DSLsofMath: Typing Mathematics

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What is “DSLs of Math”?

“Domain Specific Languages of Mathematics”
`https://github.com/DSLsofMath/`

- A BSc-level course (2016-01 CeIo, 2017 onwards: PaJa, DaSc)
- A pedagogical project to develop the course (DaHe, SoEi)
- A BSc thesis project “DSLsofMath for other courses”

Aim: “. . . improve the mathematical education of computer scientists and the computer science education of mathematicians.”

Focus on types & specifications, syntax & semantics

DSL examples: Power series, Differential equations, Linear Algebra

DSLsofMath learning outcomes

- Knowledge and understanding
  - design and implement a DSL for a new domain
  - organize areas of mathematics in DSL terms
  - explain main concepts of analysis, algebra, and lin. alg.
- Skills and abilities
  - develop adequate notation for mathematical concepts
  - perform calculational proofs
  - use power series for solving differential equations
  - use Laplace transforms for solving differential equations
- Judgement and approach
  - discuss and compare different software implementations of mathematical concepts

[https://github.com/DSLsofMath/DSLsofMath/blob/master/Course2018.md]

Case 1: limits [Adams and Essex, 2010]

> *We say that $f(x)$* ***approaches the limit*** *L as x* ***approaches*** *a, and we write*
> $$\lim_{x \to a} f(x) = L,$$
> *if the following condition is satisfied:*
> *for every number $\epsilon > 0$ there exists a number $\delta > 0$, possibly depending on ϵ, such that if $0 < |x - a| < \delta$, then x belongs to the domain of f and*
> $$|f\ (x) - L| < \epsilon.$$

Four parts: name x, point a, expr. $f(x)$, limit L.

Name + expr. combines to just f: thus three parts: a, f, and L.

$$\begin{aligned} &\mathit{lim}\ a\ f\ L = \forall\ \epsilon > 0.\ \ \exists\ \delta > 0.\ \ P\ \epsilon\ \delta \\ &\quad \textbf{where}\ P\ \epsilon\ \delta = (0 < |x - a| < \delta) \Rightarrow (x \in \mathcal{D} f \wedge |f\ x - L| < \epsilon)) \end{aligned}$$

Where did x come from?

Case 1: limits [Adams and Essex, 2010]

*We say that $f(x)$ **approaches the limit** L as x **approaches** a, and we write*

$$\lim_{x \to a} f(x) = L,$$

if the following condition is satisfied:
for every number $\epsilon > 0$ there exists a number $\delta > 0$, possibly depending on ϵ, such that if $0 < |x - a| < \delta$, then x belongs to the domain of f and

$$|f(x) - L| < \epsilon.$$

Four parts: name x, point a, expr. $f(x)$, limit L.

Name + expr. combines to just f: thus three parts: a, f, and L.

$$\begin{aligned}&\mathit{lim}\ a\ f\ L = \forall\ \epsilon > 0.\ \ \exists\ \delta > 0.\ \ \forall\ x.\ \ P\ \epsilon\ \delta\ x\\ &\quad \textbf{where}\ P\ \epsilon\ \delta\ x = (0 < |x - a| < \delta) \Rightarrow (x \in \mathcal{D}f \wedge |f\ x - L| < \epsilon))\end{aligned}$$

Case 2: derivative

[We now assume limits exist and use *lim* as a function from a and f to L.]

> *The* ***derivative*** *of a function f is another function f' defined by*
>
> $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
>
> *at all points x for which the limit exists (i.e., is a finite real number). If $f'(x)$ exists, we say that f is* ***differentiable*** *at x.*

We can write

$$D\ f\ x = lim\ 0\ g \qquad \textbf{where} \qquad g\ h = \frac{f\ (x+h) - f\ x}{h}$$

Case 2: derivative

[We now assume limits exist and use *lim* as a function from *a* and *f* to *L*.]

