Under consideration for publication in Math. Struct. in Comp. Science

The Biequivalence of Locally Cartesian Closed Categories and Martin-Löf Type Theories

PIERRE CLAIRAMBAULT[1†] and PETER DYBJER[2]

[1] *Computer Laboratory, University of Cambridge*
[2] *Department of Computer Science and Engineering, Chalmers University of Technology*

Received May 5, 2012

Seely's paper *Locally cartesian closed categories and type theory* contains a well-known result in categorical type theory: that the category of locally cartesian closed categories is equivalent to the category of Martin-Löf type theories with Π, Σ, and extensional identity types. However, Seely's proof relies on the problematic assumption that substitution in types can be interpreted by pullbacks. Here we prove a corrected version of Seely's theorem: that the Bénabou-Hofmann interpretation of Martin-Löf type theory in locally cartesian closed categories yields a biequivalence of 2-categories. To facilitate the technical development we employ categories with families as a substitute for syntactic Martin-Löf type theories. As a second result we prove that if we remove Π-types the resulting categories with families are biequivalent to left exact categories. (A short version of the present paper appears in the Proceedings of Typed Lambda Calculus and Applications, Novi Sad, Serbia, 1-3 June 2011.)

1. Introduction

It is "well-known" that locally cartesian closed categories (lcccs) are equivalent to Martin-Löf's intuitionistic type theory (Martin-Löf, 1982; Martin-Löf, 1984). But how *known* is it really? Seely's original proof (Seely, 1984) contains a flaw, and the papers by Curien (Curien, 1993) and Hofmann (Hofmann, 1994) who address this flaw only show that Martin-Löf type theory can be interpreted in locally cartesian closed categories, but not that this interpretation is an equivalence of categories provided the type theory has Π, Σ, and extensional identity types. Here we complete the work and fully rectify Seely's result except that we do not prove an equivalence of categories but a *biequivalence* of 2-categories. In fact, a significant part of the endeavour has been to find an appropriate formulation of the result, and in particular to find a suitable notion analogous to Seely's "interpretation of Martin-Löf theories".

† We would also like to acknowledge the support of the (UK) EPSRC grant RC-CM1025 and the ERC Advanced Grant ECSYM for the first author and of the (Swedish) Vetenskapsrådet grant "Types for Proofs and Programs" for the second author.

1.1. *Categories with families and democracy.*

Seely turns a given Martin-Löf theory into a category where the objects are *closed* types and the morphisms from type A to type B are functions of type $A \to B$. Such categories are the objects of Seely's "category of Martin-Löf theories".

Instead of syntactic Martin-Löf theories we shall employ *categories with families (cwfs)* (Dybjer, 1996). A cwf is a pair $(\mathbb{C}, T)$ where $\mathbb{C}$ is the category of contexts and explicit substitutions, and $T : \mathbb{C}^{op} \to \mathbf{Fam}$ is a functor, where $T(\Gamma)$ represents the family of sets of terms indexed by types in context Γ and $T(\gamma)$ performs the substitution of γ in types and terms. Cwf is an appropriate substitute for syntax for dependent types: its definition unfolds to a variable-free calculus of explicit substitutions (Dybjer, 1996), which is like Martin-Löf's (Martin-Löf, 1992; Tasistro, 1993) except that variables are encoded by projections. One advantage of this approach compared to Seely's is that we get a natural definition of morphism of cwfs, which preserves the structure of cwfs up to isomorphism. In contrast Seely's notion of "interpretation of Martin-Löf theories" is defined indirectly via the construction of an lccc associated with a Martin-Löf theory, and basically amounts to a functor preserving structure between the corresponding lcccs, rather than directly as something which preserves all the "structure" of Martin-Löf theories.

To prove our biequivalences we require that our cwfs are *democratic*. This means that each context is *represented* by a type. Our results require us to build local cartesian closed structure in the category of contexts. To this end we use available constructions on types and terms, and by democracy such constructions can be moved back and forth between types and contexts. Since Seely works with closed types only he has no need for democracy.

1.2. *The coherence problem.*

Seely interprets type substitution in Martin-Löf theories as pullbacks in lcccs. The interpretation relies on the idea that type substitution is interpreted by a choice of pullbacks in the lccc. However, type substitution is also defined syntactically by induction on the type structure, and thus fixed by the interpretation, and there is no reason why the result of this syntactic operation should coincide strictly with the previous choice of pullbacks.

In the paper *Substitution up to isomorphism* (Curien, 1993) Curien describes the fundamental nature of this problem. He sets out

> ... to solve a difficulty arising from a mismatch between syntax and semantics: in locally cartesian closed categories, substitution is modelled by pullbacks (more generally pseudo-functors), that is, only up to isomorphism, unless split fibrational hypotheses are imposed. ... but not all semantics do satisfy them, and in particular not the general description of the interpretation in an arbitrary locally cartesian closed category. In the general case, we have to show that the isomorphisms between types arising from substitution are *coherent* in a sense familiar to category theorists.

To solve the problem Curien introduces a calculus with explicit substitutions for Martin-Löf type theory, with special terms witnessing applications of the type equality rule. In this calculus type equality can be interpreted as isomorphism in lcccs. The remaining coherence problem is to show that Curien's calculus is equivalent to the usual formulation of Martin-Löf type theory, and Curien proves this result by cut-elimination.

Somewhat later, Hofmann (Hofmann, 1994) gave an alternative solution based on a technique which had been used by Bénabou (Bénabou, 1985) for constructing a *split* fibration from an arbitrary fibration. In this way Hofmann constructed a model of Martin-Löf type theory with Π-types, Σ-types, and (extensional) identity types from a locally cartesian closed category. Hofmann used categories with attributes (cwa) in the sense of Cartmell (Cartmell, 1986) as his notion of model. In fact, cwas and cwfs are closely related: the notion of cwf arises by reformulating the axioms of cwas to make the connection with the usual syntax of dependent type theory more transparent. Both cwas and cwfs are split notions of model of Martin-Löf type theory, hence the relevance of Bénabou's construction.

However, Seely wanted to prove an equivalence of categories. Hofmann conjectured (Hofmann, 1994):

> We have now constructed a cwa over $\mathcal{C}$ which can be shown to be equivalent to $\mathcal{C}$ in some suitable 2-categorical sense.

Here we spell out and prove this result, and thus fully rectify Seely's theorem. It should be apparent from what follows that this is not a trivial exercise. In our setting the result is a biequivalence analogous to Bénabou's (much simpler) result: that the 2-category of fibrations (with non-strict morphisms) is biequivalent to the 2-category of split fibrations (with non-strict morphisms).

While carrying out the proof we noticed that if we remove Π-types the resulting 2-category of cwfs is biequivalent to the 2-category of left exact (or finitely complete) categories. We present this result in parallel with the main result.

1.3. *Plan of the paper.*

An equivalence of categories consists of a pair of functors which are inverses up to natural isomorphism. Biequivalence is the appropriate notion of equivalence for bicategories (Leinster, 1999). Instead of functors we have *pseudofunctors* which only preserve identity and composition up to isomorphism. Instead of natural isomorphisms we have *pseudonatural transformations* which are inverses up to *invertible modification.*

A 2-category is a strict bicategory, and the remainder of the paper consists of constructing two biequivalences of 2-categories. In Section 2 we introduce cwfs and show how to turn a cwf into an indexed category. In Section 3 we define the 2-categories $\mathbf{CwF}^{\mathrm{I_{ext}}\Sigma}_{\mathrm{dem}}$ of democratic cwfs which support extensional identity types and Σ-types and $\mathbf{CwF}^{\mathrm{I_{ext}}\Sigma\Pi}_{\mathrm{dem}}$ which also support Π-types. We also define the notions of pseudo cwf-morphism and pseudo cwf-transformation. In Section 4 we define the 2-categories **FL** of left exact categories and **LCC** of locally cartesian closed categories. We show that there are forgetful 2-functors $U : \mathbf{CwF}^{\mathrm{I_{ext}}\Sigma}_{\mathrm{dem}} \to \mathbf{FL}$ and $U : \mathbf{CwF}^{\mathrm{I_{ext}}\Sigma\Pi}_{\mathrm{dem}} \to \mathbf{LCC}$. In section 5 we construct the pseudofunctors $H : \mathbf{FL} \to \mathbf{CwF}^{\mathrm{I_{ext}}\Sigma}_{\mathrm{dem}}$ and $H : \mathbf{LCC} \to \mathbf{CwF}^{\mathrm{I_{ext}}\Sigma\Pi}_{\mathrm{dem}}$ based on the Bénabou-Hofmann construction. In section 6 we prove that H and U give rise to the biequivalences of **FL** and $\mathbf{CwF}^{\mathrm{I_{ext}}\Sigma}_{\mathrm{dem}}$ and of **LCC** and $\mathbf{CwF}^{\mathrm{I_{ext}}\Sigma\Pi}_{\mathrm{dem}}$.

2. Categories with Families

2.1. *Definition*

Definition 1. Let **Fam** be the category of families of sets defined as follows. An object is a pair (A, B) where A is a set and $B(x)$ is a family of sets indexed by $x \in A$. A morphism with source (A, B) and target (A', B') is a pair consisting of a function $f : A \to A'$ and a family of functions $g(x) : B(x) \to B'(f(x))$ indexed by $x \in A$.

Note that **Fam** is equivalent to the arrow category $\mathbf{Set}^{\to}$.

Definition 2. A **category with families (cwf)** consists of the following data:

— A base category $\mathbb{C}$. Its objects represent *contexts* and its morphisms represent *substitutions*. The identity map is denoted by $\mathrm{id} : \Gamma \to \Gamma$ and the composition of maps $\gamma : \Delta \to \Gamma$ and $\delta : \Xi \to \Delta : \Xi \to \Gamma$ is denoted by $\gamma \circ \delta$ or more briefly by $\gamma\delta : \Xi \to \Gamma$.

— A functor $T : \mathbb{C}^{op} \to \mathbf{Fam}$. $T(\Gamma)$ is a pair, where the first component represents the set $\mathrm{Type}(\Gamma)$ of *types* in context Γ, and the second component represents the type-indexed family $(\Gamma \vdash A)_{A \in \mathrm{Type}(\Gamma)}$ of sets of *terms* in context Γ. We write $a : \Gamma \vdash A$ for a term $a \in \Gamma \vdash A$. Moreover, if γ is a morphism in $\mathbb{C}$, then $T(\gamma)$ is a pair consisting of the *type substitution* function $A \mapsto A[\gamma]$ and the type-indexed family of *term substitution* functions $a \mapsto a[\gamma]$.

— A *terminal object* $[]$ of $\mathbb{C}$ which represents the *empty context* and a terminal map $\langle\rangle : \Delta \to []$ which represents the *empty substitution*.

— A *context comprehension* which to an object Γ in $\mathbb{C}$ and a type $A \in \mathrm{Type}(\Gamma)$ associates an object $\Gamma \cdot A$ of $\mathbb{C}$, a morphism $\mathrm{p}_A : \Gamma \cdot A \to \Gamma$ of $\mathbb{C}$ and a term $\mathrm{q} \in \Gamma \cdot A \vdash A[\mathrm{p}]$ such the following universal property holds: for each object Δ in $\mathbb{C}$, morphism $\gamma : \Delta \to \Gamma$, and term $a \in \Delta \vdash A[\gamma]$, there is a unique morphism $\theta = \langle \gamma, a \rangle : \Delta \to \Gamma \cdot A$, such that $\mathrm{p}_A \circ \theta = \gamma$ and $\mathrm{q}[\theta] = a$. (We remark that a related notion of comprehension for hyperdoctrines was introduced by Lawvere (Lawvere, 1970).)

The definition of cwf can be presented as a system of axioms and inference rules for a variable-free generalized algebraic formulation of the most basic rules of dependent type theory (Dybjer, 1996). The correspondence with standard syntax is explained by Hofmann (Hofmann, 1996) and the equivalence is proved in detail by Mimram (Mimram, 2004). The easiest way to understand this correspondence might be as a translation between the standard lambda calculus based syntax of dependent type theory and the language of cwf-combinators. In one direction the key idea is to translate a variable (de Bruijn number) to a projection of the form $\mathrm{q}[\mathrm{p}^n]$. In the converse direction, recall that the cwf-combinators yield a calculus of explicit substitutions whereas substitution is a meta-operation in usual lambda calculus. When we translate cwf-combinators to lambda terms, we execute the explicit substitutions, using the equations for substitution in types and terms as rewrite rules. The equivalence proof is similar to the proof of the equivalence of cartesian closed categories and the simply typed lambda calculus.

2.2. *Categories with Families with Extra Structure*

We shall now define what it means that a cwf supports extra structure corresponding to the rules for the various type formers of Martin-Löf type theory.

Definition 3. A cwf *supports (extensional) identity types* provided the following conditions hold:
Form. If $A \in \mathrm{Type}(\Gamma)$ and $a, a' : \Gamma \vdash A$, there is $\mathrm{I}_A(a, a') \in \mathrm{Type}(\Gamma)$;
Intro. If $a : \Gamma \vdash A$, there is $\mathrm{r}_{A,a} : \Gamma \vdash \mathrm{I}_A(a, a)$;
Elim. If $c : \Gamma \vdash \mathrm{I}_A(a, a')$ then $a = a'$ and $c = \mathrm{r}_{A,a}$.
Moreover, we have stability under substitution: if $\delta : \Delta \to \Gamma$ then

$$\begin{aligned} \mathrm{I}_A(a, a')[\delta] &= \mathrm{I}_{A[\delta]}(a[\delta], a'[\delta]) \\ \mathrm{r}_{A,a}[\delta] &= \mathrm{r}_{A[\delta],a[\delta]} \end{aligned}$$

Definition 4. A cwf *supports* Σ*-types* iff the following conditions hold:
Form. If $A \in \mathrm{Type}(\Gamma)$ and $B \in \mathrm{Type}(\Gamma \cdot A)$, there is $\Sigma(A, B) \in \mathrm{Type}(\Gamma)$,
Intro. If $a : \Gamma \vdash A$ and $b : \Gamma \vdash B[\langle \mathrm{id}, a\rangle]$, there is $\mathrm{pair}(a, b) : \Gamma \vdash \Sigma(A, B)$,
Elim. If $a : \Gamma \vdash \Sigma(A, B)$, there are $\pi_1(a) : \Gamma \vdash A$ and $\pi_2(a) : \Gamma \vdash B[\langle \mathrm{id}, \pi_1(a)\rangle]$ such that

$$\begin{aligned} \pi_1(\mathrm{pair}(a, b)) &= a \\ \pi_2(\mathrm{pair}(a, b)) &= b \\ \mathrm{pair}(\pi_1(c), \pi_2(c)) &= c \end{aligned}$$

Moreover, we have stability under substitution:

$$\begin{aligned} \Sigma(A, B)[\delta] &= \Sigma(A[\delta], B[\langle \delta \circ \mathrm{p}, \mathrm{q}\rangle]) \\ \mathrm{pair}(a, b)[\delta] &= \mathrm{pair}(a[\delta], b[\delta]) \\ \pi_1(c)[\delta] &= \pi_1(c[\delta]) \\ \pi_2(c)[\delta] &= \pi_2(c[\delta]) \end{aligned}$$

Note that in a cwf which supports extensional identity types and Σ-types surjective pairing, $\mathrm{pair}(\pi_1(c), \pi_2(c)) = c$, follows from the other conditions (Martin-Löf, 1984).

Definition 5. A cwf *supports* Π*-types* iff the following conditions hold:
Form. If $A \in \mathrm{Type}(\Gamma)$ and $B \in \mathrm{Type}(\Gamma \cdot A)$, there is $\Pi(A, B) \in \mathrm{Type}(\Gamma)$.
Intro. If $b : \Gamma \cdot A \vdash B$, there is $\lambda(b) : \Gamma \vdash \Pi(A, B)$.
Elim. If $c : \Gamma \vdash \Pi(A, B)$ and $a : \Gamma \vdash A$ then there is a term $\mathrm{ap}(c, a) : \Gamma \vdash B[\langle \mathrm{id}, a\rangle]$ such that

$$\begin{aligned} \mathrm{ap}(\lambda(b), a) &= b[\langle \mathrm{id}, a\rangle] : \Gamma \vdash B[\langle \mathrm{id}, a\rangle] \\ c &= \lambda(\mathrm{ap}(c[\mathrm{p}], \mathrm{q})) : \Gamma \vdash \Pi(A, B) \end{aligned}$$

Moreover, we have stability under substitution:

$$\begin{aligned} \Pi(A, B)[\gamma] &= \Pi(A[\gamma], B[\langle \gamma \circ \mathrm{p}, \mathrm{q}\rangle]) \\ \lambda(b)[\gamma] &= \lambda(b[\langle \gamma \circ \mathrm{p}, \mathrm{q}\rangle]) \\ \mathrm{ap}(c, a)[\gamma] &= \mathrm{ap}(c[\gamma], a[\gamma]) \end{aligned}$$

Definition 6. A cwf $(\mathbb{C}, T)$ is *democratic* iff for each object Γ of $\mathbb{C}$ there is $\overline{\Gamma} \in \mathrm{Type}([])$ and an isomorphism $\Gamma \cong_{\gamma_\Gamma} [] \cdot \overline{\Gamma}$. Each substitution $\delta : \Delta \to \Gamma$ can then be represented by the term $\overline{\delta} = q[\gamma_\Gamma \delta \gamma_\Delta^{-1}] : [] \cdot \overline{\Delta} \vdash \overline{\Gamma}[p]$.

Democracy does not correspond to a rule of Martin-Löf type theory. However, a cwf generated inductively by the standard rules of Martin-Löf type theory with a one element type N_1 and Σ-types is democratic, since we can associate N_1 to the empty context and the closed type $\Sigma x_1 : A_1. \cdots .\Sigma x_n : A_n$ to a context $x_1 : A_1, \ldots, x_n : A_n$ by induction on n.

