Type Theory with Weak J*

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Judgmental equality is central to intensional type theory, and closely related to computation. In typical formulations of intensional type theory two terms are judgmentally equal if they have identical normal forms. Computation tends to make it easier to write type-correct programs, because fewer explicit casts need to be inserted into the terms. Thus judgmental equalities can make type theories more convenient to use. Our central question is whether they also change the strength of the theories. What are the consequences of replacing some judgmental equalities by propositional (internal) equalities? Do we get a weaker theory? Is the most basic form of judgmental equality based on β-equality for functions more fundamental than stronger forms that also include computation for the J rule, or various forms of η-equality (for instance as presented by Allais, McBride, and Boutillier [1])? In this talk we do not provide any answers, but we record a few observations and a conjecture.

In extensional Martin-Löf type theory [6] any propositional equality can be turned into a judgmental equality. Hofmann [3] has shown that one variant of extensional type theory is a conservative extension of an intensional type theory, i.e. if a type in the intensional theory has an inhabitant in the extensional one, then a corresponding inhabitant exists in the intensional theory. A similar statement for the Calculus of (Inductive) Constructions is due to Oury [7]. In these settings one could thus say that the additional judgmental equalities do not add additional strength (except in so far as they allow the formation of new types).

We note that a very important assumption in both Hofmann's and Oury's setting is *uniqueness of identity proofs (UIP)*, a principle which is not always assumed, and which is even rejected in homotopy type theory. UIP can be derived in extensional type theory, but is independent of some forms of intensional type theory [4].

In the absence of UIP, an additional concept distinguishes judgmental and computational equality: *coherence*. Judgmental equality is some sort of *law*, while propositional equality is *data*, and data is not automatically well-behaved (for example, associativity in a bicategory is given by 2-morphisms, i.e. data, and coherence follows from the pentagon law).

To make one of the questions that we are interested in precise, consider a suitable version of Martin-Löf type theory with an equality type, written $x = y$, and without UIP. We assume function extensionality. For a type A, let us write I_A for the type $\Sigma_{x,y:A} x = y$. Disregarding universe levels, we can write down the type of the eliminator J as

$$\mathsf{J} : (A : \mathcal{U})\ (P : I_A \to \mathcal{U})\ (d : (x : A) \to P(x, x, \mathsf{refl}))\ (q : I_A) \to P(q). \tag{1}$$

Usually the β-rule for J is assumed to hold judgmentally: $\mathsf{J}^{A,P,d}(x, x, \mathsf{refl}) \equiv d(x)$. We ask ourselves what happens if we replace this rule by a postulated term J_β:

$$\mathsf{J}_\beta : (A : \mathcal{U})\ (P : I_A \to \mathcal{U})\ (d : (x : A) \to P(x, x, \mathsf{refl}))\ (x : A) \to \mathsf{J}^{A,P,d}(x, x, \mathsf{refl}) = d(x). \tag{2}$$

One reason why this might be interesting is that cubical type theory [2] has a type of paths that satisfies all of the identity type's axioms, except that the β-rule for J does not in general hold judgmentally.

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Here is an example, first discussed in 2011 [5], that was originally intended to illustrate the lack of coherence that could arise. For simplicity we use subst (aka $\mathsf{transport}$), a non-dependent variant of J which is derivable from J. For a type A, a family $P : A \to \mathcal{U}$, and an equality $q : x = y$, we denote the term by $\mathsf{subst}^{A,P,q} : P(x) \to P(y)$. From J_β and function extensionality we can derive the equality $\mathsf{subst}_\beta^{A,P} : \mathsf{subst}^{A,P,\mathsf{refl}} = \mathsf{id}_{P(x)}$. Consider the two terms $\mathsf{subst}^{A,P,\mathsf{refl}}(\mathsf{subst}^{A,P,\mathsf{refl}}(p))$ and $\mathsf{subst}^{A,P,\mathsf{refl}}(p)$. There are two obvious ways to prove that the first term is equal to the second one: we can use $\mathsf{subst}_\beta^{A,P}$ to remove either the first or the second occurrence of $\mathsf{subst}^{A,P,\mathsf{refl}}$ in the first term. If J_β was a judgmental equality, then subst_β would just be (defined as) refl, and the two equalities between the terms would both be refl and thus be equal. In the version where J_β (and thus subst_β) is only given as a propositional equality, it is less clear whether the two equalities are equal. In fact, some of us believed for some time that they are not, and that the propositional J_β thus made the type theory weaker.

However, it turns out that the two equalities are equal. To see this, note that the pair $(\mathsf{subst}^{A,P,\mathsf{refl}}, \mathsf{subst}_\beta^{A,P})$ is an element of the *singleton type* $\Sigma_{f : P(x) \to P(x)} f = \mathsf{id}_{P(x)}$, and equal to the pair $(\mathsf{id}_{P(x)}, \mathsf{refl})$. With that replacement one can derive an equality between the two equalities.[1]

While our initial thought was that (2) is a postulated inhabitant of a not necessarily propositional type (i.e. a type which can have more than one element), and thus potentially problematic, we realised that we should consider (1) and (2) in combination—after all, J is a postulated constant as well. It turns out that the Σ-type of pairs $(\mathsf{J}, \mathsf{J}_\beta)$ is contractible, assuming that we already have instances of $(\mathsf{J}, \mathsf{J}_\beta)$ (for suitable universe levels).[1] Observe that, by exchange of Σ's and Π's, the Σ-type of pairs $(\mathsf{J}, \mathsf{J}_\beta)$ is equivalent to

$$\begin{aligned}&(A : \mathcal{U})\ (P : I_A \to \mathcal{U})\ (d : (x : A) \to P(x, x, \mathsf{refl})) \to \\ &\Sigma_{j : (q : I_A) \to P(q)}\ ((x : A) \to j(x, x, \mathsf{refl}) = d(x)).\end{aligned} \tag{3}$$

The function $\lambda x.(x, x, \mathsf{refl})$ is an equivalence between A and I_A, and using this equivalence in the second line of (3) one can see that this second line is equivalent to a singleton type. Thus the whole type (3) is contractible.

Hence, if we replace the usual judgmental computation rule for J by the constant J_β, then coherence issues do not seem to arise, and we conjecture that a conservativity result holds even in the absence of UIP.

References

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[1] At the time of writing an Agda formalisation of a very similar statement is available at `http://www.cse.chalmers.se/~nad/listings/equality/README.Weak-J.html`.