> *The **derivative** of a function f is another function f' defined by*
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> $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
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> *at all points x for which the limit exists (i.e., is a finite real number). If $f'(x)$ exists, we say that f is **differentiable** at x.*

We can write

$$D\ f\ x = lim\ 0\ g \qquad \textbf{where} \qquad g\ h = \frac{f\ (x+h) - f\ x}{h}$$

$$D\ f\ x = lim\ 0\ (\varphi\ x)\ \textbf{where} \quad \varphi\ x\ h = \frac{f\ (x+h) - f\ x}{h}$$

Case 2: derivative

[We now assume limits exist and use *lim* as a function from *a* and *f* to *L*.]

> *The* ***derivative*** *of a function f is another function f′ defined by*
>
> $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
>
> *at all points x for which the limit exists (i.e., is a finite real number). If f′(x) exists, we say that f is* ***differentiable*** *at x.*

We can write

$$D\ f\ x = lim\ 0\ g \qquad \textbf{where} \qquad g\ h = \frac{f\ (x+h) - f\ x}{h}$$

$$D\ f\ x = lim\ 0\ (\varphi\ x)\ \textbf{where} \quad \varphi\ x\ h = \frac{f\ (x+h) - f\ x}{h}$$

$$D\ f \quad = lim\ 0 \circ \psi\ f\ \textbf{where}\ \psi\ f\ \ x\ h = \frac{f\ (x+h) - f\ x}{h}$$

Derivatives, cont.

Examples:

$$
\begin{aligned}
&D : (\mathbb{R} \to \mathbb{R}) \to (\mathbb{R} \to \mathbb{R}) \\
&\mathit{sq}\ x \quad\;\; = x\hat{}2 \\
&\mathit{double}\ x = 2 * x \\
&c_2\ x \quad\;\;\; = 2 \\
&\mathit{sq}' \ = D\ \mathit{sq}\ \ = D\ (\lambda x \to x\hat{}2) = D\ (\hat{}2) = (2*) = \mathit{double} \\
&\mathit{sq}'' = D\ \mathit{sq}' = D\ \mathit{double} = c_2 = \mathit{const}\ 2
\end{aligned}
$$

Note: we cannot *implement* D in Haskell.

Given only $f : \mathbb{R} \to \mathbb{R}$ as a "black box" we cannot compute the actual derivative $f' : \mathbb{R} \to \mathbb{R}$.

We need the "source code" of f to apply rules from calculus.

Case 3: Lagrangian

From [Sussman and Wisdom, 2013]:

> *A mechanical system is described by a Lagrangian function of the system state (time, coordinates, and velocities). A motion of the system is described by a path that gives the coordinates for each moment of time. A path is allowed if and only if it satisfies the Lagrange equations. Traditionally, the Lagrange equations are written*
>
> $$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$
>
> *What could this expression possibly mean?*

Lagrangian, cont.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

- The use of notation for "partial derivative", $\partial L / \partial q$, suggests that L is a function of at least a pair of arguments:

  $$L : \mathbb{R}^{\mathrm{i}} \to \mathbb{R}, i \geqslant 2$$

  This is consistent with the description: "Lagrangian function of the system state (time, coordinates, and velocities)". So, if we let "coordinates" be just one coordinate, we can take $i = 3$:

  $$L : \mathbb{R}^3 \to \mathbb{R}$$

  The "system state" here is a triple, of type $S = (T, Q, V)$, and we can call the the three components $t : T$ for time, $q : Q$ for coordinate, and $v : V$ for velocity. $(T = Q = V = \mathbb{R}.)$

Lagrangian, cont.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

- Looking again at $\partial L / \partial q$, q is the name of a variable, one of the 3 args to L. In the context, which we do not have, we would expect to find somewhere the definition of the Lagrangian as

$$\begin{aligned} L &: (T, Q, V) \to \mathbb{R} \\ L &\ (t, q, v) = \ldots \end{aligned}$$

- therefore, $\partial L / \partial q$ should also be a function of the same triple:

$$(\partial L \,/\, \partial q) : (T, Q, V) \to \mathbb{R}$$

It follows that the equation expresses a relation between *functions*, therefore the 0 on the right-hand side is *not* the real number 0, but rather the constant function *const* 0:

$$\begin{aligned} &\mathit{const}\ 0 : (T, Q, V) \to \mathbb{R} \\ &\mathit{const}\ 0\ \ (t, q, v) \ = 0 \end{aligned}$$