2.3. *The Indexed Category of Types in Context*

We shall now define the indexed category associated with a cwf. This will play a crucial role and in particular introduce the notion of *isomorphism* of types.

Proposition 1 (The Context-Indexed Category of Types). If $(\mathbb{C}, T)$ is a cwf, then we can define a functor $\boldsymbol{T} : \mathbb{C}^{op} \to \mathbf{Cat}$ as follows:

— The objects of $\boldsymbol{T}(\Gamma)$ are types in $\mathrm{Type}(\Gamma)$. If $A, B \in \mathrm{Type}(\Gamma)$, then a morphism in $\boldsymbol{T}(\Gamma)(A, B)$ is a morphism $\delta : \Gamma \cdot A \to \Gamma \cdot B$ in $\mathbb{C}$ such that $\mathrm{p}\delta = \mathrm{p}$.
— If $\gamma : \Delta \to \Gamma$ in $\mathbb{C}$, then $\boldsymbol{T}(\gamma) : \mathrm{Type}(\Gamma) \to \mathrm{Type}(\Delta)$ maps an object $A \in \mathrm{Type}(\Gamma)$ to $A[\gamma]$ and a morphism $\delta : \Gamma \cdot A \to \Gamma \cdot B$ to $\langle \mathrm{p}, \mathrm{q}[\delta \langle \gamma \circ \mathrm{p}, \mathrm{q} \rangle] \rangle : \Delta \cdot A[\gamma] \to \Delta \cdot B[\gamma]$.

We write $A \cong_\theta B$ if $\theta : A \to B$ is an isomorphism in $\boldsymbol{T}(\Gamma)$. If $a : \Gamma \vdash A$, we write $\{\theta\}(a) = q[\theta\langle id, a\rangle] : \Gamma \vdash B$ for the *coercion* of a to type B and $a =_\theta b$ if $a = \{\theta\}(b)$. Coercions compose, as we can check by the following direct calculation.

$$
\begin{aligned}
\{\theta_2\}(\{\theta_1\}(a)) &= \mathrm{q}[\theta_2\langle \mathrm{id}, \mathrm{q}[\theta_1\langle \mathrm{id}, a\rangle]\rangle] \\
&= \mathrm{q}[\theta_2\langle \mathrm{p}\theta_1\langle \mathrm{id}, a\rangle, \mathrm{q}[\theta_2\langle \mathrm{id}, a\rangle]\rangle] \\
&= \mathrm{q}[\theta_2\langle \mathrm{p}, \mathrm{q}\rangle\theta_1\langle \mathrm{id}, a\rangle] \\
&= \mathrm{q}[\theta_2\theta_1\langle \mathrm{id}, a\rangle] \\
&= \{\theta_2\theta_1\}(a)
\end{aligned}
$$

where we use the definition of coercions and manipulation of cwf combinators,

Seely's category **ML** of Martin-Löf theories (Seely, 1984) is essentially the category of categories $\boldsymbol{T}([])$ of closed types. We have the following alternative formulation of democracy:

Proposition 2. $(\mathbb{C}, T)$ is democratic iff the functor from $\boldsymbol{T}([])$ to $\mathbb{C}$, which maps a closed type A to the context $[] \cdot A$, is an equivalence of categories.

2.4. *Fibres, slices and lcccs.*

Seely's interpretation of type theory in lcccs relies on the idea that a type $A \in \mathrm{Type}(\Gamma)$ can be interpreted as its *display map*, that is, a morphism with codomain Γ. For instance, the type $\texttt{list}(n)$ of lists of length $n : \texttt{nat}$ is interpreted as the function $l : \texttt{list} \to \texttt{nat}$ which to each list associates its length. Hence, types and terms in context Γ are interpreted in the *slice category* $\mathbb{C}/\Gamma$, since terms are interpreted as global sections. Syntactic types are connected with types-as-display-maps by the following result, an analogue of which was one of the cornerstones of Seely's paper.

Proposition 3. If $(\mathbb{C}, T)$ is democratic and supports extensional identity and Σ-types, then $\boldsymbol{T}(\Gamma)$ and $\mathbb{C}/\Gamma$ are equivalent categories for all Γ.

Proof. To each object (type) A in $\boldsymbol{T}(\Gamma)$ we associate the object p_A in $\mathbb{C}/\Gamma$. A morphism from A to B in $\boldsymbol{T}(\Gamma)$ is by definition a morphism from p_A to p_B in $\mathbb{C}/\Gamma$.

Conversely, to each object $\delta : \Delta \to \Gamma$ of $\mathbb{C}/\Gamma$ we associate a type in $\mathrm{Type}(\Gamma)$. This is the inverse image $x : \Gamma \vdash \mathrm{Inv}(\delta)(x)$ which is defined type-theoretically by

$$\mathrm{Inv}(\delta)(x) = \Sigma y : \overline{\Delta}.\mathrm{I}_{\overline{\Gamma}}(\overline{x}, \overline{\delta}(y))$$

written in ordinary notation. In cwf combinator notation it becomes

$$\mathrm{Inv}(\delta) = \Sigma(\overline{\Delta}[\langle\rangle], \mathrm{I}_{\overline{\Gamma}[\langle\rangle]}(\mathrm{q}[\gamma_\Gamma \mathrm{p}], \overline{\delta}[\langle\langle\rangle, \mathrm{q}\rangle]) \in \mathrm{Type}(\Gamma)$$

These associations yield an equivalence of categories since $\mathrm{p}_{\mathrm{Inv}(\delta)}$ and δ are isomorphic in $\mathbb{C}/\Gamma$:

$$\begin{tikzcd}
\Gamma\cdot\mathrm{Inv}(\delta) \arrow[rr, bend left, "\alpha_\delta"] \arrow[dr, swap, "\mathrm{p}_{\mathrm{Inv}(\delta)}"] & & \Delta \arrow[ll, bend left, "\alpha_\delta^{-1}"] \arrow[dl, "\delta"] \\
& \Gamma &
\end{tikzcd}$$

The isomorphism is defined as follows:

$$\begin{aligned}
\alpha_\delta &= \gamma_\Delta^{-1}\langle\langle\rangle, \pi_1(\mathrm{q})\rangle \\
\alpha_\delta^{-1} &= \langle\delta, \mathrm{pair}(\mathrm{q}[\gamma_\Delta], \mathrm{r}_{\overline{\Gamma}[\langle\rangle]})\rangle
\end{aligned}$$

It is easy to show that they have the right types, and that $\alpha_\delta\alpha_\delta^{-1} = \mathrm{id}_\Delta$. For the other equality, we have $\alpha_\delta^{-1}\alpha_\delta = \langle\delta\gamma_\Delta^{-1}\langle\langle\rangle, \pi_1(\mathrm{q})\rangle, \mathrm{pair}(\pi_1(\mathrm{q}), \mathrm{r}_{\overline{\Gamma}[\langle\rangle]})\rangle$. By the property of extensional identity types $\mathrm{q}[\gamma_\Gamma \mathrm{p}]$ and $\overline{\delta}[\langle\langle\rangle, \pi_1(\mathrm{q})\rangle]$ are equal terms in context $\Gamma\cdot\mathrm{Inv}(\delta)$, so $\gamma_\Gamma^{-1}\langle\langle\rangle, \mathrm{q}[\gamma_\Gamma \mathrm{p}]\rangle = \mathrm{p}$ and $\gamma_\Gamma^{-1}\langle\langle\rangle, \overline{\delta}[\langle\langle\rangle, \pi_1(\mathrm{q})\rangle]\rangle = \delta\gamma_\Delta^{-1}\langle\langle\rangle, \pi_1(\mathrm{q})\rangle$ are equal substitutions. Likewise, $\mathrm{r}_{\overline{\Gamma}[\langle\rangle]} = \pi_2(\mathrm{q})$ by uniqueness of identity proofs, therefore $\alpha_\delta^{-1}\alpha_\delta = \mathrm{id}_{\Gamma\cdot\mathrm{Inv}(\delta)}$. $\square$

It is easy to see that $\boldsymbol{T}(\Gamma)$ has binary products if the cwf supports Σ-types and exponentials if it supports Π-types. Simply define $A \times B = \Sigma(A, B[\mathrm{p}])$ and $B^A = \Pi(A, B[\mathrm{p}])$. Hence by Proposition 9 it follows that $\mathbb{C}/\Gamma$ has products and $\mathbb{C}$ has finite limits in any democratic cwf which supports extensional identity types and Σ-types. If it supports Π-types too, then $\mathbb{C}/\Gamma$ is cartesian closed and $\mathbb{C}$ is locally cartesian closed.

3. The 2-Category of Categories with Families

3.1. *Pseudo Cwf-Morphisms*

A notion of *strict cwf-morphism* between cwfs $(\mathbb{C}, T)$ and $(\mathbb{C}', T')$ was defined by Dybjer (Dybjer, 1996). It is a pair (F, σ), where $F : \mathbb{C} \to \mathbb{C}'$ is a functor and $\sigma : T \xrightarrow{\bullet} T'F$ is a natural transformation of family-valued functors, such that terminal objects and context comprehension are preserved on the nose. Here we need a weak version where the terminal object, context comprehension, and substitution of types and terms of a cwf are only preserved up to isomorphism. The pseudo-natural transformations needed to prove our biequivalences will be families of cwf-morphisms which do not preserve cwf-structure on the nose.

The definition of pseudo cwf-morphism will be analogous to that of *strict* cwf-morphism, but cwf-structure will only be preserved up to coherent isomorphism.

Definition 7. A **pseudo cwf-morphism** from $(\mathbb{C}, T)$ to $(\mathbb{C}', T')$ is a pair (F, σ) where:

— $F : \mathbb{C} \to \mathbb{C}'$ is a functor,

— For each context Γ in $\mathbb{C}$, σ_Γ is a **Fam**-morphism from $T\Gamma$ to $T'F\Gamma$. We will write $\sigma_\Gamma(A) : \mathrm{Type}'(F\Gamma)$ for the type component and $\sigma_\Gamma^A(a) : F\Gamma \vdash \sigma_\Gamma(A)$ for the term component of this morphism.

The following preservation properties must be satisfied:

— Substitution is preserved: For each context $\delta : \Delta \to \Gamma$ in $\mathbb{C}$ and $A \in \mathrm{Type}(\Gamma)$, there is an isomorphism of types $\theta_{A,\delta} : \sigma_\Gamma(A)[F\delta] \to \sigma_\Delta(A[\delta])$ such that substitution in terms is also preserved, that is, $\sigma_\Delta^{A[\gamma]}(a[\gamma]) =_{\theta_{A,\gamma}} \sigma_\Gamma^A(a)[F\gamma]$.

— The terminal object is preserved: $F[]$ is terminal.

— Context comprehension is preserved: $F(\Gamma{\cdot}A)$ with the projections $F(\mathrm{p}_A)$ and $\{\theta_{A,\mathrm{p}}^{-1}\}(\sigma_{\Gamma\cdot A}^{A[p]}(\mathrm{q}_A))$ is a context comprehension of $F\Gamma$ and $\sigma_\Gamma(A)$. Note that the universal property on context comprehensions provides a unique isomorphism $\rho_{\Gamma,A} : F(\Gamma\cdot A) \to F\Gamma\cdot\sigma_\Gamma(A)$ which preserves projections.

These data must satisfy naturality and coherence laws which amount to the fact that if we extend σ_Γ to a functor $\boldsymbol{\sigma}_\Gamma : \boldsymbol{T}(\Gamma) \to \boldsymbol{T'}F(\Gamma)$, then $\boldsymbol{\sigma}$ is a pseudonatural transformation from $\boldsymbol{T}$ to $\boldsymbol{T'}F$. This functor is defined by $\boldsymbol{\sigma}_\Gamma(A) = \sigma_\Gamma(A)$ on an object A and $\boldsymbol{\sigma}_\Gamma(f) = \rho_{\Gamma,B}F(f)\rho_{\Gamma,A}^{-1}$ on a morphism $f : A \to B$.

More explicitely, pseudonaturality of $\boldsymbol{\sigma}$ amounts to the following coherence and naturality laws.

— *Identity.* For all $A \in \mathrm{Type}(\Gamma)$, we have $\theta_{A,\mathrm{id}} = \mathrm{id}_{F\Gamma\sigma_\Gamma(A)}$,

— *Coherence.* For all $\delta : \Xi \to \Delta$ and $\gamma : \Delta \to \Gamma$, the following diagram commutes.

$$\begin{tikzcd}
F\Xi\cdot\sigma_\Gamma(A)[F(\gamma\delta)] \arrow[rr, "\theta_{A,\gamma\delta}"] \arrow[dr, "\boldsymbol{T'}(F\delta)(\theta_{A,\gamma})"'] & & F\Xi\cdot\sigma_\Xi(A[\gamma\delta]) \\
& F\Xi\cdot\sigma_\Delta(A[\gamma])[F(\delta)] \arrow[ur, "\theta_{A[\gamma],\delta}"'] &
\end{tikzcd}$$

— *Naturality.* For all $\delta : \Delta \to \Gamma$ in $\mathbb{C}$, $A, B \in \mathrm{Type}(\Gamma)$ and $f : A \to B$ in $\boldsymbol{T}(\Gamma)$, the following diagram commutes in $\boldsymbol{T'}(F\Delta)$.

$$\begin{tikzcd}
\sigma_\Gamma(A)[F\delta] \arrow[r, "\theta_{A,\delta}"] \arrow[d, "\boldsymbol{T'}(F\delta)(\boldsymbol{\sigma}_\Gamma(f))"] & \sigma_\Delta(A[\delta]) \arrow[d, "\boldsymbol{\sigma}_\Delta(\boldsymbol{T}(\delta)(f))"] \\
\sigma_\Gamma(B)[F\delta] \arrow[r, "\theta_{B,\delta}"] & \sigma_\Delta(B[\delta])
\end{tikzcd}$$

From this definition we can prove that all cwf structure is preserved.

Proposition 4. All pseudo cwf-morphisms (F, σ) from $(\mathbb{C}, T)$ to $(\mathbb{C}', T')$ preserve substitution extension in the following sense: For all $\delta : \Delta \to \Gamma$ in $\mathbb{C}$ and $a : \Delta \vdash A[\delta]$,

$$F(\langle \delta, a\rangle) = \rho^{-1}_{\Gamma,A}\langle F\delta, \{\theta^{-1}_{A,\delta}\}(\sigma^{A[\delta]}_{\Delta}(a))\rangle$$

Proof. First note that for each context Γ in $\mathbb{C}$ and type $A \in \mathrm{Type}(\Gamma)$, the isomorphism $\rho_{\Gamma,A}$ is defined as the unique morphism preserving projections between the two context comprehensions of $F\Gamma$ and $\sigma_\Gamma A$, that is, $\rho_{\Gamma,A} = \langle F(\mathrm{p}_A), \{\theta^{-1}_{A,\mathrm{p}}\}(\sigma^{A[\mathrm{p}]}_{\Gamma A}(\mathrm{q}_A))\rangle$, which implies that the projections are preserved in the following sense:

$$\begin{array}{rcl} F(\mathrm{p}_A) & = & \mathrm{p}_{\sigma_\Gamma A}\rho_{\Gamma,A} \\ \sigma^{A[\mathrm{p}]}_{\Gamma A}(\mathrm{q}_A) & = & \{\theta_{A,\mathrm{p}}\}(\mathrm{q}_{\sigma_\Gamma A}[\rho_{\Gamma,A}]) \end{array}$$

We now use it to prove the announced property. The required equality boils down to the following two equations.

$$\begin{array}{rcl} \mathrm{p}\rho_{\Gamma,A}F(\langle\delta,a\rangle) & = & F\delta \\ \mathrm{q}[\rho_{\Gamma,A}F(\langle\delta,a\rangle)] & = & \{\theta^{-1}_{A,\delta}\}(\sigma^{A[\delta]}_{\Delta}(a)) \end{array}$$

The proof of the first equality is straightforward:

$$\begin{array}{rcl} \mathrm{p}\rho_{\Gamma,A}F(\langle\delta,a\rangle) & = & F(\mathrm{p})F(\langle\delta,a\rangle) \\ & = & F\delta \end{array}$$

However, the proof of the second is far more subtle and relies on many properties of pseudo cwf-morphisms and cwf combinators:

$$\begin{array}{rcl} \mathrm{q}[\rho_{\Gamma,A}F(\langle\delta,a\rangle)] & =_1 & \{\theta^{-1}_{A,\mathrm{p}}\}(\sigma^{A[\mathrm{p}]}_{\Gamma A}(\mathrm{q}))[F(\langle\delta,a\rangle)] \\ & =_2 & \mathrm{q}[\theta^{-1}_{A,\mathrm{p}}\langle \mathrm{id}, \sigma^{A[\mathrm{p}]}_{\Gamma A}(\mathrm{q})\rangle][F(\langle\delta,a\rangle)] \\ & = & \mathrm{q}[\theta^{-1}_{A,\mathrm{p}}\langle F(\langle\delta,a\rangle), \sigma^{A[\mathrm{p}]}_{\Gamma A}(\mathrm{q})[F(\langle\delta,a\rangle)]\rangle] \\ & =_3 & \mathrm{q}[\theta^{-1}_{A,\mathrm{p}}\langle F(\langle\delta,a\rangle), \{\theta^{-1}_{A[\mathrm{p}],\langle\delta,a\rangle}\}(\sigma^{A[\delta]}_{\Delta}(\mathrm{q}[\langle\delta,a\rangle]))\rangle] \\ & = & \mathrm{q}[\theta^{-1}_{A,\mathrm{p}}\langle F(\langle\delta,a\rangle), \{\theta^{-1}_{A[\mathrm{p}],\langle\delta,a\rangle}\}(\sigma^{A[\delta]}_{\Delta}(a))\rangle] \\ & = & \mathrm{q}[\langle \mathrm{p}, \mathrm{q}[\theta^{-1}_{A,\mathrm{p}}\langle F(\langle\delta,a\rangle)\mathrm{p}, \mathrm{q}\rangle]\rangle\langle \mathrm{id}, \{\theta^{-1}_{A[\mathrm{p}],\langle\delta,a\rangle}\}(\sigma^{A[\delta]}_{\Delta}(a))\rangle] \\ & =_4 & \mathrm{q}[\boldsymbol{T'}(F(\langle\delta,a\rangle))(\theta^{-1}_{A,\mathrm{p}})\langle \mathrm{id}, \{\theta^{-1}_{A[\mathrm{p}],\langle\delta,a\rangle}\}(\sigma^{A[\delta]}_{\Delta}(a))\rangle] \\ & =_2 & \{\boldsymbol{T'}(F(\langle\delta,a\rangle))(\theta^{-1}_{A,\mathrm{p}})\}(\{\theta^{-1}_{A[\mathrm{p}],\langle\delta,a\rangle}\}(\sigma^{A[\delta]}_{\Delta}(a))) \\ & =_5 & \{\theta^{-1}_{A,\delta}\}(\sigma^{A[\delta]}_{\Delta}(a)) \end{array}$$

