5 LP Optimality Conditions

So, the strong duality theory yields us to an indirect way to calculate the optimal cost of an LP: by solving its dual. What about the optimal solution? Does the dual optimal solution provides us any information about the primal optimal solution? The answer is positive. Again, proving the results is beyond our ability and we just state them. The main result in this case is known as the **complementary slackness** condition, which we state in the following, but first let me make a short reminder: An inequality constraint is called active at a feasible solution if it holds at this point with equality. Let us move on to the complementary slackness conditions:

Theorem 1. Given a LP problem and its dual, consider a primal feasible solution $\mathbf{x}$ and a dual feasible solution $\mathbf{u}$. These two points are both optimal solutions of their corresponding problems *if and only if* the following two conditions hold:

1. For any primal inequality constraint, if its corresponding dual parameter u_i is nonzero, then the constraint has to be active at $\mathbf{x}$.
2. For any dual inequality constraint, if its corresponding primal parameter x_i is nonzero, then the constraint has to be active at $\mathbf{u}$.

Let us consider and example:

Example 14. Take the example in (48) with its dual in (48). Using CVX, we get $u_1 = -0.0203..., u_2 = -0.0052$. Hence, both dual parameters are non-zero and due to the complementary slackness conditions, both primal conditions are active:

$$\begin{aligned} 50x_1 + 31x_2 &= 250 \\ -3x_1 + 2x_2 &= 4 \end{aligned} \tag{49}$$

which can be solved to obtain $x_2 = \frac{50\times4+3\times250}{50\times2+3\times31} = \frac{950}{193} = 4.9223...$ and $x_1 = \frac{250\times2-31\times4}{50\times2+3\times31} = \frac{376}{193} = 1.9482$. This is what we previously obtained by directly solving the primal optimization. You can try the same idea on the dual variables and see that they can be calculated from the complementary slackness conditions and the primal solution.

This is not an accident that we could find in example (14) the primal solution from the dual one. In fact, it can be proved that *there exists* a dual solution (i.e., it is not true for every dual solution, but for at least one particular one), from which we can construct *one* primal solution (i.e., not all of them) by the following method:

1. Find all inactive dual constraints and set their corresponding primal variables to zero.
2. Find all nonzero dual parameters and write their corresponding primal constraints with equality.
3. Solve the resulting system of linear equations (the primal equality constraints and the ones obtained from step 2) for the remaining primal variables (the ones that are not set to zero in step 1).

Example 15. Minimum cost network flow problem: Consider a digraph $G = (V, A)$. A flow on G is defined as an assignment $x : A \to \mathbb{R}$, which designates to each arc a a value $x(a)$, also denoted by x_a. Suppose that there exists another assignment $c : A \to \mathbb{R}$, where $c(a) = c_a$ is the cost of unit flow on the arc a i.e., the total cost of the flow is given by

$$f(x) = \sum_{a\in A} c_a x_a \tag{50}$$

The flow should satisfy a set of balance conditions. For each node v define its sets of outgoing arcs (i.e., the ones which initiate from v) as $A^+(v)$ and the ingoing ones (i.e., the ones terminating at v) as $A^-(v)$. Also, take for each node $v \in V$, a balance value b_v. Then, we must have that

$$\forall v \in V, \quad \sum_{a\in A^+(v)} x_a - \sum_{a\in A^-(v)} x_a = b_v \tag{51}$$

Finally for each arc $a \in A$, there exists an upper bound u_a, such that

$$\forall a \in A,\ 0 \leq x_a \leq u_a \tag{52}$$

The problem is to find a flow with minimal total cost. This can be written as the optimization

$$\begin{aligned} &\min_{(x_a \in \mathbb{R})} \sum_{a \in A} c_a x_a \\ \forall v \in V,\quad &\sum_{a \in A^+(v)} x_a - \sum_{a \in A^-(v)} x_a = b_v \\ &\forall a \in A,\ 0 \leq x_a \leq u_a \end{aligned} \tag{53}$$

An example of the minimum cost flow problem is the maximum s-t flow problem that we considered in the previous lectures. In the maximum s-t flow problem, there are input (source) and output (sink) nodes v_s, v_t and we are to maximize the total flow between them. In the previous lectures, we wrote this problem in a slightly different form: the input and output nodes did not have balance constraints. One way to bring the max s-t flow into the above format is to introduce a new arc a_0 from v_t to v_s. Then, we have that $b_v = 0$ for every $v \in V$, and $c_a = 0$ for all arcs except for a_0, where $c_{a_0} = -1$. In the previous lectures, we dealt with undirected graphs, but took an arbitrary directions for the edges and assumed $-u_a \leq x_a \leq u_a$ (or equivalently $|x_a| \leq u_a$). Here, we can only have $0 \leq x_a \leq u_a$. To overcome this problem, for each edge $e = \{u, v\}$ in the previous problem, we define two arcs instead $a_1 = (u, v)$ and $a_2 = (v, u)$, and set $0 \leq x_{a_1} \leq u_a$ and $0 \leq x_{a_2} \leq u_a$. (Can you see the relation between this trick and what we did for the absolute value functions in the previous lecture?).