Lagrangian, cont.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

- We now have a problem: $d \mathbin{/} dt$ can only be applied to functions of *one* real argument t, and the result is a function of one real argument:

  $$(d \mathbin{/} dt)\,(\partial L \mathbin{/} \partial \dot{q}) : T \to \mathbb{R}$$

  Since we subtract from this the function $\partial L / \partial q$, it follows that this, too, must be of type $T \to \mathbb{R}$. But we already typed it as $(T, Q, V) \to \mathbb{R}$, contradiction!
- The expression $\partial L / \partial \dot{q}$ appears to also be malformed. We would expect a variable name where we find $\dot{q}$, but $\dot{q}$ is the same as dq/dt, a function.

Lagrangian, cont.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

- The only immediate candidate for an application of d/dt is "a path that gives the coordinates for each moment of time". Thus, the path is a function of time, let us say

  $w : T \to Q$ -- with T for time and Q for coords ($q : Q$)

  We can now guess that the use of the plural form "equations" might have something to do with the use of "coordinates". In an n-dim. space, a position is given by n coordinates. A path would then be

  $w : T \to Q$ -- with $Q = \mathbb{R}^n$

  which is equivalent to n functions of type $T \to \mathbb{R}$, each computing one coordinate as a function of time. We would then have an equation for each of them. We will use $n = 1$ for the rest of this example.

Lagrangian, cont.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

- Now that we have a path, the coordinates at any time are given by the path. And as the time derivative of a coordinate is a velocity, we can actually compute the trajectory of the full system state (T, Q, V) starting from just the path.

  $q : T \to Q$
  $q\ t = w\ t$ -- or, equivalently, $q = w$
  $\dot{q} : T \to V$
  $\dot{q}\ t = dw \mathbin{/} dt$ -- or, equivalently, $\dot{q} = D\ w$

  We combine these in the "combinator" *expand*, given by

  $expand : (T \to Q) \to (T \to (T, Q, V))$
  $expand\ w\ t = (t, w\ t, D\ w\ t)$

Lagrangian, cont.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

- With *expand* in our toolbox we can fix the typing problem.

$$(\partial L \,/\, \partial q) \circ (expand\ w) : T \to \mathbb{R}$$

- We now move to using D for $d \,/\, dt$, D_2 for $\partial \,/\, \partial q$, and D_3 for $\partial \,/\, \partial \dot{q}$. In combination with *expand w* we find these type correct combinations for the two terms in the equation:

$$\begin{aligned} D\ ((D_2\ L) \circ (expand\ w)) &: T \to \mathbb{R} \\ (D_3\ L) \circ (expand\ w) &: T \to \mathbb{R} \end{aligned}$$

The equation becomes

$$D\ ((D_3\ L) \circ (expand\ w)) - (D_2\ L) \circ (expand\ w) = const\ 0$$

or, after simplification:

$$D\ (D_3\ L \circ expand\ w) = D_2\ L \circ expand\ w$$

Case 3: Lagrangian, summary

"A path is allowed if and only if it satisfies the Lagrange equations" means that this equation is a predicate on paths:

$$Lagrange\ (L, w) = \qquad D\ (D_3\ L \circ expand\ w) == D_2\ L \circ expand\ w$$

Thus: If we can describe a mechanical system in terms of "a Lagrangian" ($L : S \to \mathbb{R}$), then we can use the predicate to check if a particular candidate path $w : T \to \mathbb{R}$ qualifies as a "motion of the system" or not. The unknown of the equation is the path w, and the equation is an example of a partial differential equation (a PDE).

DSLsofMath: Typing Mathematics

- Mathematical concepts like *lim*, *D*, *Lagrangian* can be explored and explained using typed functional programming.
- Sometimes new insights arise: Stream calculus, for example.
- Aim: "... improve the mathematical education of computer scientists and the computer science education of mathematicians."
- Focus on types & specifications, syntax & semantics
- DSL examples: Power series, Differential equations, Linear Algebra

Bibliography

R. A. Adams and C. Essex. *Calculus: a complete course*. Pearson Canada, 7th edition, 2010.

G. J. Sussman and J. Wisdom. *Functional Differential Geometry*. MIT Press, 2013.

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