DSLsofMath: Presenting Mathematical Analysis Using Functional Programming

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Paper + talk: `https://github.com/DSLsofMath/tfpie2015`

Style example

$$\forall\ \epsilon\ \in\ \mathbb{R}.\ \ (\epsilon > 0)\ \Rightarrow\ \exists\ a\ \in\ A.\ \ (|a - sup\ A| < \epsilon)$$

Background

Domain-Specific Languages of Mathematics [Ionescu and Jansson, 2015]: is a course currently developed at Chalmers in response to difficulties faced by third-year students in learning and applying classical mathematics (mainly real and complex analysis)

Main idea: encourage students to approach mathematical domains from a functional programming perspective (similar to Wells [1995]).

[Link: Summary of Wells 1995]

> "... ideally, the course would improve the mathematical education of computer scientists and the computer science education of mathematicians."

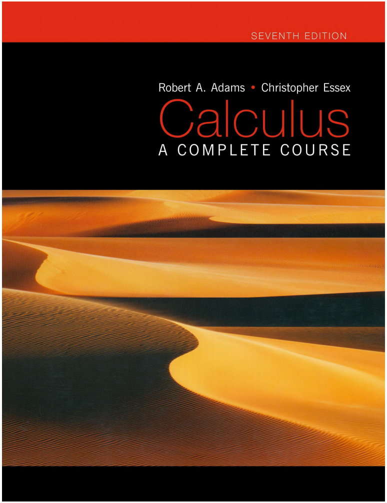
Introduction

- make functions and types explicit
- make the distinction between syntax and semantics explicit
- use types ($\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$) as carriers of semantic information, not just variable names (n, x, z)
- introduce functions and types for implicit operations such as the power series interpretation of a sequence
- use a calculational style for proofs
- organize the types and functions in DSLs

Not working code, rather working understanding of concepts

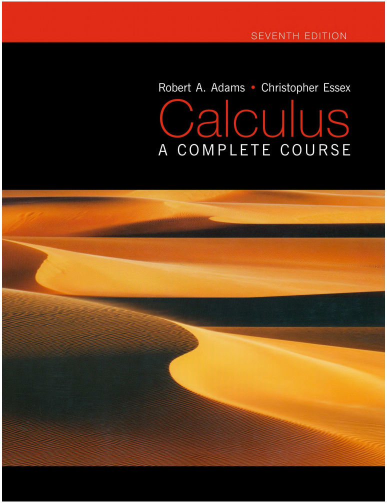
Active reading example

Calculus course book (D)



Active reading example: complex numbers

Calculus course book (D)



…Appendix I, Page 993!

Complex Numbers

Definition of Complex Numbers

We begin by defining the symbol i, called **the imaginary unit,**[1] to have the

$$i^2 = -1.$$

Thus, we could also call i the **square root of** -1 and denote it $\sqrt{-1}$. Of
not a real number; no real number has a negative square.

DEFINITION 1

A **complex number** is an expression of the form

$$a + bi \qquad \text{or} \qquad a + ib,$$

where a and b are *real numbers*, and i is the imaginary unit.

For example, $3+2i$, $\frac{7}{2} - \frac{2}{3}i$, $i\pi = 0 + i\pi$, and $-3 = -3 + 0i$ are all comple
The last of these examples shows that every real number can be regarded as
number. (We will normally use $a + bi$ unless b is a complicated expressio
case we will write $a + ib$ instead. Either form is acceptable.)

It is often convenient to represent a complex number by a single lette
are frequently used for this purpose. If a, b, x, and y are real numbers, and

$$w = a + bi \qquad \text{and} \qquad z = x + yi,$$

then we can refer to the complex numbers w and z. Note that $w = z$ if
$a = x$ and $b = y$. Of special importance are the complex numbers

$$0 = 0 + 0i, \qquad 1 = 1 + 0i, \qquad \text{and} \qquad i = 0 + 1i.$$

DEFINITION 2

If $z = x + yi$ is a complex number (where x and y are real), we call x
part of z and denote it $\mathrm{Re}\,(z)$. We call y the **imaginary part** of z and d
$\mathrm{Im}\,(z)$:

$$\mathrm{Re}\,(z) = \mathrm{Re}\,(x + yi) = x, \qquad \mathrm{Im}\,(z) = \mathrm{Im}\,(x + yi) = y.$$

Complex numbers

We begin by defining the symbol i, called **the imaginary unit**, to have the property

$$i^2 = -1$$

Thus, we could also call i the square root of -1 and denote it $\sqrt{-1}$. Of course, i is not a real number; no real number has a negative square.

(Adams and Essex [2010], Appendix I)

Complex numbers

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> $$i^2 = -1$$
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> Thus, we could also call i the square root of -1 and denote it $\sqrt{-1}$. Of course, i is not a real number; no real number has a negative square.

