

Categories with Families

Untyped, Simply Typed, Dependently Typed

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Lambek and Scott:

A quotation from their book *Introduction to higher order categorical logic*, 1986:

We also claim that intuitionistic type theories and toposes are closely related, in as much as there is a pair of adjoint functors between their respective categories. This is worked out out in Part II. The relationship between Martin-Löf type theories and locally cartesian closed categories was established too recently (by Robert Seely) to be treated here.

Categories and dependent type theory

P. Clairambault, PD, TLCA 2011, MSCS 2014:

- Two biequivalences of 2-categories

$$\mathbf{FL} \begin{array}{c} \xrightarrow{\mathbf{L}} \\ \xleftarrow{\mathbf{C}} \end{array} \mathbf{Cwf}_{\text{dem}}^{\mathbb{N}_1, \Sigma, \text{Iext}}$$

$$\mathbf{LCC} \begin{array}{c} \xrightarrow{\mathbf{L}} \\ \xleftarrow{\mathbf{C}} \end{array} \mathbf{Cwf}_{\text{dem}}^{\mathbb{N}_1, \Sigma, \Pi, \text{Iext}}$$

Democracy means there exists type $\bar{\Gamma}$ and $d_{\Gamma} : \Gamma \cong 1.\bar{\Gamma}$.

S. Castellan, P. Clairambault, PD, TLCA 2015, LMCS 2017:

- Equality of arrows in the bifree **LCC** is undecidable.
- (Construction of free cwfs with extra structure and bifree lcccs.)

Lambek and Scott

- 1 Categorical preliminaries
- 2 Simply typed λ -calculus and cccs
- 3 Untyped λ -calculus and C-monoids
- 4 Intuitionistic type theory and toposes

How to rewrite Lambek and Scott?

Instead use cwf as central notion on the *syntactic* side:

- 1 Categorical preliminaries (add lcccs, bicategories, ...)
- 2 Untyped cwfs (ucwfs) with λ -structure instead of untyped λ -calculus
- 3 Simply typed cwfs (scwfs) supporting $\rightarrow, \times, \dots$ instead of simply typed λ -calculus
- 4 Cwfs supporting Π, Σ, I, \dots instead of Martin-Löf type theory

(See arXiv:1904.00827 [cs.LO], 44 pages.)

Simply typed and untyped cwfs

Cwfs: has presheaf of types

Scwfs: has set of types (presheaf is constant)

Ucwfs: has one (constant) type

Context comprehension simplifies for scwfs and ucwfs.

Ucwfs

Three equivalences of 1-categories (structure strictly preserved)

- with cartesian operads:



- with Lawvere theories:



- with Lawvere theories of type $\lambda\beta\eta$ (Obtuloicz 1977)



In contextual ucwfs the objects of $\mathcal{C} \cong \mathbb{N}$.

Two Lambek-Scott-style equivalences of categories

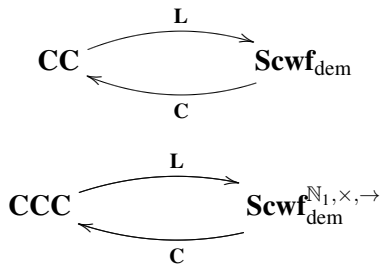
- Two equivalences of 1-categories (structure strictly preserved)

$$\begin{array}{ccc} & \text{L} & \\ \text{CCs} & \xrightarrow{\quad} & \mathbf{Scwf}_{\text{ctx}}^{\mathbb{N}_1, \times} \\ & \xleftarrow{\quad} & \\ & \text{C} & \end{array}$$

$$\begin{array}{ccc} & \text{L} & \\ \text{CCCs} & \xrightarrow{\quad} & \mathbf{Scwf}_{\text{ctx}}^{\mathbb{N}_1, \times, \rightarrow} \\ & \xleftarrow{\quad} & \\ & \text{C} & \end{array}$$

- Products and exponentials in **CCs** and **CCCs** are given as structure which is preserved *strictly*.
- **Scwfs** are *contextual*, that is, contexts are lists of types.

Two biequivalences of 2-categories



- Products and exponentials in **CCs** and **CCCs** are given as property which is preserved up to isomorphism.
- Pseudo scwf-morphisms preserve structure up to isomorphism.
- Scwfs are *democratic*, that is, context Γ is *represented* as type $\bar{\Gamma}$, such that $\Gamma \cong 1.\bar{\Gamma}$.

Initial ucwfs and scwfs with extra structure

For example: the untyped $\lambda\beta\eta$ -calculus defined as the initial object in $\mathbf{Ucwf}^{\lambda,\beta,\eta}$. Two instances:

- Turn the rules for $\lambda\beta\eta$ -ucwfs into an inductive family: yields well-scoped, name-free version of the untyped $\lambda\sigma$ -calculus.
- Construct the $\lambda\beta\eta$ -ucwf of (one of the) "usual" definitions of the λ -calculus.

An "abstract syntax perspective" of logical systems.

Contextuality

A cwf is **contextual** iff there is a length function

$$l : \mathcal{C}_0 \rightarrow \mathbb{N}$$

such that

- $l(\Gamma) = 0$ iff $\Gamma = 1$ and
- $l(\Gamma) = n + 1$ iff there are unique Δ and A such that $\Gamma = \Delta.A$ and $l(\Delta) = n$.

Cf Cartmell's 1978 **contextual categories**. Note that unlike the other parts of the definition of cwfs

- it does not correspond to an inference rule of dependent type theory;
- it is not expressed in the language of generalized algebraic theories;
- however, free cwfs are contextual.

Democracy

A cwf is **democratic** provided each context Γ is represented by a type $\bar{\Gamma}$ in the sense that there is an isomorphism

$$d_{\Gamma} : \Gamma \cong 1.\bar{\Gamma}$$

- it does not correspond to an inference rule of dependent type theory;
- however, the free cwf is democratic;
- democracy can actually be expressed in the language of generalized algebraic theories.