

DSL: Presenting Mathematical Analysis Using Functional Programming

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Paper + talk: <https://github.com/DSLsofMath/>

Style example

$$\forall \epsilon \in \mathbb{R}. (\epsilon > 0) \Rightarrow \exists a \in A. (|a - \sup A| < \epsilon)$$

Domain-Specific Languages of Mathematics [Ionescu and Jansson, 2015]:
is a course currently developed at Chalmers in response to difficulties faced
by third-year students in learning and applying classical mathematics
(mainly real and complex analysis)

Main idea: encourage students to approach mathematical domains from a
functional programming perspective (similar to Wells [1995]).

“... ideally, the course would improve the mathematical education of
computer scientists and the computer science education of
mathematicians.”

- make functions and types explicit
- make the distinction between syntax and semantics explicit
- use types (\mathbb{R} , \mathbb{C}) as carriers of semantic information, not just variable names (x , z)
- introduce functions and types for implicit operations such as the power series interpretation of a sequence
- use a calculational style for proofs
- organize the types and functions in DSLs

Not working code, rather working understanding of concepts

We begin by defining the symbol i , called **the imaginary unit**, to have the property

$$i^2 = -1$$

Thus, we could also call i the square root of -1 and denote it $\sqrt{-1}$. Of course, i is not a real number; no real number has a negative square.

(Adams and Essex [2010], Appendix I)

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(Adams and Essex [2010], Appendix I)

data $i = i$

Definition: A **complex number** is an expression of the form

$$a + bi \quad \text{or} \quad a + ib,$$

where a and b are real numbers, and i is the imaginary unit.

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```
data Complex = Plus1 ℝ ℝ I
              | Plus2 ℝ I ℝ
```

```
show : Complex → String
```

```
show (Plus1 x y i) = show x ++ " + " ++ show y ++ "i"
```

```
show (Plus2 x i y) = show x ++ " + " ++ "i" ++ show y
```

Complex numbers examples

Definition: A **complex number** is an expression of the form

$$a + bi \quad \text{or} \quad a + ib,$$

where a and b are real numbers, and i is the imaginary unit.

For example, $3 + 2i$, $\frac{7}{2} - \frac{2}{3}i$, $i\pi = 0 + i\pi$, and $-3 = -3 + 0i$ are all complex numbers. The last of these examples shows that every real number can be regarded as a complex number.

Complex numbers examples

For example, $3 + 2i$, $\frac{7}{2} - \frac{2}{3}i$, $i\pi = 0 + i\pi$, and $-3 = -3 + 0i$ are all complex numbers. The last of these examples shows that every real number can be regarded as a complex number.

```
data Complex = Plus1 ℝ ℝ |  
                | Plus2 ℝ ℝ
```

```
toComplex : ℝ → Complex  
toComplex x = Plus1 x 0 i
```

- what about i by itself?
- what about, say, $2i$?

(We will normally use $a + bi$ unless b is a complicated expression, in which case we will write $a + ib$ instead. Either form is acceptable.)

data *Complex* = *Plus* \mathbb{R} \mathbb{R} *I*

data *Complex* = *PlusI* \mathbb{R} \mathbb{R}

It is often convenient to represent a complex number by a single letter; w and z are frequently used for this purpose. If a , b , x , and y are real numbers, and $w = a + bi$ and $z = x + yi$, then we can refer to the complex numbers w and z . Note that $w = z$ if and only if $a = x$ and $b = y$.

```
newtype Complex = C (ℝ, ℝ)
```

Equality and pattern-matching

Definition: If $z = x + yi$ is a complex number (where x and y are real), we call x the **real part** of z and denote it $Re(z)$. We call y the **imaginary part** of z and denote it $Im(z)$:

$$Re(z) = Re(x + yi) = x$$

$$Im(z) = Im(x + yi) = y$$

$$Re : Complex \rightarrow \mathbb{R}$$

$$Re\ z@(C(x, y)) = x$$

$$Im : Complex \rightarrow \mathbb{R}$$

$$Im\ z@(C(x, y)) = y$$

The sum and difference of complex numbers

If $w = a + bi$ and $z = x + yi$, where a , b , x , and y are real numbers, then

$$w + z = (a + x) + (b + y) i$$

$$w - z = (a - x) + (b - y) i$$

Shallow embedding:

$(+)$: *Complex* \rightarrow *Complex* \rightarrow *Complex*

$(C(a, b)) + (C(x, y)) = C((a + x), (b + y))$

newtype *Complex* = $C(\mathbb{R}, \mathbb{R})$

Shallow vs. deep embeddings

The sum and difference of complex numbers

If $w = a + bi$ and $z = x + yi$, where a , b , x , and y are real numbers, then

$$w + z = (a + x) + (b + y) i$$

$$w - z = (a - x) + (b - y) i$$

Deep embedding:

$(+)$: *Complex* \rightarrow *Complex* \rightarrow *Complex*

$(+)$ = *Plus*

data *Complex* = *i*