Equality (1) is by preservation of q, equalities (2) by definition of coercions, equality (3) by preservation of substitution on terms, equality (4) by definition of $\boldsymbol{T'}$, equality (5) by the coherence requirement on θ and the fact that coercions compose. All the other steps are by simple manipulations on cwf combinators. $\square$

Lemma 1. Let $(F,\sigma) : (\mathbb{C}, T) \to (\mathbb{C}', T')$ be a pseudo cwf-morphism with families of isomorphisms θ and ρ. Then for any $\delta : \Delta \to \Gamma$ in $\mathbb{C}$ and type $A \in \mathrm{Type}(\Gamma)$, we have:

$$F(\langle \delta \mathrm{p}, \mathrm{q}\rangle) = \rho^{-1}_{\Gamma,A} \langle F(\delta)\mathrm{p}, \mathrm{q}\rangle \theta^{-1}_{A,\delta} \rho_{\Delta, A[\delta]}$$

Proof. Direct calculation.

$$\begin{array}{rcl}
F(\langle \delta \mathrm{p}, \mathrm{q}\rangle) & = & \rho^{-1}_{\Gamma,A}\langle F(\delta \mathrm{p}), \{\theta^{-1}_{A,\delta \mathrm{p}}\}(\sigma^{A[\delta \mathrm{p}]}_{\Delta A[\delta]}(\mathrm{q}))\rangle \\
& = & \rho^{-1}_{\Gamma,A}\langle F(\delta \mathrm{p}), \{\theta^{-1}_{A,\delta \mathrm{p}}\}(\{\theta_{A[\delta],\mathrm{p}}\}(\mathrm{q}[\rho_{\Delta,A[\delta]}]))\rangle \\
& = & \rho^{-1}_{\Gamma,A}\langle F(\delta \mathrm{p}), \{\boldsymbol{T'}(F\mathrm{p})(\theta^{-1}_{A,\delta})\}(q[\rho_{\Delta,A[\delta]}])\rangle \\
& = & \rho^{-1}_{\Gamma,A}\langle F(\delta \mathrm{p}), \mathrm{q}[\boldsymbol{T'}(F\mathrm{p})(\theta^{-1}_{A,\delta})\langle \mathrm{id}, q[\rho_{\Delta,A[\delta]}]\rangle]\rangle \\
& = & \rho^{-1}_{\Gamma,A}\langle F(\delta \mathrm{p}), \mathrm{q}[\langle \mathrm{p}, \mathrm{q}[\theta^{-1}_{A,\delta}\langle (F\mathrm{p})\mathrm{p}, \mathrm{q}\rangle]\rangle\langle \mathrm{id}, q[\rho_{\Delta,A[\delta]}]\rangle]\rangle \\
& = & \rho^{-1}_{\Gamma,A}\langle F(\delta \mathrm{p}), \mathrm{q}[\theta^{-1}_{A,\delta}\langle F\mathrm{p}, \mathrm{q}[\rho_{\Delta,A[\delta]}]\rangle]\rangle \\
& = & \rho^{-1}_{\Gamma,A}\langle F(\delta \mathrm{p}), \mathrm{q}[\theta^{-1}_{A,\delta}\rho_{\Delta,A[\delta]}]\rangle \\
& = & \rho^{-1}_{\Gamma,A}\langle F(\delta)\mathrm{p}, \mathrm{q}\rangle\theta^{-1}_{A,\delta}\rho_{\Delta,A[\delta]}
\end{array}$$

Using preservation of substitution extension and q, then coherence of θ and manipulation of cwf combinators. □

Proposition 5. If (F,σ) is a pseudo cwf-morphism from $(\mathbb{C}, T)$ to $(\mathbb{C}', T')$, then its action on terms is determined by its action on sections: for all $a \in \Gamma \vdash A$

$$\sigma^A_\Gamma(a) = \mathrm{q}[\rho_{\Gamma,A} F(\langle \mathrm{id}, a\rangle)]$$

Proof. This follows from preservation of substitution extension:

$$F(\langle \mathrm{id}, a\rangle) = \rho^{-1}_{\Gamma,A}\langle \mathrm{id}, \{\theta^{-1}_{A,\mathrm{id}}\}\sigma^A_\Gamma(a)\rangle$$

but $\theta_{A,\mathrm{id}} = \mathrm{id}$ by coherence of θ, hence the result is proved. □

Lemma 2. If (F,σ) is a pseudo cwf-morphism from $(\mathbb{C}, T)$ to $(\mathbb{C}', T')$ and $\theta : A \to B$ is a morphism in $\boldsymbol{T}(\Gamma)$, then the coercion $\{\theta\}$ commutes with σ in the following way, for each $a \in \Gamma \vdash A$:

$$\sigma^B_\Gamma(\{\theta\}(a)) = \{\boldsymbol{\sigma}_\Gamma(\theta)\}(\sigma^A_\Gamma(a))$$

Proof. Direct calculation.

$$\begin{array}{rcl}\sigma_\Gamma^B(\{\theta\}(a)) & =_1 & \mathrm{q}[\rho_{\Gamma,B}F(\langle \mathrm{id}, \{\theta\}(a)\rangle)] \\ & =_2 & \mathrm{q}[\rho_{\Gamma,B}F(\langle \mathrm{id}, \mathrm{q}[\theta\langle \mathrm{id}, a\rangle]\rangle)] \\ & =_3 & \mathrm{q}[\rho_{\Gamma,B}F(\theta\langle \mathrm{id}, a\rangle)] \\ & =_4 & \mathrm{q}[\boldsymbol{\sigma}_\Gamma(\theta)\rho_{\Gamma,A}F(\langle \mathrm{id}, a\rangle)] \\ & =_2 & \{\boldsymbol{\sigma}_\Gamma(\theta)\}(\mathrm{q}[\rho_{\Gamma,A}F(\langle \mathrm{id}, a\rangle)]) \\ & =_1 & \{\boldsymbol{\sigma}_\Gamma(\theta)\}(\sigma_\Gamma^A(a))\end{array}$$

Where (1) is by Proposition 5, (2) by definition of coercions, (3) by basic manipulation of cwf combinators and (4) by definition of $\boldsymbol{\sigma}$. □

Proposition 6. Pseudo cwf-morphisms are stable under composition.

Proof. If $(F,\sigma) : (\mathbb{C}_0, T_0) \to (\mathbb{C}_1, T_1)$ and $(G,\tau) : (\mathbb{C}_1, T_1) \to (\mathbb{C}_2, T_2)$ are two pseudo cwf-morphisms, we define their composition as $(GF, \tau\sigma)$ where:

$$\begin{array}{rcl}(\tau\sigma)_\Gamma(A) & = & \tau_{F\Gamma}(\sigma_\Gamma(A)) \\ (\tau\sigma)_\Gamma^A(a) & = & \tau_{F\Gamma}^{\sigma_\Gamma(A)}(\sigma_\Gamma^A(a))\end{array}$$

If the other components of (F,σ) are denoted by θ^F, ρ^F and those of (G,τ) by θ^G, ρ^G, we define:

$$\theta_{A,\delta} = \boldsymbol{\tau}_{F\Delta}(\theta^F_{A,\delta})\theta^G_{\sigma_\Gamma(A),F\delta}$$

All the components are now defined, and we can show that the conditions hold.

— *Preservation of substitution on terms.* Direct calculation, if $a : \Gamma \vdash A$ and $\delta : \Delta \to \Gamma$ in $\mathbb{C}_0$.

$$\begin{array}{rcl}(\tau\sigma)_\Delta^{A[\delta]}(a[\delta]) & =_1 & \tau_{F\Delta}^{\sigma_\Delta(A[\delta])}(\sigma_\Delta^{A[\delta]}(a[\delta])) \\ & =_2 & \tau_{F\Delta}^{\sigma_\Delta(A[\delta])}(\{\theta^F_{A,\delta}\}(\sigma_\Gamma^A(a)[F\delta])) \\ & =_3 & \mathrm{q}[\rho^G_{F\Delta,\sigma_\Delta(A[\delta])}G(\langle \mathrm{id}, \{\theta^F_{A,\delta}\}(\sigma_\Gamma^A(a)[F\delta])\rangle)] \\ & =_4 & \mathrm{q}[\rho^G_{F\Delta,\sigma_\Delta(A[\delta])}G(\langle \mathrm{id}, \mathrm{q}[\theta^F_{A,\delta}\langle \mathrm{id}, \sigma_\Gamma^A(a)[F\delta]\rangle]\rangle)] \\ & =_5 & \mathrm{q}[\rho^G_{F\Delta,\sigma_\Delta(A[\delta])}G(\theta^F_{A,\delta}\langle \mathrm{id}, \sigma_\Gamma^A(a)[F\delta]\rangle)] \\ & =_6 & \mathrm{q}[\boldsymbol{\tau}_{F\Delta}(\theta^F_{A,\delta})\rho^G_{F\Delta,\sigma_\Gamma(A)[F\delta]}G(\langle \mathrm{id}, \sigma_\Gamma^A(a)[F\delta]\rangle)] \\ & =_7 & \mathrm{q}[\boldsymbol{\tau}_{F\Delta}(\theta^F_{A,\delta})\langle \mathrm{id}, \mathrm{q}[\rho^G_{F\Delta,\sigma_\Gamma(A)[F\delta]}G(\langle \mathrm{id}, \sigma_\Gamma^A(a)[F\delta]\rangle)]\rangle] \\ & =_4 & \{\boldsymbol{\tau}_{F\Delta}(\theta^F_{A,\delta})\}(\mathrm{q}[\rho^G_{F\Delta,\sigma_\Gamma(A)[F\delta]}G(\langle \mathrm{id}, \sigma_\Gamma^A(a)[F\delta]\rangle)]) \\ & =_3 & \{\boldsymbol{\tau}_{F\Delta}(\theta^F_{A,\delta})\}(\tau_{F\Delta}^{\sigma_\Gamma(A)[F\delta]}(\sigma_\Gamma^A(a)[F\delta])) \\ & =_2 & \{\boldsymbol{\tau}_{F\Delta}(\theta^F_{A,\delta})\}(\{\theta^G_{\sigma_\Gamma(A),F\delta}\}(\tau_{F\Gamma}^{\sigma_\Gamma(A)}(\sigma_\Gamma^A(a))[GF\delta])) \\ & =_8 & \{\theta_{A,\delta}\}(\tau_{F\Gamma}^{\sigma_\Gamma(A)}(\sigma_\Gamma^A(a))[GF\delta]) \\ & =_1 & \{\theta_{A,\delta}\}((\tau\sigma)_\Gamma^A(a)[GF\delta])\end{array}$$

Equalities annotated by (1) come from the definition of $\tau\sigma$, (2) is preservation of substitution for σ or τ, (3) is Proposition 5, (4) is by definition of coercions, (5) uses

$\mathrm{p}\theta^F_{A,\delta} = \mathrm{p}$ and basic manipulations with cwf combinators, (6) is by definition of $\boldsymbol{\tau}$, (7) uses preservation of p by (G, τ) and basic manipulations with cwf combinators, and (8) is by definition of θ.

— *Preservation of the terminal object.* Trivial from the preservation of the terminal object by F and G.

— *Preservation of context comprehension.* Using preservation of context comprehension from (F, σ) and (G, τ) we define:

$$GF(\Gamma\cdot A) \xrightarrow{G(\rho^F_{\Gamma,A})} G(F\Gamma\cdot\sigma_\Gamma A) \xrightarrow{\rho^G_{F\Gamma,\sigma_\Gamma A}} GF\Gamma\cdot(\tau\sigma)_\Gamma(A)$$

As a composition of isomorphisms it is an isomorphism so $GF(\Gamma\cdot A)$ is also a context comprehension of $GF\Gamma$ and $(\tau\sigma)_\Gamma(A)$. We must still check that the corresponding projections are those required by the definition. It is obvious for the first projection:

$$\begin{aligned} \mathrm{p}\rho^G_{F\Gamma,\sigma_\Gamma A}G(\rho^F_{\Gamma,A}) &= G(\mathrm{p})G(\rho^F_{\Gamma,A}) \\ &= GF\mathrm{p} \end{aligned}$$

But more intricate for the second.

$$\begin{aligned} \mathrm{q}[\rho^G_{F\Gamma,\sigma_\Gamma A}G(\rho^F_{\Gamma,A})] &=_1 \{(\theta^G_{\sigma_\Gamma A,\mathrm{p}})^{-1}\}(\tau^{\sigma_\Gamma(A)[\mathrm{p}]}_{F\Gamma\sigma_\Gamma A}(\mathrm{q}))[G(\rho^F_{\Gamma,A})] \\ &=_2 \{(\theta^G_{\sigma_\Gamma A,\mathrm{p}})^{-1}\}(\{(\theta^G_{\sigma_\Gamma(A)[\mathrm{p}],\rho^F_{\Gamma,A}})^{-1}\}(\tau^{\sigma_\Gamma(A)[F\mathrm{p}]}_{F(\Gamma\cdot A)}(\mathrm{q}[\rho^F_{\Gamma,A}]))) \\ &=_3 \{(\theta^G_{\sigma_\Gamma A,F\mathrm{p}})^{-1}\}(\tau^{\sigma_\Gamma(A)[F\mathrm{p}]}_{F(\Gamma\cdot A)}(\mathrm{q}[\rho^F_{\Gamma,A}])) \\ &=_4 \{(\theta^G_{\sigma_\Gamma A,F\mathrm{p}})^{-1}\}(\tau^{\sigma_\Gamma(A)[F\mathrm{p}]}_{F(\Gamma\cdot A)}(\{(\theta^F_{A,\mathrm{p}})^{-1}\}(\sigma^{A[\mathrm{p}]}_{\Gamma\cdot A}(\mathrm{q})))) \\ &=_5 \{(\theta^G_{\sigma_\Gamma A,F\mathrm{p}})^{-1}\}(\{\boldsymbol{\tau}_{F(\Gamma\cdot A)}((\theta^F_{A,\mathrm{p}})^{-1})\}(\tau^{\sigma_{\Gamma\cdot A}(A[\mathrm{p}])}_{F(\Gamma\cdot A)}(\sigma^{A[\mathrm{p}]}_{\Gamma\cdot A}(\mathrm{q})))) \\ &=_6 \{\theta^{-1}_{A,[p]}\}((\tau\sigma)^{A[\mathrm{p}]}_{\Gamma\cdot A}(q)) \end{aligned}$$

Where (1) is preservation of the second projection by ρ^G, (2) is preservation of substitution on terms, (3) is coherence for θ^G, (4) is preservation of the second projection by ρ^F, (5) is Lemma 2 and (6) is by definition of θ and $\tau\sigma$.

Finally, by unfolding the definitions we can conclude that for all contexts Γ

$$(\boldsymbol{\tau\sigma})_\Gamma = \boldsymbol{\tau}_{F\Gamma} \circ \boldsymbol{\sigma}_\Gamma$$

Hence the necessary coherence and naturality conditions amount to the stability of pseudonatural transformations under composition. □

Since the isomorphism $(\Gamma\cdot A)\cdot B \cong \Gamma\cdot\Sigma(A, B)$ holds in an arbitrary cwf which supports Σ-types, it follows that pseudo cwf-morphisms preserve Σ-types, since they preserve context comprehension. More precisely, we have

Proposition 7. A pseudo cwf-morphism (F, σ) from $(\mathbb{C}, T)$ to $(\mathbb{C}', T')$, where both cwfs support Σ-types, also preserves them in the sense that

$$\sigma_\Gamma(\Sigma(A, B)) \cong \Sigma(\sigma_\Gamma(A), \sigma_{\Gamma\cdot A}(B)[\rho^{-1}_{\Gamma,A}])$$

Proof. We exploit the fact that in a cwf $(\mathbb{C}, T)$ with Σ-types we have $\Gamma{\cdot}\Sigma(A,B) \cong \Gamma{\cdot}A{\cdot}B$ (obvious). Therefore:

$$\begin{array}{rcl} F\Gamma\cdot\sigma_\Gamma(\Sigma(A,B)) & \cong & F(\Gamma\cdot\Sigma(A,B)) \\ & \cong & F(\Gamma\cdot A\cdot B) \\ & \cong & F(\Gamma\cdot A)\cdot\sigma_{\Gamma\cdot A}(B) \\ & \cong & F\Gamma\cdot\sigma_\Gamma(A)\cdot\sigma_{\Gamma\cdot A}(B)[\rho^{-1}_{\Gamma,A}] \\ & \cong & F\Gamma\cdot\Sigma(\sigma_\Gamma(A),\sigma_{\Gamma\cdot A}(B)[\rho^{-1}_{\Gamma,A}]) \end{array}$$

It is easy to see that the resulting morphism is a type isomorphism. $\square$

If cwfs support other structure, we need to define what it means that cwf-morphisms preserve this extra structure up to isomorphism.

Definition 8. Let (F, σ) be a pseudo cwf-morphism between cwfs $(\mathbb{C}, T)$ and $(\mathbb{C}', T')$ which support identity types, Π-types, and democracy, respectively.