Now, let us find out the dual of this optimization. It is not in one of the standard forms. So, we may use our general recipe. We have three different set of constraints:

1. The equality constraints $\sum_{a \in A^+(v)} x_a - \sum_{a \in A^-(v)} x_a = b_v$, for which we take the dual parameters y_v, respectively.

2. The inequality constraints $x_a \leq u_a$, for which we take the dual parameters $z_a \leq 0$, respectively.

3. The inequality constraints $x_a \geq u_a$, for which we take the dual parameters $s_a \geq 0$, respectively.

I use the notation $h(a)$ and $t(a)$ to refer to the head (initial node) and the tail (terminal node) of the arc a, respectively. Then, the dual can be written as

$$\begin{gathered} \max_{(y_v),(z_a,s_a)} \sum_{v \in V} b_v y_v + \sum_{a \in A} u_a z_a \\ \forall a \in A,\ y_{h(a)} - y_{t(a)} + z_a + s_a = c_a \\ \forall a \in A,\ s_a \geq 0 \\ \forall a \in A,\ z_a \leq 0 \end{gathered} \tag{54}$$

As expected, the dual parameters s_a are slack variables and can be eliminated, which leads to

$$\begin{gathered} \max_{(y_v),(z_a)} \sum_{v \in V} b_v y_v + \sum_{a \in A} u_a z_a \\ \forall a \in A,\ y_{h(a)} - y_{t(a)} + z_a \leq c_a \\ \forall a \in A,\ z_a \leq 0 \end{gathered} \tag{55}$$

In the maximum flow problem that we previously considered, the balance terms b_v are zero. Also, the costs c_a are zero, except for $c_{a_0} = -1$. In this case, the dual problem becomes

$$\begin{gathered}\max_{(y_v),(z_a)} \sum_{a\in A} u_a z_a \\ \forall a \in A\backslash\{a_0\},\ y_{h(a)} - y_{t(a)} + z_a \le 0 \\ a = a_0 \Rightarrow y_{h(a)} - y_{t(a)} \le -1 \\ \forall a \in A,\ z_a \le 0\end{gathered} \tag{56}$$

In the next lectures, we will see that this optimization has an important interpretation: It is a special type of the so-called **weighted minimum cut** problem. The strong duality shows that the cost of the maximum flow problem is *equal* to the cost of its corresponding minimum-cut problem. We will talk more about this so-called **min-cut-max-flow** theorem in the next lectures.

Another interesting observation in (55) is that for every value of $\alpha \in \mathbb{R}$, if the dual parameters y_v and h_a are feasible, so are the parameters $y_v + \alpha$ and h_a. Replacing y_v by $y_v + \alpha$ in the dual cost function, improves the cost value by $\alpha \sum\limits_{v\in V} b_v$. Hence, if $\sum\limits_{v\in V} b_v \neq 0$, the dual optimization is unbounded by letting α tend to infinity. Hence by the duality theorem, the primal optimization is infeasible in this case. We always assume that $\sum\limits_{v\in V} b_v = 0$.

Now, let us try the complementary slackness conditions:

1. Take the primal inequality $x_a \le u_a$. Its corresponding dual parameter is $z_a \le 0$. Hence at the optimal point
$$\forall a \in A,\ z_a < 0 \Rightarrow x_a = u_a \tag{57}$$

2. Take the dual inequality $y_{h(a)} - y_{t(a)} + z_a \le c_a$ its corresponding primal parameter is x_a. Hence, we get at the optimal point
$$\forall a \in A,\ x_a > 0 \Rightarrow y_{h(a)} - y_{t(a)} + z_a = c_a \tag{58}$$

   Finally, we try to find a primal solution from a proper dual solution. The first step is to identify arcs with inactive dual constraints. Let me call the set of these arcs A_1. From the second part of the complementary slackness condition the primal parameters of A_1 are zero. Next, I identify the set A_2 of arcs with nonzero dual parameters $z_a < 0$. By the first part of the complementary slackness conditions, the primal parameters of A_2 should be equal to u_a. Finally for a suitable choice of the dual solution, it is now possible to find the remaining dual parameters in $A_3 = A\backslash(A_1 \cup A_2)$ from the set of equality constraints $\sum\limits_{a\in A^+(v)} x_a - \sum\limits_{a\in A^-(v)} x_a = b_v$. Note that we replace the value of the primal variables in A_1 and A_2. So, the unknowns are only in A_3 and the system of equation has a unique solution. It turns out that this system of equations satisfies the so-called **totally uni-modularity** property, such that if all parameters u_a, b_a and c_a are integer, then its solution is also integer. Overall, I showed that if the parameters are integer, then the minimum cost network flow problem always has an integer solution, although it is essentially over all real numbers.