Erik Palmgren and the Higher Infinite in Type Theory

Peter Dybjer

with reference to work in progress with
Marc Bezem, Thierry Coquand, and Martín Hötzel Escardó
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> *In [12, 13] Martin-Löf only considers an infinite tower of universes $U_0 \in U_1 \in \cdots \in U_n \in \cdots$ all of which are closed under the same ensemble of set forming operations. The next natural step was to implement a universe operator into type theory which takes a family of sets and constructs a universe above it. Such a universe operator was formalized by Palmgren while working on a domain-theoretic interpretation of the logical framework with an infinite sequence of universes (cf. [17]).*

Rathjen, Griffor, Palmgren, Inaccessibility in constructive set theory and type theory, Annals of Pure and Applied Logic (1998).

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E. Palmgren, On universes in type theory, in: G. Sambin and J. Smith (eds.) Twenty Five Years of Constructive Type Theory. Oxford Logic Guides, Oxford University Press 1998, 191-204.

- two external universe towers, with and without cumulativity
- a universe operator building a universe above an arbitrary family of sets; the "next universe" operator.
- super universe; a universe closed under the universe operator
- higher order universe operators, and the theories $\mathbf{ML}^n$

  "suggested by the author in October 1989".
- a discussion of Setzer's Mahlo universe, elimination rule, is it "impredicative"?

Formalization in Agda

We shall show how to formalize several constructions of universes à la Tarski:

- intensional Martin-Löf type theory with one universe U, T;
- Palmgren's universe operator $U\,A\,B, T\,A\,B$;
- Palmgren's super universe V, S;
- an internal tower of universes $U'\,n$ (a minimal super universe);
- internal universe polymorphism;
- full reflection (cumulativity) and a type-checking problem.

Erik's constructions initiated the exploration of the higher infinite in type theory

- Quantifier universe (Rathjen, Griffor, Palmgren)
- Mahlo universe (Setzer)
- Induction-recursion (Dybjer, Setzer)
- Setzer unpublished: beyond the schema for induction-recursion (autonomous Mahlo, Π_3-reflection). These constructions are still inductive-recursive, but go beyond the schema for induction-recursion.

The exploration of the higher infinite and the scope of constructive validity

Universe operators and the super universe

- extend the scope of type-theoretic constructivity into the higher infinite;
- provide crucial motivating cases for induction-recursion;
- provide challenging cases for foundations; why are higher universes constructively valid?
- provide food for thought concerning the meaning of Martin-Löf's meaning explanations.

Martin-Löf's logical framework (1986)

The logical framework is the core theory you have before defining any data types ("sets")

- It has a type of Set of "sets" or "small types".
- It has built in dependent function types $\Gamma \vdash (x : A) \rightarrow B$ (in Agda style notation) provided $\Gamma \vdash A$ and $\Gamma, x : A \vdash B$ are types.

Then you define your own data types (inductively or inductive-recursively) inside Set. Examples are $\Pi A\, B$: Set and U : Set, $T : U \rightarrow$ Set.

Agda's logical framework

The logical framework is the core theory you have before defining any data types ("sets")

- It has a sequence of universes á la Russell

$$\mathsf{Set}_0 : \mathsf{Set}_1 : \mathsf{Set}_2 : \cdots$$

  and a special "kind" Setω.
- It has a special type Level of universe levels enabling *universe polymorphic* definition by quantification over it.
- There are level operations: $0, l^+, l \sqcup m$ subject to equations making it a $\sqcup$-semilattice with $l \sqcup l^+ = l^+, (l \sqcup m)^+ = (l^+ \sqcup m^+)$, and $l \sqcup 0 = l$.
- It has dependent function types written $\Gamma \vdash (x : A) \to B : \mathsf{Set}_{l \sqcup m}$ for $\Gamma \vdash A : \mathsf{Set}_l$ and $\Gamma, x : A \vdash B : \mathsf{Set}_m$

Why extend Martin-Löf's logical framework?

Escardó's library of univalent mathematics in Agda

Martin Escardó 2019: "Introduction to Univalent Foundations of Mathematics with Agda".

- Agda's approach to universe polymorphism works nicely.
- Discussion between Bezem, Coquand, Escardó and myself about the proper type-theoretic system for such universe polymorphism. (Starting at CAS in Oslo, 2019)

Universes for univalent mathematics?

- Numerous questions about the exact formulation of rules for universes arise.
  - Tarski vs Russell?
  - Open vs closed (inductive-recursive)?
  - Cumulativity or not?
  - Limiting the proof-theoretic strength?
  - Algebraic structure of universe levels?
- Ideas:
  - a theory with level-judgements, level variables;
  - a theory with level-constraints (similar to "A universe polymorphic type system" by Voevodsky 2012) and a constraint solving algorithm;
  - cwfs with universe tower structures (generalized algebraic theories).