value of a process
const v1_ini=1;      // Initial value of Process 1
const v2_ini=2;      // Initial value of Process 2
const v3_ini=1;      // Initial value of Process 3
const v4_ini=2;      // Initial value of Process 4

const not_last=1;    // Auxiliary constant to define the next process
const last=0;        // Auxiliary constant to define the next process

global v1_ext : [0..v_max] init 0;  // Message value of Process 1
global v2_ext : [0..v_max] init 0;  // Message value of Process 2
global v3_ext : [0..v_max] init 0;  // Message value of Process 3
global v4_ext : [0..v_max] init 0;  // Message value of Process 4

global w1_v2_ext : bool init false;     // Process 1 view of Process 2
global w1_v3_ext : bool init false;     // Process 1 view of Process 3
global w1_v4_ext : bool init false;     // Process 1 view of Process 4

global w2_v1_ext : bool init false;     // Process 2 view of Process 1
global w2_v3_ext : bool init false;     // Process 2 view of Process 3
global w2_v4_ext : bool init false;     // Process 2 view of Process 4

global w3_v1_ext : bool init false;     // Process 3 view of Process 1
global w3_v2_ext : bool init false;     // Process 3 view of Process 2
global w3_v4_ext : bool init false;     // Process 3 view of Process 4

global w4_v1_ext : bool init false;     // Process 4 view of Process 1
global w4_v2_ext : bool init false;     // Process 4 view of Process 2
global w4_v3_ext : bool init false;     // Process 4 view of Process 3

global token : [1..N] init 1;       // Token used to coordinate the Processs
global m_lost: [0..RN*N*(N-1)] init 0;  // Number of lost messages

formula next = N1*not_last+1;       // Define the next Process in the network

formula v1_new = max(v1,(n1_nf2?v2_ext:0),(n1_nf3?v3_ext:0),(n1_nf4?v4_ext:0));     // Process 1 compute new value
formula w1_v2_new = w1_v2 | n1_nf2 | (n1_nf3 & w3_v2_ext) | (n1_nf4 & w4_v2_ext) ;  // Process 1 update its view of Process 2
formula w1_v3_new = w1_v3 | n1_nf3 | (n1_nf2 & w2_v3_ext) | (n1_nf4 & w4_v3_ext) ;  // Process 1 update its view of Process 3
formula w1_v4_new = w1_v4 | n1_nf4 | (n1_nf2 & w2_v4_ext) | (n1_nf3 & w3_v4_ext) ;  // Process 1 update its view of Process 4

formula w1_c2_new = w1_c2 | (n1_nf2 & (w2_v1_ext & w2_v3_ext & w2_v4_ext));     // Process 1 knows that Process 2 view is complete
formula w1_c3_new = w1_c3 | (n1_nf3 & (w3_v1_ext & w3_v2_ext & w3_v4_ext));     // Process 1 knows that Process 3 view is complete
formula w1_c4_new = w1_c4 | (n1_nf4 & (w4_v1_ext & w4_v2_ext & w4_v3_ext));     // Process 1 knows that Process 4 view is complete


// Optimistic Decision
formula decision_OP = w1_v2 & w1_v3 & w1_v4; // Process 1 has complete view at RN
```

Listing 51. N>3 with Asymmetric Failures- part 1

```
// Pessimistic Decision
formula decision_PS = w1_v2 & w1_v3 & w1_v4 & w1_c2 & w1_c3 & w1_c4; // Process 1 has complete view at (RN-1) and received complete view from all processes at RN

// General Decision Formula
formula decision = ((DC=1) & decision_OP) | ((DC=3) & decision_PS); // Combine all decision criterea in a single formula

module Process_1
s1 : [0..N+2] init 1;  // Process 1 current state
RN1: [0..RN] init 0; // Current round
v1 : [0..v_max] init v1_ini; // Process 1 value
d1: bool init false; // Process 1 decision

// Status of the message of the other Processs
n1_nf2: bool init true; // Process 1 has not received the message of Process 2
n1_nf3: bool init true; // Process 1 has not received the message of Process 3
n1_nf4: bool init true; // Process 1 has not received the message of Process 4

// Process 1 view of other Processs
w1_v2 : bool init false; // Process 1 has the view of Process 2
w1_v3 : bool init false; // Process 1 has the view of Process 3
w1_v4 : bool init false; // Process 1 has the view of Process 4

// Process 1 has confirmation that other processes have complete view
w1_c2 : bool init false; // Process 1 has confirmation from Process 2
w1_c3 : bool init false; // Process 1 has confirmation from Process 3
w1_c4 : bool init false; // Process 1 has confirmation from Process 4

// Process 1 sends its message;
[] s1=1 & token=N1 & RN1<RN -> 1:(s1'=2) & (token'=next) & (v1_ext'=v1) & (w1_v2_ext'=w1_v2) & (w1_v3_ext'=w1_v3) & (w1_v4_ext'=w1_v4) & (RN1'=RN1+1);

