Reachability Analysis of Generalized Polygonal Hybrid Systems (GSPDIs)

Gerardo Schneider

Department of Informatics
University of Oslo

SAC-SV’08
March 16–20, 2008 - Fortaleza
Reachability Analysis of GSPDIs

- **Hybrid System**: combines discrete and continuous dynamics
- **Examples**: thermostat, robot, chemical reaction
Reachability Analysis of GSPDIs

- **Hybrid System:** combines discrete and continuous dynamics
- **Examples:** thermostat, robot, chemical reaction
Outline

1. Polygonal Hybrid Systems (SPDIs) and Motivation
2. Generalized Polygonal Hybrid Systems (GSPDIs)
3. Reachability Analysis of GSPDIs
Outline

1. Polygonal Hybrid Systems (SPDIs) and Motivation
2. Generalized Polygonal Hybrid Systems (GSPDIs)
3. Reachability Analysis of GSPDIs
A constant differential inclusion (angle between vectors $\mathbf{a}$ and $\mathbf{b}$):

$$\dot{x} \in \angle_{\mathbf{a}}^{\mathbf{b}}$$
Polygonal Hybrid Systems (SPDIs)

- A finite partition of (a subset of) the plane into convex polygonal sets (regions)
- Dynamics given by the angle determined by two vectors: $\dot{x} \in \angle_{\mathbf{a}}^{\mathbf{b}}$
Polygonal Hybrid Systems (SPDIs)

- A finite partition of (a subset of) the plane into convex polygonal sets (regions)
- Dynamics given by the angle determined by two vectors: $\dot{x} \in \angle_{a}^{b}$
Goodness Assumption

The dynamics of an SPDI only allows trajectories traversing any edge only in one direction.
Goodness Assumption

The dynamics of an SPD I only allows trajectories traversing any edge only in one direction.
The dynamics of an SPDI only allows trajectories traversing any edge only in one direction.
Goodness Assumption

The dynamics of an SPDI only allows trajectories traversing any edge only in one direction.

Theorem

Under the goodness assumption, reachability for SPDIs is **decidable**.
Motivation
Use of SPDIs for approximating non-linear differential equations

Example

Pendulum with friction coefficient $k$, mass $M$, pendulum length $R$ and gravitational constant $g$. Behaviour: $\dot{x} = y$ and $\dot{y} = -\frac{ky}{MR^2} - \frac{g \sin(x)}{R}$. 

Triangulation of the plane: Huge number of regions

Need to reduce the complexity without too much overhead

Relax Goodness: GSPDI

Gerardo Schneider ()
Reachability Analysis of GSPDIs
SAC-SV’08 – 20.03.2008 15 / 44
Motivation
Use of SPDIs for approximating non-linear differential equations

Example
Pendulum with friction coefficient $k$, mass $M$, pendulum length $R$ and gravitational constant $g$. Behaviour: $\dot{x} = y$ and $\dot{y} = -\frac{ky}{MR^2} - \frac{g \sin(x)}{R}$

- Triangulation of the plane: Huge number of regions
- Need to reduce the complexity ... without too much overhead
  - Relax Goodness: GSPDI
Motivation

Use of SPDIs for approximating non-linear differential equations
Outline

1. Polygonal Hybrid Systems (SPDIs) and Motivation
2. Generalized Polygonal Hybrid Systems (GSPDIs)
3. Reachability Analysis of GSPDIs
An SPDI without the goodness assumption is called a GSPDI.
Why Goodness is Good

1. $e_6 e_7 e_8 e_1 e_2 e_3$
2. $e_6 e_7 e_8 (e_1 e_2 e_3 e_4 e_5 e_6 e_7 e_8)^5 e_9$

3. $(e_6 e_7 e_8 e_1 e_2 e_3 e_4 e_5)^5 e_6 e_7 e_8 e_9$
4. $e_6 e_7 e_8 e_1 e_2 e_3$

$r_1 s_1 r_2$
Why Goodness is Good

Theorem

An edge-signature $\sigma = e_1 \ldots e_p$ can always be abstracted into types of signatures of the form $\sigma_A = r_1 s_1^* \ldots r_n s_n^* r_{n+1}$, where $r_i$ is a sequence of pairwise different edges and all $s_i$ are disjoint simple cycle.

There are finitely many type of signatures.
Why Goodness is Good

Theorem

An edge-signature $\sigma = e_1 \ldots e_p$ can always be abstracted into types of signatures of the form $\sigma_A = r_1s^*_1 \ldots r_ns^*_n r_{n+1}$, where $r_i$ is a sequence of pairwise different edges and all $s_i$ are disjoint simple cycle.

There are finitely many type of signatures.