— (F, σ) *preserves identity types* provided $\sigma_\Gamma(\mathrm{I}_A(a, a')) \cong \mathrm{I}_{\sigma_\Gamma(A)}(\sigma^A_\Gamma(a), \sigma^A_\Gamma(a))$;
— (F, σ) *preserves* Π*-types* provided $\sigma_\Gamma(\Pi(A,B)) \cong \Pi(\sigma_\Gamma(A), \sigma_{\Gamma\cdot A}(B)[\rho^{-1}_{\Gamma,A}])$;
— (F, σ) *preserves democracy* provided $\sigma_{[]}(\overline{\Gamma}) \cong_{d_\Gamma} \overline{F\Gamma}[\langle\rangle]$, and the following diagram commutes:

$$\begin{tikzcd} F\Gamma \arrow[rr, "F\gamma_\Gamma"] \arrow[d, "\gamma_{F\gamma}"'] & & F([]\cdot\overline{\Gamma}) \arrow[d, "\rho_{[],\overline{\Gamma}}"] \\ {[]\cdot\overline{F\Gamma}} & F[]\cdot\overline{F\Gamma}[\langle\rangle] \arrow[l, "{\langle\langle\rangle,\mathrm{q}\rangle}"'] & F[]\cdot\sigma_{[]}(\overline{\Gamma}) \arrow[l, "d_\Gamma"'] \end{tikzcd}$$

Lemma 3. The above preservation isomorphisms for identity types, Π-types, and democracy are all stable under composition.

Proof. This is trivial for identity types and Π-types (just apply the hypothesis on both pseudo cwf-morphism).

For democracy, we must check that the isomorphism $d^{GF}_\Gamma : GF[]\cdot(\tau\sigma)_{[]}(\overline{\Gamma}) \to GF[]\cdot \overline{GF\Gamma}[\langle\rangle]$ defined by

$$\begin{array}{rl} d^{GF}_\Gamma = & GF[]\cdot\tau_{F[]}(\sigma_{[]}(\overline{\Gamma})) \xrightarrow{(\rho^G_{F[],\sigma_{[]}\overline{\Gamma}})^{-1}} F(F[]\cdot\sigma_{[]}\overline{\Gamma}) \xrightarrow{G(d^F_\Gamma)} G(F[]\cdot\overline{F\Gamma}[\langle\rangle]) \\ & \xrightarrow{G(\langle\langle\rangle,\mathrm{q}\rangle)} G([]\cdot\overline{F\Gamma}) \xrightarrow{\rho^G_{[],\overline{F\Gamma}}} G[]\cdot\tau_{[]}\overline{F\Gamma} \\ & \xrightarrow{d^G_{F\Gamma}} G[]\cdot\overline{GF\Gamma}[\langle\rangle] \end{array}$$

satisfies the coherence law. This follows by simple diagram chasing. $\square$

3.2. *Pseudo Cwf-Transformations*

Definition 9 (Pseudo cwf-transformation). Let (F, σ) and (G, τ) be two cwf-morphisms from $(\mathbb{C}, T)$ to $(\mathbb{C}', T')$. A *pseudo cwf-transformation* from (F, σ) to (G, τ) is a pair (ϕ, ψ) where $\phi : F \xrightarrow{\bullet} G$ is a natural transformation, and for each Γ in $\mathbb{C}$ and $A \in \mathrm{Type}(\Gamma)$, a morphism $\psi_{\Gamma,A} : \sigma_\Gamma(A) \to \tau_\Gamma(A)[\phi_\Gamma]$ in $\boldsymbol{T'}(F\Gamma)$, natural in A and such that the following diagram commutes:

$$\begin{tikzcd} \sigma_\Gamma(A)[F\delta] \arrow[r, "\boldsymbol{T'}(F\delta)(\psi_{\Gamma,A})"] \arrow[d, "\theta_{A,\delta}"] & \tau_\Gamma(A)[\phi_\Gamma F(\delta)] \arrow[d, "\boldsymbol{T'}(\phi_\Delta)(\theta'_{A,\delta})"] \\ \sigma_\Delta(A[\delta]) \arrow[r, "\psi_{\Delta,A[\delta]}"'] & \tau_\Delta(A[\delta])[\phi_\Delta] \end{tikzcd}$$

where θ and θ' are the isomorphisms witnessing preservation of substitution in types in the definition of pseudo cwf-morphism.

Pseudo cwf-transformations can be composed both vertically (denoted by $(\phi', \psi') \bullet (\phi, \psi)$) and horizontally (denoted by $(\phi', \psi')(\phi, \psi)$), and these compositions are associative and satisfy the interchange law. Note that just as coherence and naturality laws for pseudo cwf-morphisms ensure that they give rise to pseudonatural transformations (hence morphisms of indexed categories) $\boldsymbol{\sigma}$ to $\boldsymbol{\tau}$, this definition means that pseudo cwf-transformations from (F, σ) to (F, τ) correspond to modifications from $\boldsymbol{\sigma}$ to $\boldsymbol{\tau}$.

3.3. *2-Categories of Cwfs with Extra Structure*

As a consequence of the preservation properties in Lemma 3 we have several different 2-categories of structure-preserving pseudo cwf-morphisms.

Definition 10. Let $\mathbf{CwF}_{\mathrm{dem}}^{\mathrm{I_{ext}}\Sigma}$ be the 2-category of small democratic categories with families which support extensional identity types and Σ-types. The 1-cells are cwf-morphisms preserving democracy and extensional identity types (and Σ-types automatically) and the 2-cells are pseudo cwf-transformations.

Moreover, let $\mathbf{CwF}_{\mathrm{dem}}^{\mathrm{I_{ext}}\Sigma\Pi}$ be the sub-2-category of $\mathbf{CwF}_{\mathrm{dem}}^{\mathrm{I_{ext}}\Sigma}$ where also Π-types are supported and preserved.

4. Forgetting Types and Terms

Definition 11. Let **FL** be the 2-category of small categories with finite limits (left exact categories). The 1-cells are functors preserving finite limits (up to isomorphism) and the 2-cells are natural transformations.

Let **LCC** be the 2-category of small locally cartesian closed categories. The 1-cells are functors preserving local cartesian closed structure (up to isomorphism), and the 2-cells are natural transformations.

FL is a sub(2-)category of the 2-category of categories: we do not provide a choice of finite limits. Similarly, **LCC** is a sub(2-)category of **FL**. The first component of our biequivalences will be *forgetful* 2-functors.

Proposition 8. The forgetful 2-functors

$$\begin{aligned} U &: \mathbf{CwF}_{\mathrm{dem}}^{\mathrm{I_{ext}}\Sigma} \to \mathbf{FL} \\ U &: \mathbf{CwF}_{\mathrm{dem}}^{\mathrm{I_{ext}}\Sigma\Pi} \to \mathbf{LCC} \end{aligned}$$

defined as follows on 0-, 1-, and 2-cells

$$\begin{array}{rcl} U(\mathbb{C},T) & = & \mathbb{C} \\ U(F,\sigma) & = & F \\ U(\phi,\psi) & = & \phi \end{array}$$

are well-defined.

By definition U is a 2-functor from **CwF** to **Cat**, it remains to prove that it sends a cwf in $\mathbf{CwF}_{\mathrm{dem}}^{\mathrm{I_{ext}}\Sigma}$ to **FL** and a cwf in $\mathbf{CwF}_{\mathrm{dem}}^{\mathrm{I_{ext}}\Sigma\Pi}$ to **LCC**, along with the corresponding properties for 1-cells and 2-cells.

For 0-cells we already proved as corollaries of Proposition 3 that if $(\mathbb{C},T)$ supports Σ-types, identity types and democracy, then $\mathbb{C}$ has finite limits; and if $(\mathbb{C},T)$ also supports Π-types, then $\mathbb{C}$ is an lccc.

For 1-cells we shall prove in Proposition 9 that if (F,σ) preserves identity types and democracy, then F preserves finite limits; and if (F,σ) also preserves Π-types then F preserves local exponentiation. Since finite limits and local exponentiation in $\mathbb{C}$ and $\mathbb{C}'$ can be defined by the inverse image construction, these two statements boil down to the fact that if (F,σ) preserves identity types and democracy then inverse images are preserved. In Lemma 5 we shall construct the isomorphism $F(\Gamma\cdot\mathrm{Inv}(\delta)) \cong F\Gamma\cdot\mathrm{Inv}(F\delta)$ needed for this purpose. To prove this lemma, we prove that identity types and Σ-types preserve isomorphisms in Lemma 4. To prove the local exponentiation part of Proposition 9 we also prove that Π-types preserve isomorphisms in Lemma 6.

There is nothing to prove for 2-cells.

Lemma 4 (Propagation of isomorphisms). Isomorphisms propagate through types in several different ways. Suppose that you have $A, A' \in \mathit{Type}(\Gamma)$, $B, B' \in \mathit{Type}(\Gamma\cdot A)$, then

(1) If $B \cong B'$, then $\Sigma(A,B) \cong \Sigma(A,B')$
(2) If $A \cong_\theta A'$, then $\Sigma(A,B) \cong \Sigma(A',B[\theta^{-1}])$
(3) If $A \cong_\theta A'$ and $a, a' \in \Gamma \vdash A$, then $\mathrm{I}_A(a,a') \cong \mathrm{I}_{A'}(\{\theta\}(a),\{\theta\}(a'))$

Proof. (1) is obvious, since $\Gamma\cdot\Sigma(A,B)$ is isomorphic to $\Gamma\cdot A\cdot B$. For (2), we give the following two isomorphisms:

$$\begin{array}{rcl} \langle p, \mathrm{pair}(\mathrm{q}[\theta\langle \mathrm{p}, \pi_1(\mathrm{q})\rangle], \pi_2(\mathrm{q}))\rangle & : & \Gamma\cdot\Sigma(A,B) \to \Gamma\cdot\Sigma(A',B[\theta^{-1}]) \\ \langle p, \mathrm{pair}(\mathrm{q}[\theta^{-1}\langle \mathrm{p}, \pi_1(\mathrm{q})\rangle], \pi_2(\mathrm{q}))\rangle & : & \Gamma\cdot\Sigma(A',B[\theta^{-1}]) \to \Gamma\cdot\Sigma(A,B) \end{array}$$

A simple calculation shows that they have the right types and that they are inverse of one another. It is obvious that they are isomorphisms of types. (3) is also obvious since by extensionality, $\langle \mathrm{p}, \mathrm{r}\rangle$ typechecks in both directions and is its own inverse. □

Lemma 5 (Preservation of inverse image). Let $(\mathbb{C},T)$ and $(\mathbb{C}',T')$ be cwfs supporting democracy, Σ-types and identity types and let (F,σ) be a pseudo cwf-morphism preserving them. Moreover, suppose that $\delta : \Delta \to \Gamma$ is a morphism in $\mathbb{C}$, then there is an isomorphism in $\mathbb{C}'$:

$$F(\Gamma\cdot\mathrm{Inv}(\delta)) \quad\cong\quad F\Gamma\cdot\mathrm{Inv}(F\delta)$$

Proof. Exploiting Lemma 4 and preservation of substitution on types and terms, a careful (but straightforward) calculation derives the following type isomorphism:

$$\sigma_\Gamma(\mathrm{Inv}(\delta)) \cong \Sigma(\overline{F\Delta}[\langle\rangle], \mathrm{I}_{\overline{F\Gamma}[\langle\rangle]}(C(\sigma^{\overline{\Gamma}[\langle\rangle]}_{\Gamma\cdot\overline{\Delta}[\langle\rangle]}(\overline{\delta}[\langle\langle\rangle, \mathrm{q}\rangle])), C(\sigma^{\overline{\Gamma}[\langle\rangle]}_{\Gamma\cdot\overline{\Delta}[\langle\rangle]}(\mathrm{q}[\gamma_\Gamma \mathrm{p}]))))$$

where $C(-)$ is an invertible context given by:

$$C(M) = \{\boldsymbol{T'}(\iota\langle\rangle)(d_\Gamma)\theta^{-1}_{\overline{\Gamma},\langle\rangle}\}(M)[\rho_{\Gamma,\overline{\Delta}[\langle\rangle]}\theta_{\overline{\Delta},\langle\rangle}\boldsymbol{T'}(\iota\langle\rangle)(d^{-1}_{\Delta})]$$

Here, ι denotes the inverse of the terminal morphism $\langle\rangle : F[] \to []$ is part of the definition of a pseudo cwf-morphism. Hence, it remains to show the following equalities:

$$\sigma^{\overline{\Gamma}[\langle\rangle]}_{\Gamma\cdot\overline{\Delta}[\langle\rangle]}(\overline{\delta}[\langle\langle\rangle, \mathrm{q}\rangle]) \quad=\quad C^{-1}(\overline{F\delta}[\langle\langle\rangle, \mathrm{q}\rangle]) \qquad (1)$$

$$\sigma^{\overline{\Gamma}[\langle\rangle]}_{\Gamma\cdot\overline{\Delta}[\langle\rangle]}(\mathrm{q}[\gamma_\Gamma \mathrm{p}]) \quad=\quad C^{-1}(\mathrm{q}[\gamma_{F\Gamma}\mathrm{p}]) \qquad (2)$$

Let us focus on (1). Using preservation of substitution on terms, coherence of θ and the basic computation laws in cwfs, we derive:

$$\begin{aligned}
\sigma^{\overline{\Gamma}[\langle\rangle]}_{\Gamma\cdot\overline{\Delta}[\langle\rangle]}(\overline{\delta}[\langle\langle\rangle, \mathrm{q}\rangle]) &= \sigma^{\overline{\Gamma}[\mathrm{p}][\langle\langle\rangle,\mathrm{q}\rangle]}_{\Gamma\cdot\overline{\Delta}[\langle\rangle]}(\overline{\delta}[\langle\langle\rangle, \mathrm{q}\rangle]) \\
&= \{\theta_{\overline{\Gamma}[\mathrm{p}],\langle\langle\rangle,\mathrm{q}\rangle}\}(\sigma^{\overline{\Gamma}[\mathrm{p}]}_{[]\cdot\overline{\Delta}}(\overline{\delta})[F(\langle\langle\rangle, \mathrm{q}\rangle)]) \\
&= \{\theta_{\overline{\Gamma},\langle\rangle}\}(\{\boldsymbol{T'}(\langle\langle\rangle, \mathrm{q}\rangle)(\theta^{-1}_{\overline{\Gamma},\mathrm{p}})\}(\sigma^{\overline{\Gamma}[\mathrm{p}]}_{[]\cdot\overline{\Delta}}(\overline{\delta})[F(\langle\langle\rangle, \mathrm{q}\rangle)])) \\
&= \{\theta_{\overline{\Gamma},\langle\rangle}\}(\{\theta^{-1}_{\overline{\Gamma},\mathrm{p}}\}(\sigma^{\overline{\Gamma}[\mathrm{p}]}_{[]\cdot\overline{\Delta}}(\overline{\delta}))[F(\langle\langle\rangle, \mathrm{q}\rangle)])
\end{aligned}$$

Let us now focus on $\sigma^{\overline{\Gamma}[\mathrm{p}]}_{[]\cdot\overline{\Delta}}(\overline{\delta})$, to see how terms created from substitution using democracy are transformed by the action of the cwf-morphism. Here, we are only need the coherence of θ, preservation of q and the basic computation laws for cwfs.

$$\begin{aligned}
\sigma^{\overline{\Gamma}[\mathrm{p}]}_{[]\cdot\overline{\Delta}}(\overline{\delta}) &= \sigma^{\overline{\Gamma}[\mathrm{p}]}_{[]\cdot\overline{\Delta}}(\mathrm{q}[\gamma_\Gamma\delta\gamma^{-1}_\Delta]) \\
&= \sigma^{\overline{\Gamma}[\mathrm{p}][\gamma_\Gamma\delta\gamma^{-1}_\Delta]}_{[]\cdot\overline{\Delta}}(\mathrm{q}[\gamma_\Gamma\delta\gamma^{-1}_\Delta]) \\
&= \{\theta_{\overline{\Gamma}[\mathrm{p}],\gamma_\Gamma\delta\gamma^{-1}_\Delta}\}(\sigma^{\overline{\Gamma}[\mathrm{p}]}_{[]\cdot\overline{\Gamma}}(\mathrm{q})[F(\gamma_\Gamma\delta\gamma^{-1}_\Delta)]) \\
&= \{\theta_{\overline{\Gamma}[\mathrm{p}],\gamma_\Gamma\delta\gamma^{-1}_\Delta}\}(\{\theta_{\overline{\Gamma},\mathrm{p}}\}(\mathrm{q}[\rho_{[],\overline{\Gamma}}])[F(\gamma_\Gamma\delta\gamma^{-1}_\Delta)]) \\
&= \{\theta_{\overline{\Gamma},\mathrm{p}}\}(\{\boldsymbol{T'}(F(\gamma_\Gamma\delta\gamma^{-1}_\Delta))(\theta^{-1}_{\overline{\Gamma},\mathrm{p}})\}(\{\theta_{\overline{\Gamma},\mathrm{p}}\}(\mathrm{q}[\rho_{[],\overline{\Gamma}}])[F(\gamma_\Gamma\delta\gamma^{-1}_\Delta)])) \\
&= \{\theta_{\overline{\Gamma},\mathrm{p}}\}(\mathrm{q}[\rho_{[],\overline{\Gamma}}F(\gamma_\Gamma\delta\gamma^{-1}_\Delta)])
\end{aligned}$$