(Adams and Essex [2010], Appendix I)

data $I = i$

Complex numbers

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Complex numbers

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where a and b are real numbers, and i is the imaginary unit.

$$\begin{array}{lcl} \textbf{data}\ \mathit{Complex} & = & \mathit{Plus}_1\ \mathbb{R}\ \mathbb{R}\ I \\ & \mid & \mathit{Plus}_2\ \mathbb{R}\ I\ \mathbb{R} \end{array}$$

$$\begin{array}{lcl} \mathit{show}\ :\ \mathit{Complex} & \rightarrow & \mathit{String} \\ \mathit{show}\ (\mathit{Plus}_1\ x\ y\ i) & = & \mathit{show}\ x \mathbin{+\!\!+} \texttt{" + "} \mathbin{+\!\!+} \mathit{show}\ y \mathbin{+\!\!+} \texttt{"i"} \\ \mathit{show}\ (\mathit{Plus}_2\ x\ i\ y) & = & \mathit{show}\ x \mathbin{+\!\!+} \texttt{" + "} \mathbin{+\!\!+} \texttt{"i"} \mathbin{+\!\!+} \mathit{show}\ y \end{array}$$

Complex numbers examples

Definition: A **complex number** is an expression of the form

$$a + bi \qquad \text{or} \qquad a + ib,$$

where a and b are real numbers, and i is the imaginary unit.

For example, $3 + 2\ i$, $\frac{7}{2} - \frac{2}{3}\ i$, $i\ \pi\ =\ 0 + i\ \pi$, and $-3\ =\ -3 + 0\ i$ are all complex numbers. The last of these examples shows that every real number can be regarded as a complex number.

Complex numbers examples

For example, $3 + 2\,i$, $\frac{7}{2} - \frac{2}{3}\,i$, $i\,\pi \;=\; 0 + i\,\pi$, and $-3 \;=\; -3 + 0\,i$ are all complex numbers. The last of these examples shows that every real number can be regarded as a complex number.

$$\begin{array}{lcl}\textbf{data}\ \mathit{Complex} & = & \mathit{Plus}_1\ \mathbb{R}\ \mathbb{R}\ \mathit{I} \\ & | & \mathit{Plus}_2\ \mathbb{R}\ \mathit{I}\ \mathbb{R}\end{array}$$

$$\begin{array}{l}\mathit{toComplex} : \mathbb{R} \to \mathit{Complex} \\ \mathit{toComplex}\ x = \mathit{Plus}_1\ x\ 0\ i\end{array}$$

- what about i by itself?
- what about, say, $2\,i$?

Complex numbers version 2.0

(We will normally use $a + bi$ unless b is a complicated expression, in which case we will write $a + ib$ instead. Either form is acceptable.)

$$\textbf{data}\ \mathit{Complex} \ = \ \mathit{Plus}\ \mathbb{R}\ \mathbb{R}\ \mathit{I}$$

$$\textbf{data}\ \mathit{Complex} \ = \ \mathit{PlusI}\ \mathbb{R}\ \mathbb{R}$$

Name and reuse

It is often convenient to represent a complex number by a single letter; w and z are frequently used for this purpose. If a, b, x, and y are real numbers, and $w = a + bi$ and $z = x + yi$, then we can refer to the complex numbers w and z. Note that $w = z$ if and only if $a = x$ and $b = y$.

$$\textbf{newtype}\ \mathit{Complex} = C\ (\mathbb{R}, \mathbb{R})$$

Equality and pattern-matching

Definition: If $z = x + yi$ is a complex number (where x and y are real), we call x the **real part** of z and denote it $Re\ (z)$. We call y the **imaginary part** of z and denote it $Im\ (z)$:

$$\begin{aligned} Re\ (z) &= Re\ (x + yi) = x \\ Im\ (z) &= Im\ (x + yi) = y \end{aligned}$$

$$\begin{aligned} Re &: Complex \rightarrow \mathbb{R} \\ Re\ z@(C\ (x, y)) &= x \end{aligned}$$

$$\begin{aligned} Im &: Complex \rightarrow \mathbb{R} \\ Im\ z@(C\ (x, y)) &= y \end{aligned}$$

The sum and difference of complex numbers

If $w = a + bi$ and $z = x + yi$, where a, b, x, and y are real numbers, then

$$w + z = (a + x) + (b + y)\,i$$
$$w - z = (a - x) + (b - y)\,i$$

Shallow embedding:

$$(+) : \mathit{Complex} \to \mathit{Complex} \to \mathit{Complex}$$
$$(C\,(a, b)) + (C\,(x, y)) = C\,((a + x)\,,\ (b + y))$$
$$\textbf{newtype}\ \mathit{Complex} = C\,(\mathbb{R}\,,\ \mathbb{R})$$