Many proofs (decidability, soundess, completeness) depend on the goodness assumption.
Problems when Relaxing Goodness

Finiteness argument for types of signature is broken for GSPDIs

Type of signature:

\[ d \ (abcd)^* \ (dcba)^* \ (abcd)^* \ a \]
Problems when Relaxing Goodness

Finiteness argument for types of signature is broken for GSPDIs

Type of signature:
\[ d \ (abcd)^* \ (dcba)^* \ (abcd)^* \ a \]

Challenge: Reachability analysis of GSPDIs

- Reduce GSPDI reachability to SPDI reachability; or
- Provide a completely new decidability proof for GSPDI.
1. Polygonal Hybrid Systems (SPDIs) and Motivation

2. Generalized Polygonal Hybrid Systems (GSPDIs)

3. Reachability Analysis of GSPDIs
Getting a Decision Algorithm for GSPDIs Based on that of SPDIs

1. It is enough to consider trajectories without self-crossing
2. It is possible to eliminate all inout edges, preserving reachability
3. It is possible to eliminate all sliding edges, preserving reachability
4. Re-state and prove some results on SPDI reachability useful to GPSDI reachability analysis
5. Prove soundness and termination
Getting a Decision Algorithm for GSPDIs Based on that of SPDIs

1. It is enough to consider trajectories without self-crossing
2. It is possible to eliminate all inout edges, preserving reachability
3. It is possible to eliminate all sliding edges, preserving reachability
4. Re-state and prove some results on SPDI reachability useful to GPSDI reachability analysis
5. Prove soundness and termination
Getting a Decision Algorithm for GSPDIs Based on that of SPDIs

1. It is enough to consider trajectories without self-crossing
2. It is possible to eliminate all inout edges, preserving reachability
3. It is possible to eliminate all sliding edges, preserving reachability
4. Re-state and prove some results on SPDI reachability useful to GPSDI reachability analysis
5. Prove soundness and termination

No decision algorithm for reachability of GSPDIs...
We will give a semi-test algorithm!
It is not possible to eliminate inout edges.
It is not possible to eliminate inout edges

There is no structure-preserving reduction from the GSPDI reachability problem to the SPDI reachability problem.
$\mathcal{H}_{red} = \{\mathcal{H}_1, \ldots, \mathcal{H}_n\}$: all possible underlying SPDIs, after fixing all the inout edges of $\mathcal{H}$ as entry-only or exit-only

**Algorithm**

1. Detect all the inout edges;
2. Generate the set of SPDIs $\mathcal{H}_{red} = \{\mathcal{H}_1, \ldots, \mathcal{H}_n\}$;
3. Apply the reachability algorithm for SPDIs to each $\mathcal{H}_i$ ($1 \leq i \leq n$), $\text{Reach}_{SPDI}(\mathcal{H}_i, x_0, x_f)$.
4. If there exists at least one SPDI $\mathcal{H}_i \in \mathcal{H}_{red}$ such that $\text{Reach}_{SPDI}(\mathcal{H}_i, x_0, x_f) = \text{Yes}$ then $\text{Reach}(\mathcal{H}, x_0, x_f) = \text{Yes}$, otherwise we do not know.
A Semi-Test Algorithm for GSPDIs

1. It is enough to consider trajectories without self-crossing

- Idem as for SPDIs
A Semi-Test Algorithm for GSPDIs

3. It is possible to eliminate all sliding edges, preserving reachability

**Theorem**

> If there exists a sliding trajectory segment from points $x_0 \in e_0$ to $x_f \in e_f$ then there always exists a non-sliding trajectory segment between them.
Theorem

If there exists a sliding trajectory segment from points $x_0 \in e_0$ to $x_f \in e_f$ then there always exists a non-sliding trajectory segment between them.
4. Re-state and prove some results on SPDI reachability useful to GPSDI reachability analysis

1. Redefine the edge-to-edge successor function
2. Rephrase topologically results using contiguity between entry-only and exit-only edges
3. Re-prove soundness of some algorithms
5. Soundness and termination

**Theorem**

Given a GSPDI $\mathcal{H}$, $\text{Reach}(\mathcal{H}, x_0, x_f) = \text{Yes}$ if $\text{Reach}_{\text{SPDI}}(\mathcal{H}_i, x_0, x_f) = \text{Yes}$ for some $\mathcal{H}_i \in \mathcal{H}_{\text{red}}$. On the other hand, $\text{Reach}(\mathcal{H}, x_0, x_f)$ is inconclusive if for all $\mathcal{H}_i \in \mathcal{H}_{\text{red}}$, $\text{Reach}_{\text{SPDI}}(\mathcal{H}_i, x_0, x_f) = \text{No}$.

*The algorithm always terminate.*
Final Remarks

News at submission to SAC-SV

- A semi-test for reachability analysis of GSPDIs (this presentation)

Later news after acceptance to SAC-SV

Reachability for GSPDIs is decidable (submitted, not published yet)

Latest news at SAC-SV

Implementation of reachability algorithm and application (current work)
Final Remarks

News at submission to SAC-SV

- A semi-test for reachability analysis of GSPDIs (this presentation)

Later news after acceptance to SAC-SV

- Reachability for GSPDIs is decidable (submitted, not published yet)
Final Remarks

News at submission to SAC-SV

- A semi-test for reachability analysis of GSPDIs (this presentation)

Later news after acceptance to SAC-SV

- Reachability for GSPDIs is decidable (submitted, not published yet)

Latest news at SAC-SV

- Implementation of reachability algorithm and application (current work)