Using preservation of democracy and the terminal object, we can now conclude:

$$\begin{aligned}
\sigma^{\overline{\Gamma}[\langle\rangle]}_{\Gamma\cdot\overline{\Delta}[\langle\rangle]}(\overline{\delta}[\langle\langle\rangle, \mathrm{q}\rangle]) &= \{\theta_{\overline{\Gamma},\langle\rangle}\}(\mathrm{q}[\rho_{[],\overline{\Gamma}}F(\gamma_\Gamma \delta \gamma_{\Delta}^{-1}\langle\langle\rangle, \mathrm{q}\rangle)]) \\
&= \{\theta_{\overline{\Gamma},\langle\rangle}\}(\mathrm{q}[d_r^{-1}\langle \iota\mathrm{p}, \mathrm{q}\rangle \gamma_{F\Gamma} F\delta\gamma_{F\Delta}^{-1}\langle\langle\rangle, \mathrm{q}\rangle \boldsymbol{T'}(\iota\langle\rangle)(d_\Delta)\theta^{-1}_{\overline{\Delta},\langle\rangle}\rho_{\Gamma,\overline{\Delta}[\langle\rangle]}]) \\
&= \{\theta_{\overline{\Gamma},\langle\rangle}\boldsymbol{T'}(\iota\langle\rangle)(d_\Gamma^{-1})\}(\overline{F\delta}[\langle\langle\rangle, \mathrm{q}\rangle])[\boldsymbol{T'}(\iota\langle\rangle)(d_\Delta)\theta^{-1}_{\overline{\Delta},\langle\rangle}\rho_{\Gamma,\overline{\Delta}[\langle\rangle]}] \\
&= C^{-1}(\overline{F\delta}[\langle\langle\rangle, \mathrm{q}\rangle])
\end{aligned}$$

We get the required expression. The case of Equation (2) is similar but less intricate, so we skip the details. □

Lemma 6 (Propagation of isomorphisms under Π). Suppose that we have $A, A' \in Type(\Gamma)$ and $B, B' \in Type(\Gamma \cdot A)$, then

1 If $B \cong_\theta B'$, then $\Pi(A, B) \cong \Pi(A, B')$
2 If $A \cong_\theta A'$, then $\Pi(A, B) \cong \Pi(A', B[\theta^{-1}])$

Proof. (1). Using θ and combinators for Π-types, we construct a morphism:

$$\langle \mathrm{p}, \lambda(\{\boldsymbol{T}(\langle \mathrm{pp}, \mathrm{q}\rangle)(\theta)\}(\mathrm{ap}(\mathrm{q}[\mathrm{p}], \mathrm{q})))\rangle : \Gamma \cdot \Pi(A, B) \to \Gamma \cdot \Pi(A, B')$$

Its inverse is the corresponding expression with θ^{-1} in place of θ. (2). Likewise, the following expression provides the required isomorphism:

$$\lambda(\mathrm{ap}(\mathrm{q}[\mathrm{pp}], \{\boldsymbol{T}(\mathrm{pp})(\theta^{-1})\}(\mathrm{q}))) : \Gamma \cdot \Pi(A, B) \to \Gamma \cdot \Pi(A', B[\theta^{-1}])$$

□

Proposition 9. Let (F, σ) be a pseudo cwf-morphism between $(\mathbb{C}, T)$ and $(\mathbb{C}', T')$ supporting Σ-types and democracy. Then:

— If $(\mathbb{C}, T)$ and $(\mathbb{C}', T')$ both support identity types and (F, σ) preserves them, then F preserves finite limits.
— If $(\mathbb{C}, T)$ and $(\mathbb{C}', T')$ both support Π-types and (F, σ) preserves them, then F preserves local exponentiation.

Proof. Since finite limits and local exponentiation can be defined using Σ-types, Π-types and the inverse image type, it follows from Lemmas 4, 5 and 6 that they are preserved by F. □

5. Rebuilding Types and Terms

Now we show how to construct a pseudofunctor in the opposite direction. We use the Bénabou-Hofmann construction to build a cwf from a category with finite limits. This construction is generalized to operations on functors and natural transformations, and thus we get a pseudofunctor. The resulting cwf supports identity types and Σ-types, and is democratic. If we start with an lccc the resulting category also supports Π-types.

Proposition 10. There are pseudofunctors

$$
\begin{array}{rcl}
H & : & \mathbf{FL} \to \mathbf{CwF}_{\mathrm{dem}}^{\mathrm{I_{ext}}\Sigma} \\
H & : & \mathbf{LCC} \to \mathbf{CwF}_{\mathrm{dem}}^{\mathrm{I_{ext}}\Sigma\Pi}
\end{array}
$$

defined by

$$
\begin{array}{rcl}
H\mathbb{C} & = & (\mathbb{C}, T_{\mathbb{C}}) \\
HF & = & (F, \sigma_F) \\
H\phi & = & (\phi, \psi_\phi)
\end{array}
$$

on 0-cells, 1-cells, and 2-cells, respectively, where $T_{\mathbb{C}}, \sigma_F$, and ψ_ϕ are defined in the following three subsections.

Proof. The remainder of this Section contains the proof. We will in turn show the action on 0-cells, 1-cells, 2-cells, and then prove pseudofunctoriality of H. □

5.1. *Action on 0-Cells*

As explained before, in categorical semantics of dependent types (going back to Cartmell (Cartmell, 1986)) a type-in-context $A \in \mathrm{Type}(\Gamma)$ is represented by a *display map* (Taylor, 1999), that is, as an object p_A in $\mathbb{C}/\Gamma$. A term $\Gamma \vdash A$ is then represented as a section of the display map for A, that is, a morphism a such that $\mathrm{p}_A \circ a = \mathrm{id}_\Gamma$. Substitution in types is then represented by pullback. This is essentially the technique used by Seely for interpreting Martin-Löf type theory in lcccs. However, as we already mentioned, it leads to a coherence problem.

To solve this problem Hofmann (Hofmann, 1994) used a construction due to Bénabou (Bénabou, 1985), which from any fibration builds an equivalent *split* fibration. Hofmann used it to build a category with attributes (cwa) (Cartmell, 1986) from a locally cartesian closed category. He then showed that this cwa supports Π, Σ, and extensional identity types. This technique essentially amounts to associating to a type A, not only a display map, but a whole family of display maps, one for each substitution instance $A[\delta]$. In other words, we choose a pullback square for every possible substitution and this choice is split, hence solving the coherence problem. As we shall explain below this family takes the form of a functor, and we refer to it as a *functorial family.*

Here we reformulate Hofmann's construction (Hofmann, 1994) using cwfs. See Dybjer (Dybjer, 1996) for the correspondence between cwfs and cwas.

Proposition 11. Let $\mathbb{C}$ be a category with terminal object. Then we can build a democratic cwf $(\mathbb{C}, T_{\mathbb{C}})$ which supports Σ-types. If $\mathbb{C}$ has finite limits, then $(\mathbb{C}, T_{\mathbb{C}})$ also supports extensional identity types. If $\mathbb{C}$ is locally cartesian closed, then $(\mathbb{C}, T_{\mathbb{C}})$ also supports Π-types.

Proof. A *type* in $\mathrm{Type}_{\mathbb{C}}(\Gamma)$ is a *functorial family*, that is, a functor $\overrightarrow{A} : \mathbb{C}/\Gamma \to \mathbb{C}^{\to}$ such that $\mathrm{cod} \circ \overrightarrow{A} = \mathrm{dom}$ and if $\begin{tikzcd}[column sep=small] \Omega \arrow[rr, "\alpha"] \arrow[dr, "\delta\alpha"'] & & \Delta \arrow[dl, "\delta"] \\ & \Gamma & \end{tikzcd}$ is a morphism in $\mathbb{C}/\Gamma$, then $\overrightarrow{A}(\alpha)$ is a pullback square:

$$\begin{tikzcd}
{} \arrow[r, "\overrightarrow{A}(\delta,\alpha)"] \arrow[d, "\overrightarrow{A}(\delta\alpha)"'] & {} \arrow[d, "\overrightarrow{A}(\delta)"] \\
\Omega \arrow[r, "\alpha"'] & \Delta
\end{tikzcd}$$

Following Hofmann, we denote the upper arrow of the square by $\overrightarrow{A}(\delta, \alpha)$.

A *term* $a : \Gamma \vdash \overrightarrow{A}$ is a section of $\overrightarrow{A}(\mathrm{id}_\Gamma)$, that is, a morphism $a : \Gamma \to \Gamma \cdot \overrightarrow{A}$ such that $\overrightarrow{A}(\mathrm{id}_\Gamma)a = \mathrm{id}_\Gamma$, where we have defined context extension by $\Gamma \cdot \overrightarrow{A} = \mathrm{dom}(\overrightarrow{A}(\mathrm{id}_\Gamma))$.

Substitution in types. Let $\gamma : \Delta \to \Gamma$ in $\mathbb{C}$ and $\overrightarrow{A} \in \mathit{Type}(\Gamma)$. We define $\overrightarrow{A}[\gamma] \in \mathit{Type}(\Delta)$ as follows.

$$\begin{array}{rcl}
\overrightarrow{A}[\gamma](\delta) & = & \overrightarrow{A}(\gamma\delta) \\
\overrightarrow{A}[\gamma](\delta, \alpha) & = & \overrightarrow{A}(\gamma\delta, \alpha)
\end{array}$$

where $\delta : \Omega \to \Delta$ and $\alpha : \Xi \to \Omega$. It is an easy verification that $\overrightarrow{A}[\gamma]$ satisfies the two conditions for types.

Substitution in terms. Let $\delta : \Delta \to \Gamma$, and $a : \Gamma \vdash \overrightarrow{A}$, that is, $a : \Gamma \to \Gamma \cdot \overrightarrow{A}$ such that $\overrightarrow{A}(\mathrm{id}_\Gamma) \circ a = \mathrm{id}_\Gamma$. Then $a[\delta]$ is defined as the unique mediating arrow in the following diagram:

$$\begin{tikzcd}
\Delta \arrow[dr, dotted, "{a[\delta]}"] \arrow[drr, bend left, "a \circ \delta"] \arrow[ddr, bend right, "\mathrm{id}_\Delta"'] & & \\
& \Delta \circ \overrightarrow{A}[\delta] \arrow[r, "\overrightarrow{A}(\mathrm{id}_\Gamma{,}\delta)"] \arrow[d, "\overrightarrow{A}[\delta](\mathrm{id}_\Delta)"] & \Gamma \cdot \overrightarrow{A} \arrow[d, "\overrightarrow{A}(\mathrm{id}_\Gamma)"] \\
& \Delta \arrow[r, "\delta"'] & \Gamma
\end{tikzcd}$$

It is a term of type $\overrightarrow{A}[\delta]$ by commutativity of the left triangle.

Functoriality. Since substitution in types is defined by composition, the cwf-laws for it follow immediately. Functoriality follows from the split choice of pullbacks of $\overrightarrow{A}$. Putting all this together, we now have built a functor $T_{\mathbb{C}} : \mathbb{C}^{op} \to \mathbf{Fam}$.

Context comprehension. Let $\Gamma \in \mathbb{C}$, and $\overrightarrow{A} \in \mathit{Type}(\Gamma)$. As mentioned above, we define:

$$\Gamma \cdot \overrightarrow{A} = \mathit{dom}(\overrightarrow{A}(\mathrm{id}_\Gamma))$$

The first projection is $\mathrm{p}_{\overrightarrow{A}} = \overrightarrow{A}(\mathrm{id}_\Gamma) : \Gamma\cdot\overrightarrow{A} \to \Gamma$. The second projection $\mathrm{q}_{\overrightarrow{A}}$ is defined as the unique mediating arrow of the following pullback diagram:

$$\begin{tikzcd}
\Gamma\cdot\overrightarrow{A} \arrow[dr, dotted, "\mathrm{q}_{\overrightarrow{A}}"] \arrow[drr, bend left, "\mathrm{id}_{\Gamma\cdot\overrightarrow{A}}"] \arrow[ddr, bend right, "\mathrm{id}_{\Gamma\cdot\overrightarrow{A}}"'] & & \\
& \Gamma\cdot\overrightarrow{A}\cdot\overrightarrow{A}[\mathrm{p}_{\overrightarrow{A}}] \arrow[r, "\overrightarrow{A}(\mathrm{id}_\Gamma{,}\mathrm{p}_{\overrightarrow{A}})"] \arrow[d, "\overrightarrow{A}[\mathrm{p}_{\overrightarrow{A}}](\mathrm{id}_{\Gamma\cdot\overrightarrow{A}})"] & \Gamma\cdot\overrightarrow{A} \arrow[d, "\overrightarrow{A}(\mathrm{id}_\Gamma)"] \\
& \Gamma\cdot\overrightarrow{A} \arrow[r, "\mathrm{p}_{\overrightarrow{A}}"'] & \Gamma
\end{tikzcd}$$

Suppose now we have $\delta : \Delta \to \Gamma$ and $a : \Delta \vdash \overrightarrow{A}[\delta]$. By definition of terms we have in fact $a : \Delta \to \Delta\cdot\overrightarrow{A}[\delta]$. We define:

$$\langle \delta, a\rangle = \overrightarrow{A}(id_\Gamma, \delta) \circ a : \Delta \to \Gamma\cdot\overrightarrow{A}$$

It can be checked that for all $\overrightarrow{A} \in \mathit{Type}(\Gamma)$, $\delta : \Delta \to \Gamma$ and $a : \Delta \vdash \overrightarrow{A}[\delta]$ we have $\mathrm{q}_{\overrightarrow{A}} \circ \langle\delta, a\rangle = \langle\langle\delta, a\rangle, a\rangle$, from which it follows easily that the cwf-laws for context comprehension are satisfied.

Democracy. The cwf $(\mathbb{C}, T_{\mathbb{C}})$ is democratic: the idea is that a context Γ is represented by a functorial family having its terminal projection $\langle\rangle : \Gamma \to I$ as display map. We can easily build such a functorial family by $\overline{\Gamma} = \widehat{\langle\rangle} \in \mathit{Type}(I)$. We have then $I\cdot\overline{\Gamma} = \mathit{dom}(\widehat{\langle\rangle}(\mathrm{id})) = \Gamma$. Thus the isomorphism between them is trivial.

Σ-types. Let $A \in \mathit{Type}(\Gamma)$ and $B \in \mathit{Type}(\Gamma\cdot A)$. At each $s : \Delta \to \Gamma$, the image of $\Sigma(A, B)$ is given by composing the images of A and B. More formally, we define:

$$\begin{aligned}
\Sigma(A, B)(s) &= A(s) \circ B(A(\mathrm{id}, s)) \\
\Sigma(A, B)(s, \alpha) &= B(A(\mathrm{id}, s), A(s, \alpha))
\end{aligned}$$

The construction of the corresponding pullback square can be illustrated by the following diagram. Intuitively, the chosen pullbacks for $\Sigma(A, B)$ are directly obtained by composing the chosen pullbacks for A and for B.

$$\begin{tikzcd}
{} \arrow[r, "B(A(\mathrm{id}{,}s){,}A(s{,}\alpha))"] \arrow[d] & {} \arrow[r, "B(\mathrm{id}{,}A(\mathrm{id}{,}s))"] \arrow[d, "B(A(\mathrm{id}{,}s))"] & \Gamma\cdot A\cdot B \arrow[d, "B(\mathrm{id})"] \\
{} \arrow[r, "A(s{,}\alpha)"] \arrow[d, "A(s\alpha)"'] & {} \arrow[r, "A(\mathrm{id}{,}s)"] \arrow[d, "A(s)"] & \Gamma\cdot A \arrow[d, "A(\mathrm{id})"] \\
{} \arrow[r, "\alpha"'] & B \arrow[r, "s"'] & \Gamma
\end{tikzcd}$$

It is easy to check that this defines a functor $\Sigma(A, B) : \mathbb{C}/\Gamma \to \mathbb{C}^{\to}$ and that the necessary equations are satisfied so that we get a type $\Sigma(A, B) \in \mathit{Type}(\Gamma)$ which satisfies the corresponding introduction rules, elimination rules and equations.

Extensional identity types. To improve readability, we will sometimes omit the subscripts of the projections, when they can be recovered from the context. To build identity types, we require that the base category has finite limits. Let $\Gamma \in \mathbb{C}$, $A \in \mathit{Type}(\Gamma)$, and $a, a' : \Gamma \vdash A$. If $s : \Delta \to \Gamma$, we define $\mathrm{I}_A(a, a')(s)$ as the equalizer of $a[s]$ and $a'[s]$ (seen as morphisms $\Delta \to \Delta{\cdot}A[s]$). If
$$\begin{tikzcd} \Delta' \arrow[rr, "\delta"] \arrow[dr, "s'"'] & & \Delta \arrow[dl, "s"] \\ & \Gamma & \end{tikzcd}$$
is a morphism in $\mathbb{C}/\Gamma$, we define $\mathrm{I}_A(a, a')(\delta)$ as the upper square in the following diagram:

$$\begin{tikzcd} \cdot \arrow[r, dotted, "\gamma"] \arrow[d, "\mathrm{I}_A(a,a')(s\delta)"] & \cdot \arrow[d, "\mathrm{I}_A(a,a')(s)"] \\ \Delta' \arrow[r, "\delta"] \arrow[d, shift right, "a[s\delta]"'] \arrow[d, shift left, "a'[s\delta]"] & \Delta \arrow[d, shift right, "a[s]"'] \arrow[d, shift left, "a'[s]"] \\ \Delta'{\cdot}A[s\delta] \arrow[r, "\langle \delta \mathrm{p}, \mathrm{q} \rangle"'] & \Delta{\cdot}A[s] \end{tikzcd}$$

where γ is yet to be defined. For this purpose, and to prove that the obtained square is a pullback, we need the following:

Lemma 7. In the diagram above, if $f : \mathrm{dom}(f) \to \Delta'$, then f equalizes $a[s\delta]$ and $a'[s\delta]$ iff δf equalizes $a[s]$ and $a'[s]$.