Shallow vs. deep embeddings

The sum and difference of complex numbers

If $w = a + bi$ and $z = x + yi$, where a, b, x, and y are real numbers, then

$$
\begin{aligned}
w + z &= (a + x) + (b + y)\, i \\
w - z &= (a - x) + (b - y)\, i
\end{aligned}
$$

Deep embedding:

$$
\begin{aligned}
(+) &: \mathit{Complex} \to \mathit{Complex} \to \mathit{Complex} \\
(+) &= \mathit{Plus}
\end{aligned}
$$

$$
\begin{aligned}
\textbf{data}\ \mathit{Complex} \ =\ & i \\
\mid\ & \mathit{ToComplex}\ \mathbb{R} \\
\mid\ & \mathit{Plus}\ \ \mathit{Complex}\ \mathit{Complex} \\
\mid\ & \mathit{Times}\ \mathit{Complex}\ \mathit{Complex} \\
\mid\ & \ldots
\end{aligned}
$$

Abstraction barriers

Example: continuity defined in terms of limits.

Definition (Adams and Essex [2010], page 78)

We say that a function f is ***continuous*** *at an interior point c of its domain if*

$$\lim_{x \to c} (f\ x) \ = \ f\ c$$

If either $\lim_{x \to c} (f\ x)$ fails to exist or it exists but is not equal to $f\ c$, then we will say that f is ***discontinuous*** *at c.*

Abstraction barriers

Differentiability defined in terms of limits.

Definition (Adams and Essex [2010], page 99)

The derivative of a function f is another function f' defined by

$$f'\ x \ = \ \lim_{h \to 0} \frac{f(x+h) - f\ \ x}{h}$$

at all points x for which the limit exists (i.e., is a finite real number). If $f'(x)$ exists, we say that f is ***differentiable*** *at x.*

Abstraction barriers

Alternative: differentiability in terms of continuity.

Definition (Adapted from Pickert [1969])

Let $X \subseteq \mathbb{R}$, $a \in X$ and $f : X \to \mathbb{R}$. If there exists a function $\phi_f : X \to X \to \mathbb{R}$ such that, for all $x \in X$

$$f\ x = f\ a + (x - a) * \phi_f\ a\ x$$

*such that $\phi_f\ a : X \to \mathbb{R}$ continuous at a, then f is **differentiable** at a. The value $\phi_f\ a\ a$ is called the **derivative** of f at a and is denoted $f'\ a$.*

Note that for $X \subseteq \mathbb{R}$ we can define ϕ_f for $x \not\equiv a$ as follows:

$$\phi_f\ a\ x = (f\ x - f\ a)\ /\ (x - a)$$

but the definition above also works for vectors and matrices (when division is not available).

A calculational proof

$(f\ x) * (g\ x)$

A calculational proof

$$
\begin{aligned}
& (f\ x) * (g\ x) \\
= & \quad \{ \text{differentiability} \} \\
& (f\ a + (x - a) * \phi_f\ a\ x) * (g\ a + (x - a) * \phi_g\ a\ x)
\end{aligned}
$$

A calculational proof

$$
\begin{aligned}
& (f\ x) * (g\ x) \\
= \quad & \{\ \text{differentiability}\ \} \\
& (f\ a + (x - a) * \phi_f\ a\ x) * (g\ a + (x - a) * \phi_g\ a\ x) \\
= \quad & \{\ \text{arithmetic}\ \} \\
& f\ a * g\ a \qquad\qquad\qquad + \quad f\ a * (x - a) * \phi_g\ a\ x\ + \\
& (x - a) * \phi_f\ a\ x * g\ a \quad + \quad (x - a) * \phi_f\ a\ x * (x - a) * \phi_g\ a\ x
\end{aligned}
$$

A calculational proof

$$
\begin{array}{ll}
& (f\ x) * (g\ x) \\
= & \{\text{ differentiability }\} \\
& (f\ a + (x - a) * \phi_f\ a\ x) * (g\ a + (x - a) * \phi_g\ a\ x) \\
= & \{\text{ arithmetic }\} \\
& \begin{array}{lcl} f\ a * g\ a & + & f\ a * (x - a) * \phi_g\ a\ x\ + \\ (x - a) * \phi_f\ a\ x * g\ a & + & (x - a) * \phi_f\ a\ x * (x - a) * \phi_g\ a\ x \end{array} \\
= & \{\text{ factor out } (x - a) \text{ to get } h\ a + (x - a) * \phi_h\ a\ x\ \} \\
& f\ a * g\ a + (x - a) * (f\ a * \phi_g\ a\ x + \phi_f\ a\ x * g\ a\ + \\
& \qquad\qquad\qquad\qquad\qquad \phi_f\ a\ x * (x - a) * \phi_g\ a\ x)
\end{array}
$$