```
| ToComplex  $\mathbb{R}$   
| Plus   Complex Complex  
| Times Complex Complex  
| ...
```

Example: continuity defined in terms of limits.

Definition (Adams and Essex [2010], page 78)

We say that a function f is **continuous** at an interior point c of its domain if

$$\lim_{x \rightarrow c} (f x) = f c$$

If either $\lim_{x \rightarrow c} (f x)$ fails to exist or it exists but is not equal to $f c$, then we will say that f is **discontinuous** at c .

Differentiability defined in terms of limits.

Definition (Adams and Essex [2010], page 99)

The derivative of a function f is another function f' defined by

$$f' x = \lim_{h \rightarrow 0} \frac{f(x+h) - f x}{h}$$

*at all points x for which the limit exists (i.e., is a finite real number). If $f'(x)$ exists, we say that f is **differentiable** at x .*

Alternative: differentiability in terms of continuity.

Definition (Adapted from Pickert [1969])

Let $X \subseteq \mathbb{R}$, $a \in X$ and $f : X \rightarrow \mathbb{R}$. If there exists a function $\phi_f : X \rightarrow \mathbb{R}$ such that, for all $x \in X$

$$f(x) = f(a) + (x - a) * \phi_f(a, x)$$

such that $\phi_f(a, \cdot) : X \rightarrow \mathbb{R}$ continuous at a , then f is **differentiable** at x .
The value $\phi_f(a, a)$ is called the **derivative** of f at a and is denoted $f'(a)$.

A calculational proof

$$(f\ x) * (g\ x)$$

A calculational proof

$$\begin{aligned} & (f\ x) * (g\ x) \\ = & \{ \text{differentiability} \} \\ & (f\ a + (x - a) * \phi_f\ a\ x) * (g\ a + (x - a) * \phi_g\ a\ x) \end{aligned}$$

A calculational proof

$$\begin{aligned} & (f\ x) * (g\ x) \\ = & \{ \text{differentiability} \} \\ & (f\ a + (x - a) * \phi_f\ a\ x) * (g\ a + (x - a) * \phi_g\ a\ x) \\ = & \{ \text{arithmetic} \} \\ & f\ a * g\ a \quad + \quad f\ a * (x - a) * \phi_g\ a\ x + \\ & (x - a) * \phi_f\ a\ x * g\ a \quad + \quad (x - a) * \phi_f\ a\ x * (x - a) * \phi_g\ a\ x \end{aligned}$$

A calculational proof

$$\begin{aligned} & (f\ x) * (g\ x) \\ = & \{ \text{differentiability} \} \\ & (f\ a + (x - a) * \phi_f\ a\ x) * (g\ a + (x - a) * \phi_g\ a\ x) \\ = & \{ \text{arithmetic} \} \\ & f\ a * g\ a + f\ a * (x - a) * \phi_g\ a\ x + \\ & (x - a) * \phi_f\ a\ x * g\ a + (x - a) * \phi_f\ a\ x * (x - a) * \phi_g\ a\ x \\ = & \{ \text{factor out } (x - a) \text{ to get } h\ a + (x - a) * \phi_h\ a\ x \} \\ & f\ a * g\ a + (x - a) * (f\ a * \phi_g\ a\ x + \phi_f\ a\ x * g\ a + \\ & \phi_f\ a\ x * (x - a) * \phi_g\ a\ x) \end{aligned}$$

A calculational proof

$$\begin{aligned} & (f\ x) * (g\ x) \\ = & \{ \text{differentiability} \} \\ & (f\ a + (x - a) * \phi_f\ a\ x) * (g\ a + (x - a) * \phi_g\ a\ x) \\ = & \{ \text{arithmetic} \} \\ & f\ a * g\ a + f\ a * (x - a) * \phi_g\ a\ x + \\ & (x - a) * \phi_f\ a\ x * g\ a + (x - a) * \phi_f\ a\ x * (x - a) * \phi_g\ a\ x \\ = & \{ \text{factor out } (x - a) \text{ to get } h\ a + (x - a) * \phi_h\ a\ x \} \\ & f\ a * g\ a + (x - a) * (f\ a * \phi_g\ a\ x + \phi_f\ a\ x * g\ a + \\ & \qquad \qquad \qquad \phi_f\ a\ x * (x - a) * \phi_g\ a\ x) \\ = & \{ \text{"pattern-matching"} \} \\ & h\ a + (x - a) * \phi_h\ a\ x \\ & \text{where } h\ x = f\ x * g\ x \\ & \qquad \phi_h\ a\ x = f\ a * \phi_g\ a\ x + \phi_f\ a\ x * g\ a + \\ & \qquad \qquad \qquad \phi_f\ a\ x * (x - a) * \phi_g\ a\ x \end{aligned}$$

A calculational proof

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Therefore, by continuity of composition and differentiability:

$$h'\ a = \phi_h\ a\ a = f\ a * g'\ a + f'\ a * g\ a$$

Conclusions

- make functions and types explicit: $Re : Complex \rightarrow \mathbb{R}$,
 $\phi_f : X \rightarrow X \rightarrow \mathbb{R}$
- make the distinction between syntax and semantics explicit
- use types (\mathbb{R} , \mathbb{C}) as carriers of semantic information, not just variable names (x , z)
- pay attention to abstraction barriers (such as limits, continuity, differentiability)
- introduce functions and types for implicit operations such as
 $toComplex : \mathbb{R} \rightarrow Complex$
- use a calculational style for proofs
- organize the types and functions in DSLs (for *Complex*, limits, power series, etc.)

Partial implementation in Agda:

- errors caught by formalization (but no “royal road”)
 - Mixing up names of the same type
 - *choice* function
- subsets and coercions
 - $\epsilon : \mathbb{R}_{>0}$, different type from $\mathbb{R}_{\geq 0}$ and \mathbb{R} and \mathbb{C}
 - what is the type of $|\cdot|$? ($\mathbb{C} \rightarrow \mathbb{R}_{\geq 0}$?)
 - other subsets of \mathbb{R} or \mathbb{C} are common, but closure properties unclear

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