// Process 1 receives or loses the message of each other process
[] s1=2 & token=N1 & RN1<=RN & (m_lost<(RN*N*(N-1))) -> (1-Q): (s1'=3) & (n1_nf2'=true) + Q: (s1'=3) & (n1_nf2'=false) & (m_lost'=m_lost+1);
[] s1=3 & token=N1 & RN1<=RN & (m_lost<(RN*N*(N-1))) -> (1-Q): (s1'=4) & (n1_nf3'=true) + Q: (s1'=4) & (n1_nf3'=false) & (m_lost'=m_lost+1);
[] s1=4 & token=N1 & RN1<=RN & (m_lost<(RN*N*(N-1))) -> (1-Q): (s1'=5) & (n1_nf4'=true) + Q: (s1'=5) & (n1_nf4'=false) & (m_lost'=m_lost+1);

// Not last round, Process 1 computes the messages of other processes: updates its value, views and confirmations;
[] s1=N+1 & token=N1 & RN1<RN -> 1: (s1'=1) & (v1'=v1_new) & (w1_v2'=w1_v2_new) & (w1_v3'=w1_v3_new) & (w1_v4'=w1_v4_new) & (w1_c2'=w1_c2_new) & (w1_c3'=w1_c3_new) &
       (w1_c4'=w1_c4_new) & (token'=next);

// Last round, Process 1 computes the messages of other processes: updates its confirmations and, only for OP, updates value and views;
[] s1=N+1 & token=N1 & RN1=RN -> 1: (s1'=N+2) & (w1_c2'=w1_c2_new) & (w1_c3'=w1_c3_new) & (w1_c4'=w1_c4_new) & (v1'=(DC=1)?v1_new:v1) & (w1_v2'=(DC=1)?w1_v2_new:w1_v2)
       & (w1_v3'=(DC=1)?w1_v3_new:w1_v3) & (w1_v4'=(DC=1)?w1_v4_new:w1_v4);

// Process 1 decides -> agree or abort
[] s1=N+2 & token=N1 -> 1: (s1'=0) & (token'=next) & (d1'= decision);

endmodule

module Process_2=Process_1 [N1=N2, s1=s2, v1=v2, d1=d2, RN1=RN2, n1_nf2=n2_nf3, n1_nf3=n2_nf4, n1_nf4=n2_nf1, w1_v2=w2_v3, w1_v3=w2_v4, w1_v4=w2_v1, w1_c2=w2_c3,
       w1_c3=w2_c4, w1_c4=w2_c1, v1_ext=v2_ext, v2_ext=v3_ext, v3_ext=v4_ext, v4_ext=v1_ext, w1_v2_ext=w2_v3_ext, w1_v3_ext=w2_v4_ext , w1_v4_ext=w2_v1_ext,
       w2_v1_ext=w3_v2_ext, w2_v3_ext=w3_v4_ext, w2_v4_ext=w3_v1_ext, w3_v1_ext=w4_v2_ext, w3_v2_ext=w4_v3_ext, w3_v4_ext=w4_v1_ext, w4_v1_ext=w1_v2_ext,
       w4_v2_ext=w1_v3_ext, w4_v3_ext=w1_v4_ext, v1_ini=v2_ini] endmodule
module Process_3=Process_1 [N1=N3, s1=s3, v1=v3, d1=d3, RN1=RN3, n1_nf2=n3_nf4, n1_nf3=n3_nf1, n1_nf4=n3_nf2, w1_v2=w3_v4, w1_v3=w3_v1, w1_v4=w3_v2, w1_c2=w3_c4,
       w1_c3=w3_c1, w1_c4=w3_c2, v1_ext=v3_ext, v2_ext=v4_ext, v3_ext=v1_ext, v4_ext=v2_ext, w1_v2_ext=w3_v4_ext, w1_v3_ext=w3_v1_ext , w1_v4_ext=w3_v2_ext,
       w2_v1_ext=w4_v3_ext, w2_v3_ext=w4_v1_ext, w2_v4_ext=w4_v2_ext, w3_v1_ext=w1_v3_ext, w3_v2_ext=w1_v4_ext, w3_v4_ext=w1_v2_ext, w4_v1_ext=w2_v3_ext,
       w4_v2_ext=w2_v4_ext, w4_v3_ext=w2_v1_ext, v1_ini=v3_ini] endmodule
module Process_4=Process_1 [N1=N4, s1=s4, v1=v4, d1=d4, RN1=RN4, n1_nf2=n4_nf1, n1_nf3=n4_nf2, n1_nf4=n4_nf3, w1_v2=w4_v1, w1_v3=w4_v2, w1_v4=w4_v3, w1_c2=w4_c1,
       w1_c3=w4_c2, w1_c4=w4_c3, v1_ext=v4_ext, v2_ext=v1_ext, v3_ext=v2_ext, v4_ext=v3_ext, w1_v2_ext=w4_v1_ext, w1_v3_ext=w4_v2_ext , w1_v4_ext=w4_v3_ext,
       w2_v1_ext=w1_v4_ext, w2_v3_ext=w1_v2_ext, w2_v4_ext=w1_v3_ext, w3_v1_ext=w2_v4_ext, w3_v2_ext=w2_v1_ext, w3_v4_ext=w2_v3_ext, w4_v1_ext=w3_v4_ext,
       w4_v2_ext=w3_v1_ext, w4_v3_ext=w3_v2_ext, v1_ini=v4_ini, not_last=last] endmodule
```

Listing 52. N>3 with Asymmetric Failures- part 2