A calculational proof

$$
\begin{aligned}
& (f\ x) * (g\ x) \\
= \quad & \{ \text{ differentiability } \} \\
& (f\ a + (x - a) * \phi_f\ a\ x) * (g\ a + (x - a) * \phi_g\ a\ x) \\
= \quad & \{ \text{ arithmetic } \} \\
& f\ a * g\ a \qquad + \quad f\ a * (x - a) * \phi_g\ a\ x + \\
& (x - a) * \phi_f\ a\ x * g\ a \quad + \quad (x - a) * \phi_f\ a\ x * (x - a) * \phi_g\ a\ x \\
= \quad & \{ \text{ factor out } (x - a) \text{ to get } h\ a + (x - a) * \phi_h\ a\ x \ \} \\
& f\ a * g\ a + (x - a) * (f\ a * \phi_g\ a\ x + \phi_f\ a\ x * g\ a + \\
& \qquad \qquad \phi_f\ a\ x * (x - a) * \phi_g\ a\ x) \\
= \quad & \{ \text{ ``pattern-matching'' } \} \\
& h\ a + (x - a) * \phi_h\ a\ x \\
& \quad \textbf{where} \ \ h\ x \ = \ f\ x * g\ x \\
& \qquad \qquad \phi_h\ a\ x \ = \ f\ a * \phi_g\ a\ x + \phi_f\ a\ x * g\ a + \\
& \qquad \qquad \qquad \phi_f\ a\ x * (x - a) * \phi_g\ a\ x
\end{aligned}
$$

A calculational proof

$$
\begin{aligned}
& (f\ x) * (g\ x) \\
= & \quad \{\ \text{differentiability}\ \} \\
& (f\ a + (x - a) * \phi_f\ a\ x) * (g\ a + (x - a) * \phi_g\ a\ x) \\
= & \quad \{\ \text{arithmetic}\ \} \\
& f\ a * g\ a \qquad\qquad\qquad + \quad f\ a * (x - a) * \phi_g\ a\ x\ + \\
& (x - a) * \phi_f\ a\ x * g\ a \quad + \quad (x - a) * \phi_f\ a\ x * (x - a) * \phi_g\ a\ x \\
= & \quad \{\ \text{factor out } (x - a) \text{ to get } h\ a + (x - a) * \phi_h\ a\ x\ \} \\
& f\ a * g\ a + (x - a) * (f\ a * \phi_g\ a\ x + \phi_f\ a\ x * g\ a\ + \\
& \qquad\qquad\qquad\qquad\qquad \phi_f\ a\ x * (x - a) * \phi_g\ a\ x) \\
= & \quad \{\ \text{``pattern-matching''}\ \} \\
& h\ a + (x - a) * \phi_h\ a\ x \\
& \quad \textbf{where}\ \ h\ x \quad = f\ x * g\ x \\
& \qquad\qquad\quad \phi_h\ a\ x = f\ a * \phi_g\ a\ x + \phi_f\ a\ x * g\ a\ + \\
& \qquad\qquad\qquad\qquad\qquad \phi_f\ a\ x * (x - a) * \phi_g\ a\ x
\end{aligned}
$$

Therefore, by continuity of composition and differentiability:

$$h'\ a\ =\ \phi_h\ a\ a\ =\ f\ a * g'\ a + f'\ a * g\ a$$

Conclusions

- make functions and types explicit: $Re : Complex \to \mathbb{R}$, $\phi_f : X \to X \to \mathbb{R}$
- make the distinction between syntax and semantics explicit
- use types ($\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$) as carriers of semantic information, not just variable names (n, x, z)
- pay attention to abstraction barriers (such as limits, continuity, differentiability)
- introduce functions and types for implicit operations such as $toComplex : \mathbb{R} \to Complex$
- use a calculational style for proofs
- organize the types and functions in DSLs (for *Complex*, limits, power series, etc.)

Future work

Partial implementation in Agda:

- errors caught by formalization (but no "royal road")
  - Mixing up names of the same type
  - *choice* function
- subsets and coercions
  - $\epsilon : \mathbb{R}_{>0}$, different type from $\mathbb{R}_{\geq 0}$ and $\mathbb{R}$ and $\mathbb{C}$
  - what is the type of $|\cdot|$? ($\mathbb{C} \rightarrow \mathbb{R}_{\geq 0}$?)
  - other subsets of $\mathbb{R}$ or $\mathbb{C}$ are common, but closure properties unclear

Bibliography

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