Categories with Families

Unityped, Simply Typed, Dependently Typed

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Lambek and Scott:

A quotation from their book *Introduction to higher order categorical logic*, 1986:

> *We also claim that intuitionistic type theories and toposes are closely related, in as much as there is a pair of adjoint functors between their respective categories. This is worked out out in Part II. The relationship between Martin-Löf type theories and locally cartesian closed categories was established too recently (by Robert Seely) to be treated here.*

Categories and dependent type theory

P. Clairambault, PD, TLCA 2011, MSCS 2014:

- Two biequivalences of 2-categories

$$\mathbf{FL} \underset{\mathbf{C}}{\overset{\mathbf{L}}{\rightleftarrows}} \mathbf{Cwf}_{\mathrm{dem}}^{\mathbb{N}_1,\Sigma,\mathrm{I}_{\mathrm{ext}}}$$

$$\mathbf{LCC} \underset{\mathbf{C}}{\overset{\mathbf{L}}{\rightleftarrows}} \mathbf{Cwf}_{\mathrm{dem}}^{\mathbb{N}_1,\Sigma,\Pi,\mathrm{I}_{\mathrm{ext}}}$$

Democracy means there exists type $\overline{\Gamma}$ and $\mathrm{d}_\Gamma : \Gamma \cong 1.\overline{\Gamma}$.

S. Castellan, P. Clairambault, PD, TLCA 2015, LMCS 2017:

- Equality of arrows in the bifree $\mathbf{LCC}$ is undecidable.
- (Construction of free cwfs with extra structure and bifree lcccs.)

Lambek and Scott

1. Categorical preliminaries
2. Simply typed λ-calculus and cccs
3. Untyped λ-calculus and C-monoids
4. Intuitionistic type theory and toposes

How to rewrite Lambek and Scott?

Instead use cwf as central notion on the *syntactic* side:

1. Categorical preliminaries (add lcccs, bicategories, ...)
2. Unityped cwfs (ucwfs) with λ-structure
   instead of untyped λ-calculus
3. Simply typed cwfs (scwfs) supporting $\rightarrow, \times, \ldots$
   instead of simply typed λ-calculus
4. Cwfs supporting $\Pi, \Sigma, \mathrm{I}, \ldots$
   instead of Martin-Löf type theory

(See arXiv:1904.00827 [cs.LO], 44 pages.)

Simply typed and unityped cwfs

Cwfs: has presheaf of types

Scwfs: has set of types (presheaf is constant)

Ucwfs: has one (constant) type

Context comprehension simplifies for scwfs and ucwfs.

Ucwfs

Three equivalences of 1-categories (structure strictly preserved)

- with cartesian operads:

$$\mathbf{CartOperad} \rightleftarrows \mathbf{Ucwf}_{\mathrm{ctx}}$$

- with Lawvere theories:

$$\mathbf{LawTh} \rightleftarrows \mathbf{Ucwf}_{\mathrm{ctx}}$$

- with Lawvere theories of type $\lambda\beta\eta$ (Obtułowicz 1977)

$$\mathbf{LawTh}^{\lambda,\beta,\eta} \rightleftarrows \mathbf{Ucwf}_{\mathrm{ctx}}^{\lambda,\beta,\eta}$$

In contextual ucwfs the objects of $\mathcal{C} \cong \mathbb{N}$.

Two Lambek-Scott-style equivalences of categories

- Two equivalences of 1-categories (structure strictly preserved)

$$\mathbf{CCs} \underset{\mathbf{C}}{\overset{\mathbf{L}}{\rightleftarrows}} \mathbf{Scwf}^{\mathbb{N}_1,\times}_{\mathrm{ctx}}$$

$$\mathbf{CCCs} \underset{\mathbf{C}}{\overset{\mathbf{L}}{\rightleftarrows}} \mathbf{Scwf}^{\mathbb{N}_1,\times,\to}_{\mathrm{ctx}}$$

- Products and exponentials in $\mathbf{CCs}$ and $\mathbf{CCCs}$ are given as structure which is preserved *strictly*.
- Scwfs are *contextual*, that is, contexts are lists of types.

Two biequivalences of 2-categories

$$\mathbf{CC} \underset{\mathbf{C}}{\overset{\mathbf{L}}{\rightleftarrows}} \mathbf{Scwf}_{\mathrm{dem}}$$

$$\mathbf{CCC} \underset{\mathbf{C}}{\overset{\mathbf{L}}{\rightleftarrows}} \mathbf{Scwf}_{\mathrm{dem}}^{\mathbb{N}_1,\times,\to}$$

- Products and exponentials in $\mathbf{CC}$s and $\mathbf{CCC}$s are given as property which is preserved up to isomorphism.
- Pseudo scwf-morphisms preserve structure up to isomorphism.
- Scwfs are *democratic*, that is, context Γ is *represented* as type $\overline{\Gamma}$, such that $\Gamma \cong 1.\overline{\Gamma}$.

Initial ucwfs and scwfs with extra structure

For example: the untyped $\lambda\beta\eta$-calculus defined as the initial object in $\mathbf{Ucwf}^{\lambda,\beta,\eta}$. Two instances:

- Turn the rules for $\lambda\beta\eta$-ucwfs into an inductive family: yields well-scoped, name-free version of the untyped $\lambda\sigma$-calculus.
- Construct the $\lambda\beta\eta$-ucwf of (one of the) "usual" definitions of the λ-calculus.

An "abstract syntax perspective" of logical systems.

Contextuality

A cwf is **contextual** iff there is a length function

$$l : \mathcal{C}_0 \to \mathbb{N}$$

such that

- $l(\Gamma) = 0$ iff $\Gamma = 1$ and
- $l(\Gamma) = n + 1$ iff there are unique Δ and A such that $\Gamma = \Delta.A$ and $l(\Delta) = n$.

Cf Cartmell's 1978 **contextual categories**. Note that unlike the other parts of the definition of cwfs

- it does not correspond to an inference rule of dependent type theory;
- it is not expressed in the language of generalized algebraic theories;
- however, free cwfs are contextual.

Democracy

A cwf is **democratic** provided each context Γ is represented by a type $\overline{\Gamma}$ in the sense that there is an isomorphism

$$\mathrm{d}_\Gamma : \Gamma \cong 1.\overline{\Gamma}$$

- it does not correspond to an inference rule of dependent type theory;
- however, the free cwf is democratic;
- democracy can actually be expressed in the language of generalized algebraic theories.