6 Exercise Set No. 6

1. In the gradient descent method, when the exact solution of line search in Equation 102 (in the lecture notes) is used, the algorithm is called **steepest descent**. Consider the following optimization

$$\min_{(x_1,x_2)\in\mathbb{R}^2} x_1^2 + 2x_2^2 + 4x_1 + 4x_2 \tag{3}$$

Show by induction that starting from $\mathbf{x}_0 = (x_1^{(0)}, x_2^{(0)}) = (0,0)$ and using the steepest descent method, we have that

$$\mathbf{x}_n = (x_1^{(n)}, x_2^{(n)}) = (\frac{2}{3^n} - 2, (-\frac{1}{3})^n - 1) \tag{4}$$

what is the convergence point $\bar{\mathbf{x}}$? Prove or disprove that $\bar{\mathbf{x}}$ is the global optimal point.

2. Sensor array antennas are frequently used for estimating the direction θ of a target that sends electromagnetic signals. The output of the sensors is a vector of complex numbers $\mathbf{y} = (y_1, y_2, \ldots, y_m) \in \mathbb{C}^m$, where m is the number of sensors. For a special type of antennas, the direction of the target is found by solving

$$\max_{\theta\in\mathbb{R}} |y_1 + y_2 e^{-j\theta} + y_3 e^{-2j\theta} + \ldots + y_m e^{-(m-1)j\theta}|_2^2 \tag{5}$$

where $j = \sqrt{-1}$ is the imaginary unit and $|x+jy| = \sqrt{x^2+y^2}$ for $x, y \in \mathbb{R}$ is the absolute value function. Suppose that in one experiment $m = 4$ sensors obtained the following output

$$\mathbf{y} = (y_1 = 1,\ y_2 = 0.6 + 0.8j,\ y_3 = -0.3 + 1j,\ y_4 = -1 + 0.3j) \tag{6}$$

   (a) Substitute the $\mathbf{y}$ vector of (6) into (5) and calculate the gradient with respect to θ. (Hint: you may use Euler's formula: $e^{j\phi} = \cos(\phi) + j\sin(\phi)$ for $\phi \in \mathbb{R}$.

   (b) Write a MATLAB code which implements the gradient descent method with the step size $\mu = 10^{-2}$ and runs for 100 iterations. Hand in your MATLAB code. (Notice that it is maximization problem)

   (c) Run your code for two different initial points $\theta_0 = 0$ and $\theta_0 = -1$. Plot by MATLAB the objective value $f(\theta_n)$ and the optimal point θ_n versus the iteration n for both cases (4 plots in total. Include axis labels and captions for better readability.)

   (d) Plot by MATLAB the objective function for $\theta \in [-\pi\ \pi]$. What is the global optimal point? Which of the two starting points above leads to the global optimal point?