Proof. Follows by equational reasoning, exploiting the fact that in this cwf, for any term $a : \Gamma \vdash A$, we have the rather surprising equality $a = \langle \mathrm{id}_\Gamma, a \rangle$ (since $\langle \mathrm{id}_\Gamma, a \rangle = A(\mathrm{id}_\Gamma, \mathrm{id}_\Gamma) \circ a = a$). $\square$

We use this lemma as follows. We know that $\mathrm{I}_A(a, a')(s\delta)$ equalizes $a[s\delta]$ and $a'[s\delta]$, thus $\delta \circ \mathrm{I}_A(a, a')(s\delta)$ equalizes $a[s]$ and $a'[s]$. Thus by the equalizer property, $\delta \circ \mathrm{I}_A(a, a')(s\delta)$ factors in a unique way through $\mathrm{I}_A(a, a')(s)$, and we define γ to be the unique morphism. It follows directly from Lemma 7 that this is a pullback square. This construction is functorial: both conditions (for id_s and $\delta_1 \circ \delta_2$) follow immediately by uniqueness of the factorisation through the equalizer. Thus we have shown that $\mathrm{I}_A(a, a') \in \mathit{Type}(\Gamma)$. Introduction and elimination rules, stability under substitution and extensionality all follow from standard properties of equalizers.

Π-types. If $\mathbb{C}$ is a lccc, then the cwf $H(\mathbb{C})$ supports Π-types. Let $\overrightarrow{A}$ be a functorial family over Γ and $\overrightarrow{B}$ over $\Gamma{\cdot}\overrightarrow{A}$. Then the value of the family $\Pi(\overrightarrow{A}, \overrightarrow{B})$ at substitution $\delta : \Delta \to \Gamma$ is defined by $\Pi_{\overrightarrow{A}(\delta)}(\overrightarrow{B}(\overrightarrow{A}(\mathrm{id}, \delta)))$, where Π_f is the right adjoint of f^* obtained by the lccc structure. If $\alpha : \Omega \to \Delta$ and $\delta : \Delta \to \Gamma$, we have to define a morphism $\Pi(\overrightarrow{A}, \overrightarrow{B})(\delta, \alpha)$ yielding a pullback diagram. For this purpose, first consider the following chain of isomorphisms in $\mathbb{C}/\Omega$:

$$\begin{aligned} \Pi_{\overrightarrow{A}(\delta\alpha)} \overrightarrow{B}(\overrightarrow{A}(\mathrm{id}, \delta\alpha)) &= \Pi_{\overrightarrow{A}(\delta\alpha)} \overrightarrow{B}(\overrightarrow{A}(\mathrm{id}, \delta) \overrightarrow{A}(\delta, \alpha)) \\ &\cong \Pi_{\overrightarrow{A}(\delta\alpha)} (\overrightarrow{A}(\delta, \alpha))^* (\overrightarrow{B}(\overrightarrow{A}(\mathrm{id}, \delta))) \\ &\cong \alpha^* (\Pi_{\overrightarrow{A}(\delta)} \overrightarrow{B}(\overrightarrow{A}(\mathrm{id}, \delta))) \end{aligned}$$

The first isomorphism is by uniqueness of the pullback of $\overrightarrow{B}(\mathrm{id},\delta)$ along $\overrightarrow{A}(\delta,\alpha)$, while the second is by the Beck-Chevalley condition applied to the pullback square of $\overrightarrow{A}(\delta,\alpha)$. Let us call this isomorphism ϕ. The action of α^* also gives a canonical morphism h : $\mathrm{dom}(\alpha^*(\Pi_{\overrightarrow{A}(\delta)}\overrightarrow{B}(\overrightarrow{A}(\mathrm{id},\delta)))) \to \mathrm{dom}(\Pi_{\overrightarrow{A}(\delta)}\overrightarrow{B}(\overrightarrow{A}(\mathrm{id},\delta)))$, thus we define:

$$\Pi(\overrightarrow{A},\overrightarrow{B})(\delta,\alpha) = h\phi : \mathrm{dom}(\Pi(\overrightarrow{A},\overrightarrow{B})(\delta)) \to \mathrm{dom}(\Pi(\overrightarrow{A},\overrightarrow{B})(\delta\alpha))$$

This defines a pullback square since it is obtained from an isomorphism and a pullback. Hence the definition of the functorial family $\Pi(\overrightarrow{A},\overrightarrow{B})$ is complete, since the equations come from the universal property of the pullback. The fact that $\Pi(\overrightarrow{A},\overrightarrow{B})[\delta]$ and $\Pi(\overrightarrow{A}[\delta],\overrightarrow{B}[\langle\delta\mathrm{p},\mathrm{q}\rangle])$ coincide on objects (of $\mathbb{C}/\Gamma$) is a straightforward calculation, from which the fact that they coincide on morphisms can be directly deduced.

The combinators λ and ap come from natural applications of the adjunction $(\overrightarrow{A}(\mathrm{id}))^* \dashv \Pi_{\overrightarrow{A}(\mathrm{id})}$, and the computation rules follow from the properties of adjunctions. As in (Hofmann, 1994), the behaviour of the combinators λ and ap under substitution is obtained by reworking the proof of the Beck-Chevalley conditions for lcccs.

□

5.2. *Action on 1-Cells*

Suppose that $\mathbb{C}$ and $\mathbb{C}'$ have finite limits and that $F : \mathbb{C} \to \mathbb{C}'$ preserves them. As described in the previous section, $\mathbb{C}$ and $\mathbb{C}'$ give rise to cwfs $(\mathbb{C}, T_{\mathbb{C}})$ and $(\mathbb{C}', T_{\mathbb{C}'})$. In order to extend F to a pseudo cwf-morphism, we need to define, for each object Γ in $\mathbb{C}$, a **Fam**-morphism $(\sigma_F)_\Gamma : T_{\mathbb{C}}(\Gamma) \to T_{\mathbb{C}'}F(\Gamma)$. Unfortunately, unless F is full, it does not seem possible to embed faithfully a functorial family $\overrightarrow{A} : \mathbb{C}/\Gamma \to \mathbb{C}^{\to}$ into a functorial family over $F\Gamma$ in $\mathbb{C}'$. However, there is such an embedding for display maps (just apply F) from which we will freely regenerate a functorial family from the obtained display map.

The "hat" construction. As remarked by Hofmann, any morphism $f : \Delta \to \Gamma$ in a category $\mathbb{C}$ with a (not necessarily split) choice of finite limits generates a functorial family $\widehat{f} : \mathbb{C}/\Gamma \to \mathbb{C}^{\to}$. If $\delta : \Delta \to \Gamma$ then $\widehat{f}(\delta) = \delta^*(f)$, where $\delta^*(f)$ is obtained by taking the pullback of f along δ (δ^* is known as the *pullback functor*):

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "\delta^*(f)"'] & \cdot \arrow[d, "f"] \\ \Delta \arrow[r, "\delta"'] & \Gamma \end{tikzcd}$$

Note that we can always choose pullbacks such that $\widehat{f}(\mathrm{id}_\Gamma) = \mathrm{id}_\Gamma^*(f) = f$. If $\begin{tikzcd}[column sep=small] \Omega \arrow[rr, "\alpha"] \arrow[dr, "\delta\alpha"'] & & \Delta \arrow[dl, "\delta"] \\ & \Gamma & \end{tikzcd}$ is a morphism in $\mathbb{C}/\Gamma$, we define $\widehat{f}(\alpha)$ as the left square in the following diagram:

$$\begin{tikzcd} \cdot \arrow[r, "\widehat{f}(\delta,\alpha)"] \arrow[d, "\widehat{f}(\delta\alpha)"'] & \cdot \arrow[r] \arrow[d, "\widehat{f}(\delta)"] & \cdot \arrow[d, "f"] \\ \Delta' \arrow[r, "\alpha"'] & \Delta \arrow[r, "\delta"'] & \Gamma \end{tikzcd}$$

This is a pullback, since both the outer square and the right square are pullbacks.

Translation of types. The hat construction can be used to extend F to types:

$$\sigma_F(\overrightarrow{A}) = \widehat{F(\overrightarrow{A}(\mathrm{id}))}$$

Note that $F(\Gamma{\cdot}\overrightarrow{A}) = F(\mathrm{dom}(\overrightarrow{A}(\mathrm{id}))) = \mathrm{dom}(F(\overrightarrow{A}(\mathrm{id}))) = \mathrm{dom}(\sigma_\Gamma(\overrightarrow{A})(\mathrm{id})) = F\Gamma\sigma_\Gamma(\overrightarrow{A})$, so context comprehension is preserved on the nose. However, substitution on types is *not* preserved on the nose. Hence we have to define a coherent family of isomorphisms $\theta_{\overrightarrow{A},\delta}$.

Completion of cwf-morphisms. Fortunately, whenever F preserves finite limits there is a canonical way to generate all the remaining data.

Lemma 8 (Generation of isomorphisms). Let $(\mathbb{C}, T)$ and $(\mathbb{C}', T')$ be two cwfs, $F : \mathbb{C} \to \mathbb{C}'$ a functor preserving finite limits, $\sigma_\Gamma : \mathrm{Type}(\Gamma) \to \mathrm{Type}'(F\Gamma)$ a family of functions, and $\rho_{\Gamma,A} : F(\Gamma{\cdot}A) \to F\Gamma \cdot \sigma_\Gamma(A)$ a family of isomorphisms such that $\mathrm{p}\rho_{\Gamma,A} = F\mathrm{p}$. Then there exists a unique choice of functions σ_Γ^A on terms and of isomorphisms $\theta_{A,\delta}$ such that (F, σ) is a pseudo cwf-morphism.

Proof. By item *(2)* of Proposition 5, the unique way to extend σ to terms is to set $\sigma_\Gamma^A(a) = \mathrm{q}[\rho_{\Gamma,A}F(\langle \mathrm{id}, a\rangle)]$. To generate θ, we use the two squares below:

$$\begin{tikzcd} F\Delta{\cdot}\sigma_\Gamma(A)[F\delta] \arrow[r, "\langle (F\delta)\mathrm{p},\mathrm{q}\rangle"] \arrow[d, "\mathrm{p}"'] & F\Gamma{\cdot}\sigma_\Gamma(A) \arrow[d, "\mathrm{p}"] \\ F\Delta \arrow[r, "F\delta"'] & F\Gamma \end{tikzcd} \quad \begin{tikzcd} F\Delta{\cdot}\sigma_\Delta(A[\delta]) \arrow[r, "\rho_{\Gamma,A}F(\langle\delta\mathrm{p},\mathrm{q}\rangle)\rho^{-1}_{\Delta,A[\delta]}"] \arrow[d, "\mathrm{p}"'] & F\Gamma{\cdot}\sigma_\Gamma(A) \arrow[d, "\mathrm{p}"] \\ F\Delta \arrow[r, "F\delta"'] & F\Gamma \end{tikzcd}$$

The first square is a substitution pullback. The second is a pullback because F preserves finite limits and $\rho_{\Gamma,A}$ and $\rho_{\Delta,A[\delta]}$ are isomorphisms. The isomorphism $\theta_{A,\delta}$ is defined as the unique mediating morphism from the first to the second. It follows from the universal property of pullbacks that the family θ satisfies the necessary naturality and coherence conditions. There is no other choice for $\theta_{A,\delta}$, because if (F, σ) is a pseudo cwf-morphism with families of isomorphisms θ and ρ, then $\rho_{\Gamma,A}F(\langle\delta\mathrm{p},\mathrm{q}\rangle)\rho^{-1}_{\Delta,A[\delta]}\theta_{A,\delta} = \langle (F\delta)\mathrm{p}, \mathrm{q}\rangle$. Hence if F preserves finite limits, $\theta_{A,\delta}$ must coincide with the mediating morphism.

We must now check that θ, defined as above, satisfies the necessary coherence and naturality conditions. Clearly, θ commutes with the projections and satisfies $\theta_{A,\mathrm{id}} = \mathrm{id}$. We must also check that it satisfies the necessary coherence and naturality conditions, which will be a consequence of the universal property of the pullback above. For the coherence condition, assume that we have $\alpha : \Omega \to \Delta$ and $\delta : \Delta \to \Gamma$. Then $\theta_{A[\delta],\alpha}\boldsymbol{T'}(F\alpha)(\theta_{A,\delta})$ is an morphism from $\Omega\cdot\sigma_\Gamma(A)[F(\delta\alpha)]$ to $\Omega\cdot\sigma_\Omega(A[\delta\alpha])$. Recall that $\theta_{A,\delta\alpha}$ has been defined by the universal property of the pullback

$$\begin{tikzcd}
F\Omega\cdot\sigma_\Omega(A[\delta\alpha]) \arrow[r, "\rho_{\Gamma,A}F(\langle\delta\alpha\mathrm{p},\mathrm{q}\rangle)\rho^{-1}_{\Omega,A[\delta\alpha]}"] \arrow[d, "\mathrm{p}"'] & F\Gamma\cdot\sigma_\Gamma(A) \arrow[d, "\mathrm{p}"] \\
F\Delta \arrow[r, "F\delta"'] & F\Gamma
\end{tikzcd}$$

and more precisely as the unique morphism $h : F\Omega\cdot\sigma_\Gamma(A)[F(\delta\alpha)] \to F\Omega\cdot\sigma_\Omega(A[\delta\alpha])$ such that $\mathrm{p}h = \mathrm{p}$ and $\rho_{\Gamma,A}F(\langle\delta\alpha\mathrm{p},\mathrm{q}\rangle)\rho^{-1}_{\Omega,A[\delta\alpha]}h = \langle(F(\delta\alpha))\mathrm{p},\mathrm{q}\rangle$. Therefore, the required equality boils down to the two equations:

$$\begin{aligned}
\mathrm{p}\theta_{A[\delta],\alpha}\boldsymbol{T'}(F\alpha)(\theta_{A,\delta}) &= \mathrm{p} & (3)\\
\rho_{\Gamma,A}F(\langle\delta\alpha\mathrm{p},\mathrm{q}\rangle)\rho^{-1}_{\Omega,A[\delta\alpha]}\theta_{A[\delta],\alpha}\boldsymbol{T'}(F\alpha)(\theta_{A,\delta}) &= \langle(F(\delta\alpha))\mathrm{p},\mathrm{q}\rangle & (4)
\end{aligned}$$

Equation (3) is clear by property of $\theta_{A[\delta],\alpha}$ and construction of $(F\alpha)^*\theta_{A,\delta}$, while equation (4) is a consequence of Lemma 1.

It remains to prove that θ satisfies the naturality condition. Let $f : A \to B$ be a morphism in $\boldsymbol{T}(\Gamma)$. We need to establish the following equality:

$$\boldsymbol{\sigma}_\Delta(\boldsymbol{T}(\delta)(f))\theta_{A,\delta} = \theta_{B,\delta}\boldsymbol{T'}(F\delta)(\boldsymbol{\sigma}_\Gamma(f))$$

It follows from the fact that both sides of this equation make the two triangles commute in the following diagram, which is a pullback diagram because F preserves finite limits.

$$\begin{tikzcd}
F\Delta\cdot\sigma_\Gamma(A)[F\delta] \arrow[dr, dotted] \arrow[drr, bend left, "\boldsymbol{\sigma}_\Gamma(f)\langle(F\delta)\mathrm{p},\mathrm{q}\rangle"] \arrow[ddr, bend right, "\mathrm{p}"'] & & \\
& F\Delta\cdot\sigma_\Delta(B[\delta]) \arrow[r, "\rho_{\Gamma,B}F(\langle\delta\mathrm{p},\mathrm{q}\rangle)\rho^{-1}_{\Delta,B[\delta]}"] \arrow[d, "\mathrm{p}"] & F\Gamma\cdot\sigma_\Gamma(B) \arrow[d, "\mathrm{p}"] \\
& F\Delta \arrow[r, "F\delta"'] & F\Gamma
\end{tikzcd}$$

The proof that the two triangles commute only involves the definition of $\theta_{A,\delta}$ and $\theta_{B,\delta}$, along with manipulation of cwf combinators. This ends the proof that (F,σ) is a weak cwf-morphism. □

Preservation of additional structure. As a pseudo cwf-morphism, (F,σ_F) automatically preserves Σ-types.

Since the democratic structure of $(\mathbb{C}, T_\mathbb{C})$ and $(\mathbb{C}', T_{\mathbb{C}'})$ is trivial it is easy to prove that it is preserved by (F,σ_F):

Proposition 12. If $F : \mathbb{C} \to \mathbb{C}'$ preserves finite limits, then σ_F preserves democracy.

Proof. The functor F preserves finite limits and thus preserves the terminal object. Let $\iota : [] \to F[]$ denote the inverse to the terminal projection. Note that since the two involved cwfs have been built with Hofmann's construction, their democratic structure is trivial; we have $[]\cdot\overline{\Gamma} = \Gamma$ and $\gamma_\Gamma = \mathrm{id}$. In particular, we have $F([]\cdot\overline{\Gamma}) = F(\Gamma) = []\cdot\overline{F\Gamma}$. Thus to get preservation of the democratic structure, it is natural to choose:

$$d_\Gamma = \langle \iota, \mathrm{q}\rangle \rho^{-1}_{[],\overline{\Gamma}} : []\cdot\sigma_{[]}(\overline{\Gamma}) \to []\cdot\overline{F\Gamma}[\langle\rangle]$$

which makes the coherence condition essentially trivial. □

Moreover, type constructors are also preserved.

Proposition 13. If $(F,\sigma) : (\mathbb{C},T) \to (\mathbb{C}',T')$ such that $(\mathbb{C},T)$ and $(\mathbb{C}',T')$ supports identity types and F preserves finite limits, then σ preserves identity types.

Proof. Let $A \in \mathit{Type}(\Gamma)$, and $a, a' \in \Gamma \vdash A$, then $\Gamma\cdot \mathrm{I}_A(a,a')$ along with its projection to Γ is an equalizer of $\langle \mathrm{id}, a\rangle$ and $\langle \mathrm{id}, a'\rangle$. Indeed if $\delta : \Delta \to \Gamma$ such that $\langle \mathrm{id}, a\rangle\delta = \langle \mathrm{id}, a'\rangle\delta$, it is straightforward to see that the morphism $h = \langle \delta, \mathrm{r}_{A[\delta],a[\delta]}\rangle$ typechecks and satisfies $\mathrm{p}h = \delta$. It is also the unique such morphism because of the uniqueness of identity proofs. But F is left exact and in particular preserves equalizers, hence the pair $(F(\Gamma\mathrm{I}_A(a,a')), F(\mathrm{p}))$ defines an equalizer of $F(\langle \mathrm{id}, a\rangle) = \rho^{-1}_{\Gamma,A}\langle \mathrm{id}, \sigma^A_\Gamma(a)\rangle$ and $F(\langle \mathrm{id}, a'\rangle) = \rho^{-1}_{\Gamma,A}\langle \mathrm{id}, \sigma^A_\Gamma(a')\rangle$. From this it is obvious that the pair $(F\Gamma\cdot\sigma_\Gamma(A), \mathrm{p})$ is an equalizer of $\langle \mathrm{id}, \sigma^A_\Gamma(a)\rangle$ and $\langle \mathrm{id}, \sigma^A_\Gamma(a')\rangle$. But for the same reason as in the beginning of the proof, the pair $(F\Gamma\cdot\mathrm{I}_{\sigma_\Gamma(A)}(\sigma^A_\Gamma(a), \sigma^A_\Gamma(a')), \mathrm{p})$ is already such an equalizer, therefore they must be isomorphic and (F,σ) preserves identity types. □

To prove that (F,σ) preserves Π-types, whenever F preserves lccc structure, we shall introduce the notion of a Π-*object*.

Definition 12. If $(\mathbb{C},T)$ is a cwf (not necessarily supporting Π-types), Γ a context in $\mathbb{C}$ and $A \in \mathrm{Type}(\Gamma)$ and $B \in \mathrm{Type}(\Gamma\cdot A)$, a Π-*object* of A and B is a type $\Pi(A,B)$ such that for all terms $b : \Gamma\cdot A \vdash B$ there is $\lambda(b) : \Gamma \vdash \Pi(A,B)$, and for all $c : \Gamma \vdash \Pi(A,B)$ and $a : \Gamma \vdash A$ there is $\mathrm{ap}(c,a) : \Gamma \vdash B[\langle \mathrm{id}, a\rangle]$ satisfying the computation rules for Π-types (but no requirements w.r.t. substitution). Then it is straightforward to check that just as exponentials $A \Rightarrow B$ of A and B are unique up to isomorphism, Π-objects of A and B are unique up to type isomorphism.

This notion extends to categories by relating them to the cwf built with Hofmann's construction: if $\mathbb{C}$ is a category with a terminal object, if $g : A \to B$ and $f : B \to C$ are morphisms in $\mathbb{C}$, then a Π-*object* of f and g is a morphism $\Pi(f,g) : D \to C$ such that $\widehat{\Pi(f,g)}$ is a Π-object of $\widehat{f}$ and $\widehat{g}$ in $(\mathbb{C}, T_\mathbb{C})$.

Lemma 9. The two notions of Π-objects coincide: if $(\mathbb{C},T)$ is a cwf, Γ a context in $\mathbb{C}$ and $A \in \mathrm{Type}(\Gamma)$ and $B \in \mathrm{Type}(\Gamma\cdot A)$, then a type C is a Π-object of A and B if and only if p_C is a Π-object of p_A and p_B.

Proof. This follows from the correspondence between terms of type A and sections of p_A. □

Lemma 10. If $\mathbb{C}$ is a category with a terminal object, if $g : A \to B$ and $f : B \to C$ are morphisms in $\mathbb{C}$, then there is only one Π-object of f and g up to isomorphism in $\mathbb{C}/C$.

Proof. The proof mimics the proof of uniqueness of exponential objects. □

Lemma 11. If $\mathbb{C}$ and $\mathbb{C}'$ are lcccs and $F : \mathbb{C} \to \mathbb{C}'$ is a functor preserving finite limits, then if F preserves the lccc structure, then it preserves Π-objects.

Proof. Recall that if $\mathbb{C}$ is locally cartesian closed and if $g : A \to B$ and $f : B \to C$ are morphisms in $\mathbb{C}$, then there is a morphism $\Pi_f(g) : - \to \Gamma$, obtained by the following pullback in $\mathbb{C}/C$:

$$\begin{tikzcd} \Pi_f(g) \arrow[r] \arrow[d] & (gf)^f \arrow[d, "g^f"] \\ 1 \arrow[r, "\Lambda(\mathrm{id})"] & f^f \end{tikzcd}$$

this extends to a functor $\Pi_f : \mathbb{C}/B \to \mathbb{C}/C$, which is right adjoint to the pullback functor f^*. Exploiting this adjunction, it is straightforward to prove that $\Pi_f(f)$ is a Π-object of f and g in $\mathbb{C}$. By uniqueness, it is *the* Π-object of f and g, up to isomorphism. But since F preserves lccc structure it preserves pullbacks and local exponentiation, thus it maps (up to isomorphism) this pullback diagram into the following pullback:

$$\begin{tikzcd} F(\Pi_f(g)) \arrow[r] \arrow[d] & (F(g)F(f))^{F(f)} \arrow[d, "F(g)^{F(f)}"] \\ 1 \arrow[r, "\Lambda(\mathrm{id})"] & F(f)^{F(f)} \end{tikzcd}$$

So $F(\Pi_f(g))$ is as required a Π-object of $F(f)$ and $F(g)$. □

Proposition 14. If $(F, \sigma) : (\mathbb{C}, T) \to (\mathbb{C}', T')$ such that $(\mathbb{C}, T)$ and $(\mathbb{C}', T')$ supports Π-types and F preserves lccc structure, then σ preserves Π-types.

Proof. Let Γ be a context of $\mathbb{C}$, $A \in \mathrm{Type}(\Gamma)$ and $B \in \mathrm{Type}(\Gamma \cdot A)$, obviously $\Pi(A, B)$ is a Π-object of A and B. Hence, $\mathrm{p}_{\Pi(A,B)}$ is a Π-object of p_A and p_B by Lemma 9. But F preserves Π-objects by Lemma 11, so $F(\mathrm{p}_{\Pi(A,B)})$ is a Π-object of $F(\mathrm{p}_A)$ and $F(\mathrm{p}_B)$. But $F(\mathrm{p}_A)$ is isomorphic to $\mathrm{p}_{\sigma_\Gamma(A)}$ (the isomorphism being $\rho_{\Gamma,A}$) and $F(\mathrm{p}_B)$ is isomorphic to $\mathrm{p}_{\sigma_{\Gamma\cdot A}(B)[\rho_{\Gamma,A}]}$ (the isomorphism being $\langle \rho^{-1}_{\Gamma,A}\mathrm{p}, q\rangle \rho^{-1}_{\Gamma\cdot A,B}$), therefore $F(\mathrm{p}_{\Pi(A,B)})$ is a Π-object of $\mathrm{p}_{\sigma_\Gamma(A)}$ and $\mathrm{p}_{\sigma_{\Gamma\cdot A}(B)[\rho_{\Gamma,A}]}$, hence it must be isomorphic to $\mathrm{p}_{\Pi(\sigma_\Gamma(A), \sigma_{\Gamma\cdot A}(B)[\rho_{\Gamma,A}])}$ by Lemma 10, so we have the required isomorphism $F(\Gamma \cdot \Pi(A, B)) \to F\Gamma \cdot \Pi(\sigma_\Gamma(A), \sigma_{\Gamma\cdot A}(B)[\rho_{\Gamma,A}])$. □

5.3. *Action on 2-Cells*

In a similar way as for 1-cells, we shall show that under certain conditions a natural transformation $\phi : F \xrightarrow{\bullet} G$, where (F, σ) and (G, τ) are pseudo cwf-morphisms, can be completed to a pseudo cwf-transformation (ϕ, ψ_ϕ).

Lemma 12 (Completion of pseudo cwf-transformations). Suppose (F, σ) and (G, τ) are pseudo cwf-morphisms from $(\mathbb{C}, T)$ to $(\mathbb{C}', T)$ such that F and G preserve finite limits and $\phi : F \xrightarrow{\bullet} G$ is a natural transformation, then there exists a family of morphisms $(\psi_\phi)_{\Gamma,A} : \sigma_\Gamma(A) \to \tau_\Gamma(A)[\phi_\Gamma]$ such that (ϕ, ψ_ϕ) is a pseudo cwf-transformation from (F, σ) to (G, τ).

Proof. We set $\psi_{\Gamma,A} = \langle \mathrm{p}, \mathrm{q}[\rho'_{\Gamma,A}\phi_{\Gamma\cdot A}\rho^{-1}_{\Gamma,A}]\rangle : F\Gamma \cdot \sigma_\Gamma A \to F\Gamma \cdot \tau_\Gamma(A)[\phi_\Gamma]$. To check the coherence law, we apply the universal property of a well-chosen pullback square (exploiting the fact that G preserves finite limits).

$$\begin{tikzcd}
F\Delta\cdot\tau_\Delta(A[\delta])[\phi_\Delta] \arrow[r, "\langle\phi_\Delta \mathrm{p},\mathrm{q}\rangle"] \arrow[d, "\mathrm{p}"] & G\Delta\cdot\tau_\Delta(A[\delta]) \arrow[r, "\rho'_{\Gamma,A}G(\langle\delta\mathrm{p},\mathrm{q}\rangle)(\rho'_{\Delta,A[\delta]})^{-1}"] \arrow[d, "\mathrm{p}"] & G\Gamma\cdot\tau_\Gamma(A) \arrow[d, "\mathrm{p}"] \\
F\Delta \arrow[r, "\phi_\Delta"'] & G\Delta \arrow[r, "G\delta"'] & G\Gamma
\end{tikzcd}$$

The two paths $\boldsymbol{T'}(\phi_\Delta)(\theta'_{A,\delta})\boldsymbol{T'}(F\delta)(\psi_{\Gamma,A})$ and $\psi_{\Delta,A[\delta]}\theta_{A,\delta}$ of the coherence diagram behave in the same way with respect to this pullback. Here is the calculation for the first path of the coherence diagram:

$$\begin{array}{rl}
& \rho'_{\Gamma,A}G(\langle\delta\mathrm{p},\mathrm{q}\rangle)(\rho'_{\Delta,A[\delta]})^{-1}\langle\phi_\Delta\mathrm{p},\mathrm{q}\rangle\boldsymbol{T'}(\phi_\Delta)(\theta'_{A,\delta})\boldsymbol{T'}(F\delta)(\psi_{\Gamma,A}) \\
= & \langle(G\delta)\mathrm{p},\mathrm{q}\rangle{\theta'_{A,\delta}}^{-1}\langle\phi_\Delta\mathrm{p},\mathrm{q}\rangle\boldsymbol{T'}(\phi_\Delta)(\theta'_{A,\delta})\boldsymbol{T'}(F\delta)(\psi_{\Gamma,A}) \\
= & \langle(G\delta)\mathrm{p},\mathrm{q}\rangle{\theta'_{A,\delta}}^{-1}\langle\phi_\Delta\mathrm{p},\mathrm{q}\rangle\langle\mathrm{p},\mathrm{q}[\theta'_{A,\delta}\langle\phi_\Delta\mathrm{p},\mathrm{q}\rangle\rangle\boldsymbol{T'}(F\delta)(\psi_{\Gamma,A}) \\
= & \langle(G\delta)\mathrm{p},\mathrm{q}\rangle{\theta'_{A,\delta}}^{-1}\langle\phi_\Delta p, q[\theta'_{A,\delta}\langle\phi_\Delta\mathrm{p},\mathrm{q}\rangle\rangle\boldsymbol{T'}(F\delta)(\psi_{\Gamma,A}) \\
= & \langle(G\delta)\mathrm{p},\mathrm{q}\rangle{\theta'_{A,\delta}}^{-1}\langle\mathrm{p}\theta'_{A,\delta}\langle\phi_\Delta\mathrm{p},\mathrm{q}\rangle, q[\theta'_{A,\delta}\langle\phi_\Delta\mathrm{p},\mathrm{q}]\rangle\rangle\boldsymbol{T'}(F\delta)(\psi_{\Gamma,A}) \\
= & \langle(G\delta)\mathrm{p},\mathrm{q}\rangle\langle\phi_\Delta\mathrm{p},\mathrm{q}\rangle\boldsymbol{T'}(F\delta)(\psi_{\Gamma,A}) \\
= & \langle(G\delta)\mathrm{p},\mathrm{q}\rangle\langle\phi_\Delta\mathrm{p},\mathrm{q}\rangle\langle\mathrm{p},\mathrm{q}[\psi_{\Gamma,A}\langle(F\delta)\mathrm{p},\mathrm{q}\rangle]\rangle \\
= & \langle(G\delta)\phi_\Delta\mathrm{p},\mathrm{q}[\rho'_{\Gamma,A}\phi_{\Gamma\cdot A}\rho^{-1}_{\Gamma,A}\langle(F\delta)\mathrm{p},\mathrm{q}\rangle]\rangle \\
= & \langle\phi_\Gamma(F\delta)\mathrm{p},\mathrm{q}[\rho'_{\Gamma,A}\phi_{\Gamma\cdot A}\rho^{-1}_{\Gamma,A}\langle(F\delta)\mathrm{p},\mathrm{q}\rangle]\rangle \\
= & \langle\phi_\Gamma\mathrm{p},\mathrm{q}[\rho'_{\Gamma,A}\phi_{\Gamma\cdot A}\rho^{-1}_{\Gamma,A}]\rangle\langle(F\delta)\mathrm{p},\mathrm{q}\rangle \\
= & \rho'_{\Gamma,A}\phi_{\Gamma\cdot A}\rho^{-1}_{\Gamma,A}\langle(F\delta)\mathrm{p},\mathrm{q}\rangle
\end{array}$$

where we use Lemma 1, then only the definition of $\psi_{\Gamma,A}$, naturality of ϕ and manipulation of cwf combinators. The calculation for the other path follows:

$$\begin{aligned}
& \rho'_{\Gamma,A} G(\langle \delta \mathrm{p}, \mathrm{q}\rangle) {\rho'_{\Delta,A[\delta]}}^{-1} \langle \phi_\Delta \mathrm{p}, \mathrm{q}\rangle \psi_{\Delta,A[\delta]} \theta_{A,\delta} \\
= \; & \rho'_{\Gamma,A} G(\langle \delta \mathrm{p}, \mathrm{q}\rangle) {\rho'_{\Delta,A[\delta]}}^{-1} \langle \phi_\Delta \mathrm{p}, \mathrm{q}\rangle \langle \mathrm{p}, \mathrm{q}[\rho'_{\Delta,A[\delta]} \phi_{\Delta\cdot A[\delta]} \rho^{-1}_{\Delta,A[\delta]}]\rangle \theta_{A,\delta} \\
= \; & \rho'_{\Gamma,A} G(\langle \delta \mathrm{p}, \mathrm{q}\rangle) {\rho'_{\Delta,A[\delta]}}^{-1} \langle \phi_\Delta \mathrm{p}, \mathrm{q}[\rho'_{\Delta,A[\delta]} \phi_{\Delta\cdot A[\delta]} \rho^{-1}_{\Delta,A[\delta]}]\rangle \theta_{A,\delta} \\
= \; & \rho'_{\Gamma,A} G(\langle \delta \mathrm{p}, \mathrm{q}\rangle) {\rho'_{\Delta,A[\delta]}}^{-1} \langle \mathrm{p} \rho'_{\Delta,A[\delta]} \phi_{\Delta\cdot A[\delta]} \rho^{-1}_{\Delta,A[\delta]}, \mathrm{q}[\rho'_{\Delta,A[\delta]} \phi_{\Delta\cdot A[\delta]} \rho^{-1}_{\Delta,A[\delta]}]\rangle \theta_{A,\delta} \\
= \; & \rho'_{\Gamma,A} G(\langle \delta \mathrm{p}, \mathrm{q}\rangle) {\rho'_{\Delta,A[\delta]}}^{-1} \rho'_{\Delta,A[\delta]} \phi_{\Delta\cdot A[\delta]} \rho^{-1}_{\Delta,A[\delta]} \theta_{A,\delta} \\
= \; & \rho'_{\Gamma,A} G(\langle \delta \mathrm{p}, \mathrm{q}\rangle) \phi_{\Delta\cdot A[\delta]} \rho^{-1}_{\Delta,A[\delta]} \theta_{A,\delta} \\
= \; & \rho'_{\Gamma,A} \phi_{\Gamma\cdot A} F(\langle \delta \mathrm{p}, \mathrm{q}\rangle) \rho^{-1}_{\Delta,A[\delta]} \theta_{A,\delta} \\
= \; & \rho'_{\Gamma,A} \phi_{\Gamma\cdot A} \rho^{-1}_{\Gamma,A} \rho_{\Gamma,A} F(\langle \delta \mathrm{p}, \mathrm{q}\rangle) \rho^{-1}_{\Delta,A[\delta]} \theta_{A,\delta} \\
= \; & \rho'_{\Gamma,A} \phi_{\Gamma\cdot A} \rho^{-1}_{\Gamma,A} \langle (F\delta)\mathrm{p}, \mathrm{q}\rangle
\end{aligned}$$

We have used naturality of ϕ, preservation of the first projection by (F, σ) and (G, τ) and manipulations on cwf combinators. □

This completion operation on 2-cells commutes with units and both notions of composition, as will be crucial to prove pseudofunctoriality of H:

Lemma 13. Completion of pseudo cwf-transformations commutes with both notions of composition. More precisely, if $\phi : F \overset{\bullet}{\to} G$ and $\phi' : G \overset{\bullet}{\to} H$, then

$$(\phi', \psi_{\phi'}) \bullet (\phi, \psi_\phi) = (\phi' \bullet \phi, \psi_{\phi' \bullet \phi})$$

Likewise if $\phi : F \overset{\bullet}{\to} G$ and $\phi' : F' \to G'$,

$$(\phi', \psi_{\phi'})(\phi, \psi_\phi) = (\phi'\phi, \psi_{\phi'\phi})$$

Finally, for all pseudo cwf-morphism (F, σ) we have $1_{(F,\sigma)} = (1_F, \psi_{1_F})$.

Proof. The first equality is a straightforward verification. The second requires a more involved calculation similar to the one used to prove Lemma 12 and we only show the third case in detail. Assume that we have the following situation:

$$\begin{tikzcd}
(\mathbb{C}, T) \arrow[r, shift left, "{(F,\sigma)}"] \arrow[r, shift right, "{(G,\tau)}"'] & (\mathbb{C}', T') \arrow[r, shift left, "{(F',\sigma')}"] \arrow[r, shift right, "{(G',\tau')}"'] & (\mathbb{C}'', T'')
\end{tikzcd}$$

Let us call θ and ρ the components of (F, σ), θ' and ρ' the components of (F', σ'), t and r the components of (G, τ) and t' and r' the components of (G', τ'). Let us also consider natural transformations $\phi : F \overset{\bullet}{\to} G$ and $\phi' : F' \overset{\bullet}{\to} G'$. Let us recall that the vertical composition of pseudo cwf-transformations follow those of 2-cells in the 2-category of indexed categories over arbitrary bases, which means $(\phi, \psi_\phi)(\phi', \psi_{\phi'}) = (\phi\phi', m)$, where $m_{\Gamma,A}$ is obtained by:

$$\sigma'_{F\Gamma}(\sigma_\Gamma A) \xrightarrow{\boldsymbol{\sigma'}_{F\Gamma}((\psi_\phi)_{\Gamma,A})} \sigma'_{F\Gamma}(\tau_\Gamma A[\phi_\Gamma]) \xrightarrow{{\theta'_{\tau_\Gamma A,\phi_\Gamma}}^{-1}} \sigma'_{G\Gamma}(\tau_\Gamma A)[F'\phi_\Gamma] \xrightarrow{\boldsymbol{T''}(F'\phi_\Gamma)((\psi_{\phi'})_{G\Gamma,\tau_\Gamma A})} \tau'_{G\Gamma}(\tau_\Gamma A)[\phi'_{G\Gamma} F'(\phi_\Gamma)]$$

which the following calculation relates to $(\psi_{\phi\phi'})_{\Gamma,A}$:

$$
\begin{aligned}
m_{\Gamma,A} &= \boldsymbol{T''}(F'\phi_\Gamma)((\psi_{\phi'})_{G\Gamma,\tau_\Gamma A})\theta'^{-1}_{\tau_\Gamma A,\phi_\Gamma}\boldsymbol{\sigma'}_{F\Gamma}((\psi_\phi)_{\Gamma,A})\\
&= \langle \mathrm{p}, \mathrm{q}[(\psi_{\phi'})_{G\Gamma,\tau_\Gamma A}\langle (F'\phi_\Gamma)\mathrm{p}, \mathrm{q}\rangle]\rangle \theta'^{-1}_{\tau_\Gamma A,\phi_\Gamma}\rho'_{F\Gamma,\tau_\Gamma A[\phi_\Gamma]}F'((\psi_\phi)_{\Gamma,A})\rho'^{-1}_{F\Gamma,\sigma_\Gamma A}\\
&= \langle \mathrm{p}, \mathrm{q}[(\psi_{\phi'})_{G\Gamma,\tau_\Gamma A}\langle (F'\phi_\Gamma)\mathrm{p}, \mathrm{q}\rangle \theta'^{-1}_{\tau_\Gamma A,\phi_\Gamma}\rho'_{F\Gamma,\tau_\Gamma A[\phi_\Gamma]}F'((\psi_\phi)_{\Gamma,A})\rho'^{-1}_{F\Gamma,\sigma_\Gamma A}]\rangle\\
&= \langle \mathrm{p}, \mathrm{q}[(\psi_{\phi'})_{G\Gamma,\tau_\Gamma A}\rho'_{G\Gamma,\tau_\Gamma A}F'(\langle \phi_\Gamma \mathrm{p}, \mathrm{q}\rangle)\rho'^{-1}_{F\Gamma,\tau_\Gamma A[\phi_\Gamma]}\rho'_{F\Gamma,\tau_\Gamma A[\phi_\Gamma]}F'((\psi_\phi)_{\Gamma,A})\rho'^{-1}_{F\Gamma,\sigma_\Gamma A}]\rangle\\
&= \langle \mathrm{p}, \mathrm{q}[(\psi_{\phi'})_{G\Gamma,\tau_\Gamma A}\rho'_{G\Gamma,\tau_\Gamma A}F'(\langle \phi_\Gamma \mathrm{p}, \mathrm{q}\rangle)F'((\psi_\phi)_{\Gamma,A})\rho'^{-1}_{F\Gamma,\sigma_\Gamma A}]\rangle\\
&= \langle \mathrm{p}, \mathrm{q}[\langle \mathrm{p}, \mathrm{q}[r'_{G\Gamma,\tau_\Gamma A}\phi'_{G\Gamma\tau_\Gamma A}\rho'^{-1}_{G\Gamma,\tau_\Gamma A}\rho'_{G\Gamma,\tau_\Gamma A}F'(\langle \phi_\Gamma \mathrm{p}, \mathrm{q}\rangle)F'((\psi_\phi)_{\Gamma,A})\rho'^{-1}_{F\Gamma,\sigma_\Gamma A}]\rangle]\rangle\\
&= \langle \mathrm{p}, \mathrm{q}[r'_{G\Gamma,\tau_\Gamma A}\phi'_{G\Gamma\tau_\Gamma A}F'(\langle \phi_\Gamma \mathrm{p}, \mathrm{q}\rangle)F'((\psi_\phi)_{\Gamma,A})\rho'^{-1}_{F\Gamma,\sigma_\Gamma A}]\rangle\\
&= \langle \mathrm{p}, \mathrm{q}[r'_{G\Gamma,\tau_\Gamma A}\phi'_{G\Gamma\tau_\Gamma A}F'(\langle \phi_\Gamma \mathrm{p}, \mathrm{q}\rangle r_{\Gamma,A}\phi_{\Gamma\cdot A}\rho^{-1}_{\Gamma,A})\rho'^{-1}_{F\Gamma,\sigma_\Gamma A}]\rangle\\
&= \langle \mathrm{p}, \mathrm{q}[r'_{G\Gamma,\tau_\Gamma A}\phi'_{G\Gamma\tau_\Gamma A}F'(r_{\Gamma,A}\phi_{\Gamma\cdot A}\rho^{-1}_{\Gamma,A})\rho'^{-1}_{F\Gamma,\sigma_\Gamma A}]\rangle\\
&= \langle \mathrm{p}, \mathrm{q}[r'_{G\Gamma,\tau_\Gamma A}G'(r_{\Gamma,A})\phi'_{G(\Gamma\cdot A)}F'(\phi_{\Gamma\cdot A})F'(\rho^{-1}_{\Gamma,A})\rho'^{-1}_{F\Gamma,\sigma_\Gamma A}]\rangle\\
&= \langle \mathrm{p}, \mathrm{q}[\rho^{G'G}_{\Gamma,A}(\phi\phi')_{\Gamma\cdot A}\rho^{F'F^{-1}}_{\Gamma,A}]\rangle\\
&= (\psi_{\phi\phi'})_{\Gamma,A}
\end{aligned}
$$

We have first unfolded the action of $\boldsymbol{T''}$ and $\boldsymbol{\sigma'}$, then applied Lemma 1, unfolded the definition of $\psi_{\phi'}$ and ψ_ϕ, then used naturality of ϕ' and ϕ. Of course there are many simplification steps, involving the preservation of the first projection by all the present pseudo cwf-morphisms and manipulation of cwf combinators.

Finally, the third equality follows from the remark that by definition of ψ_{1_F}, we have $(\psi_{1_F})_{\Gamma,A} = \mathrm{id}_{\Gamma\sigma_\Gamma A}$ for all Γ, A. □

5.4. *Pseudofunctoriality of H*

First note that H is *not* a functor, because for $F : \mathbb{C} \to \mathbb{D}$ with finite limits and functorial family $\overrightarrow{A}$ over Γ (in $\mathbb{C}$), $\sigma_\Gamma(\overrightarrow{A})$ forgets all information on $\overrightarrow{A}$ except its display map $\overrightarrow{A}(\mathrm{id})$, and later extends $F(\overrightarrow{A}(\mathrm{id}))$ to an independent functorial family.

However, we shall prove that $H : \mathbf{FL} \to \mathbf{CwF}^{\mathrm{I_{ext}}\Sigma}_{\mathrm{dem}}$ and $H : \mathbf{LCC} \to \mathbf{CwF}^{\mathrm{I_{ext}}\Sigma\Pi}_{\mathrm{dem}}$ are pseudo functors.

Proof. First, note that as proved in Lemma 13, H is functorial on 2-cells.

For each $\mathbb{C}$ we need an invertible 2-cell $H_\mathbb{C} : Id_{(\mathbb{C},T_\mathbb{C})} \to H(Id_\mathbb{C})$, this will be the identity 2-cell since we have in fact $H(Id_\mathbb{C}) = (Id_\mathbb{C}, \sigma_{Id_\mathbb{C}}) = Id_{(\mathbb{C},T_\mathbb{C})}$ by construction of $\sigma_{Id_\mathbb{C}}$.

For each two functors $F : \mathbb{C} \to \mathbb{D}$ and $G : \mathbb{D} \to \mathbb{E}$ we need an isomorphism $H_{F,G} : HG \circ HF \to H(G \circ F)$, natural in F and G. It is given by $H_{F,G} = (1_{GF}, \psi_{1_{GF}})$. The naturality condition amounts to the commutativity of the following square:

$$\begin{tikzcd}
(G,\sigma_G)(F,\sigma_F) \arrow[r, "(1_{GF},\psi_{1_{GF}})"] \arrow[d, "(\phi,\psi_\phi)(\phi',\psi_{\phi'})"] & (GF,\sigma_{GF}) \arrow[d, "(\phi'\phi,\psi_{\phi'\phi})"] \\
(G',\sigma_{G'})(F',\sigma_{F'}) \arrow[r, "(1_{GF},\psi_{1_{GF}})"] & (G'F',\sigma_{G'F'})
\end{tikzcd}$$

This is a direct consequence of Lemma 13. The coherence laws w.r.t. associativity of composition and identities also follow from Lemma 13. In fact, Lemma 12 implies that to check the validity of any equation involving vertical and horizontal compositions of pseudo cwf-transformations built with Lemma 12 and identity pseudo cwf-transformations, it suffices to check the equality of the corresponding base natural transformation, ignoring the modifications. □

6. The Biequivalences

Theorem 1. We have the following biequivalences of 2-categories.

$$\mathbf{FL} \underset{U}{\overset{H}{\rightleftarrows}} \mathbf{CwF}^{\mathrm{I_{ext}}\Sigma}_{\mathrm{dem}} \qquad\qquad \mathbf{LCC} \underset{U}{\overset{H}{\rightleftarrows}} \mathbf{CwF}^{\mathrm{I_{ext}}\Sigma\Pi}_{\mathrm{dem}}$$

Proof. Since $UH = \mathrm{Id}$ (the identity 2-functor) it suffices to construct pseudonatural transformations of pseudofunctors:

$$\mathrm{Id} \underset{\epsilon}{\overset{\eta}{\rightleftarrows}} HU$$

which are inverse up to invertible modifications. Since $HU(\mathbb{C},T) = (\mathbb{C},T^{\mathbb{C}})$, these pseudonatural transformations are families of equivalences of cwfs:

$$(\mathbb{C},T) \underset{\epsilon_{(\mathbb{C},T)}}{\overset{\eta_{(\mathbb{C},T)}}{\rightleftarrows}} (\mathbb{C},T^{\mathbb{C}})$$

which satisfy the required conditions for pseudonatural transformations.

Construction of $\eta_{(\mathbb{C},T)}$. Using Lemma 8, we just need to define a base functor, which will be $\mathrm{Id}_{\mathbb{C}}$, and a family σ^η_Γ which translates types (in the sense of T) to functorial families. This is easy, since types in the cwf $(\mathbb{C},T)$ come equipped with a chosen behaviour under substitution. Given $A \in \mathrm{Type}(\Gamma)$, we define:

$$\begin{array}{rcl}
\sigma^\eta_\Gamma(A)(\delta) & = & \mathrm{p}_{A[\delta]} \\
\sigma^\eta_\Gamma(A)(\delta,\gamma) & = & \langle \gamma\mathrm{p}, \mathrm{q}\rangle
\end{array}$$

For each pseudo cwf-morphism (F,σ), the pseudonaturality square relates two pseudo cwf-morphisms whose base functor is F. Hence, the necessary invertible pseudo cwf-transformation is obtained using Lemma 12 from the identity natural transformation on F. The coherence conditions are straightforward consequences of Lemma 12.

Construction of $\epsilon_{(\mathbb{C},T)}$. As for η, the base functor for $\epsilon_{(\mathbb{C},T)}$ is $\mathrm{Id}_{\mathbb{C}}$. Using Lemma 8 again we need, for each context Γ, a function $\sigma^{\epsilon}_{\Gamma}$ which given a functorial family $\overrightarrow{A}$ over Γ will build a syntactic type $\sigma^{\epsilon}_{\Gamma}(\overrightarrow{A}) \in \mathrm{Type}(\Gamma)$. In other terms, we need to find a syntactic representative of an arbitrary display map, that is, an arbitrary morphism in $\mathbb{C}$. We use the inverse image:

$$\sigma^{\epsilon}_{\Gamma}(\overrightarrow{A}) = \mathrm{Inv}(\overrightarrow{A}(\mathrm{id})) \in \mathrm{Type}(\Gamma)$$

The family ϵ is pseudonatural for the same reason as η above.

Invertible modifications. For each cwf $(\mathbb{C}, T)$, we need to define invertible pseudo cwf-transformations $m_{(\mathbb{C},T)} : (\epsilon\eta)_{(\mathbb{C},T)} \to \mathrm{id}_{(\mathbb{C},T)}$ and $m'_{(\mathbb{C},T)} : (\eta\epsilon)_{(\mathbb{C},T)} \to \mathrm{id}_{(\mathbb{C},T)}$. As pseudo cwf-transformations between pseudo cwf-morphisms with the same base functor, their first component will be the identity natural transformation, and the second will be generated by Lemma 12. The coherence law for modifications is a consequence of Lemma 12. □

7. Conclusion

The cwf morphism $\eta_{(\mathbb{C},T)}$ describes the *interpretation* of the cwf $(\mathbb{C}, T)$ into the cwf $(\mathbb{C}, T^{\mathbb{C}})$ obtained by the Bénabou-Hofmann construction. It is analogous to Hofmann's interpretation of a category with attributes in a lccc (Hofmann, 1994). Note that $\eta_{(\mathbb{C},T)}$ is a *strict* cwf morphism, although morphisms in the categories $\mathbf{CwF}^{\mathrm{I_{ext}}\Sigma}_{\mathrm{dem}}$ and $\mathbf{CwF}^{\mathrm{I_{ext}}\Sigma\Pi}_{\mathrm{dem}}$ in general are only required to be *pseudo* cwf-morphisms. The strictness of η is important for an interpretation, since it means that the laws of cwfs are preserved strictly and not only up to isomorphism.

In order to prove our result we were forced to consider categories of cwfs with pseudo cwf morphisms. We were unable to prove a (bi)equivalence with categories of cwfs and strict cwf morphisms. For example, there is no obvious candidate for a strict replacement for $\epsilon_{(\mathbb{C},T)}$ since we must then construct a syntactic type over Γ for each semantic type over Γ (that is, an object of the slice category over Γ). The need to consider pseudo cwf morphisms indicates that the connection between Martin-Löf type theory and locally cartesian closed categories is not as tight as for example the connection between the simply typed lambda calculus and cartesian closed categories, or between syntactically defined Martin-Löf type theories and cwfs. So is Martin-Löf type theory with extensional identity types, Σ- and Π-types, an internal language for lcccs? Yes, it is an internal language "up to isomorphism".

References

Bénabou, J. (1985). Fibred categories and the foundation of naive category theory. *Journal of Symbolic Logic*, 50:10–37.

Cartmell, J. (1986). Generalized algebraic theories and contextual categories. *Annals of Pure and Applied Logic*, 32:209–243.

Curien, P.-L. (1993). Substitution up to isomorphism. *Fundamenta Informaticae*, 19(1,2):51–86.

Dybjer, P. (1996). Internal type theory. In *TYPES '95, Types for Proofs and Programs*, number 1158 in Lecture Notes in Computer Science, pages 120–134. Springer.

Hofmann, M. (1994). On the interpretation of type theory in locally cartesian closed categories. In Pacholski, L. and Tiuryn, J., editors, *CSL*, volume 933 of *Lecture Notes in Computer Science*. Springer.

Hofmann, M. (1996). Syntax and semantics of dependent types. In Pitts, A. and Dybjer, P., editors, *Semantics and Logics of Computation*. Cambridge University Press.

Lawvere, F. W. (1970). Equality in hyperdoctrines and comprehension schema as an adjoint functor. In Heller, A., editor, *Applications of Categorical Algebra, Proceedings of Symposia in Pure Mathematics*. AMS.

Leinster, T. (1999). Basic bicategories. `arXiv:math/9810017v1`.

Martin-Löf, P. (1982). Constructive mathematics and computer programming. In *Logic, Methodology and Philosophy of Science, VI, 1979*, pages 153–175. North-Holland.

Martin-Löf, P. (1984). *Intuitionistic Type Theory*. Bibliopolis.

Martin-Löf, P. (1992). Substitution calculus. Notes from a lecture given in Göteborg.

Mimram, S. (2004). Decidability of equality in categories with families. Report, Magistère d'Informatique et Modelisation, École Normale Superieure de Lyon, `http://www.pps.jussieu.fr/~smimram/`.

Seely, R. (1984). Locally cartesian closed categories and type theory. *Math. Proc. Cambridge Philos. Soc.*, 95(1):33–48.

Tasistro, A. (1993). Formulation of Martin-Löf's theory of types with explicit substitutions. Technical report, Department of Computer Sciences, Chalmers University of Technology and University of Göteborg. Licentiate Thesis.

Taylor, P. (1999). *Practical Foundations of Mathematics*. Cambridge University Press.