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Computing Invariance Kernels of Polygonal Hybrid Systems

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Abstract. Polygonal hybrid systems are a subclass of planar hybrid automata which can be represented by piecewise constant differential inclusions. One way of analysing such systems (and hybrid systems in general) is through the study of their phase portrait, which characterise the systems' qualitative behaviour. In this paper we identify and compute an important object of polygonal hybrid systems' phase portrait, namely *invariance kernels*. An *invariant set* is a set of points such that any trajectory starting in such point keep necessarily rotating in the set forever and the *invariance kernel* is the largest of such sets. We show that this kernel is a non-convex polygon and we give a non-iterative algorithm for computing the coordinates of its vertexes and edges. Moreover, we show some properties of such systems' simple cycles.

Key words: Hybrid systems, differential inclusions, invariance kernel, algorithmic analysis.

CR Classification: C.3, D.2.4, J.2

1. Introduction

In its current meaning the word *hybrid* denotes anything that is composed by two or more things of different nature. In our context, a *hybrid system* is a system with both continuous and discrete behaviours interacting with each other. A typical example is given by a discrete program that interacts with a continuous physical environment. The analysis of such systems poses an interesting challenge mainly because of their hybrid nature, implying the use of discrete and continuous mathematics, domain of expertise of different academic communities.

Traditionally, the main preoccupation of computer scientists has been the study of discrete systems, using logic and discrete mathematics as a basis for reasoning. On the other hand, continuous models have been the subject of study of mathematicians and physicists, even though differential equations with discontinuous right hand side has also been considered in these communities (see [Filippov 1988] and reference therein). Later on, control theoreticians developed theories and methods to solve problems on *Control Theory* about "switching systems", in which digital control is applied to switch between continuous laws. Hybrid systems are nowadays studied, from different points of view and using different approaches and methods, in Computer Science, Control Theory and Mathematics.

One of the main contributions of Computer Science to the hybrid system community is related to the study of the (un)decidability of a variety of problems. When

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showing that a given problem is decidable it is desirable to have a constructive proof of it, providing an algorithm. In the last decade many (un)decidability results for a variety of problems concerning classes of hybrid systems have been given (e.g., [Asarin *et al.* 2000; Alur *et al.* 1995; Dang and Maler 1998; Greenstreet and Mitchell 1999; Kurzhanski and Varaiya 2000]). Besides reachability analysis, which is one of the main research areas in hybrid systems (see for instance [Alur and Dill 1994; Asarin *et al.* 1995; Asarin *et al.* 2001; Cerāns and Vīksna 1996; Henzinger *et al.* 1995; Lafferriere *et al.* 1999; Maler and Pnueli 1993]) another important issue in the analysis of a (hybrid) dynamical system is the study of its qualitative behaviour, namely the construction of its *phase portrait*. Some typical questions on this sense are "does every trajectory except the equilibrium point in the origin converge to a limit cycle which is the unit circle?", or "what is the biggest set such that any point on it is reachable from any other point on the set?". There have been few results on the qualitative properties of trajectories of hybrid systems [Asarin *et al.* 2002b; Aubin 2001; Deshpande and Varaiya 1995; Kourjanski and Varaiya 1995; Kowalczyk and di Bernardo 2001; Matveev and Savkin 2000; Simić *et al.* 2000]. In particular, the question of defining and constructing phase portraits of hybrid systems has not been directly addressed except by Matveev and Savkin [2000], where phase portraits of deterministic systems with piecewise constant derivatives are explored and by Asarin *et al.* [2002b], where viability and controllability kernels for polygonal differential inclusion systems (SPDIs) have been computed. Moreover, a characterisation of viability and invariance kernels was given by Aubin *et al.* [2001] for impulsive differential inclusions.

In this paper we show how to compute another important object of phase portraits of SPDIs, namely the *invariance kernel*. In general, an *invariant set* is a set of points such that for any point in the set it exists an infinite trajectory starting in such point and every such trajectory remains in the set forever and the *invariance kernel* is the largest of such sets. An invariance kernel is then a kind of "sink" from where it is impossible to escape. We show that, for SPDIs, this kernel is a non-convex polygon and we give a non-iterative algorithm for exactly computing the coordinates of its vertexes and edges. Notice that since SPDIs are partially defined over the plane, their invariance kernels are in general different from the whole plane. Clearly, such kernels provide useful insight about the behaviour of the SPDI around simple cycles.

Invariance kernels for SPDIs have been first introduced in [Schneider 2003], which is a preliminary version of this work. In [Pace and Schneider 2003] it has been shown that such kernels play a key role for proving termination of a model checking algorithm for SPDIs. In the last-mentioned paper, however, only some theorems showing how to compute them are stated without a complete proof. In addition to proving in detail how to compute invariance kernels for SPDIs we prove here many other useful properties about simple cycles.

The paper is organised as follows. In section 2 we recall some mathematical definitions needed to define SPDIs which are defined in section 3. Section 4 is concerned with the proof of properties of a class of simple cycles and the computation of invariance kernels for such cycles of SPDIs. In the last section we present some concluding remarks.

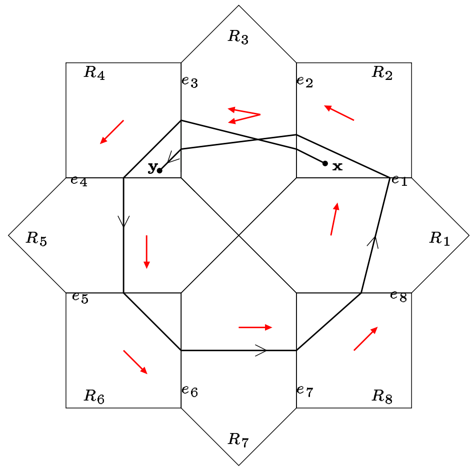

Fig. 1: An SPDI and its trajectory segment.

2. Preliminaries

Even though an SPDI may be seen as a hybrid automaton [Henzinger 1996], for the specific purposes of this paper it is better to define it in a more mathematical way. We need then the following mathematical concepts.

Let $\mathbf{a} = (a_1, a_2), \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$. The *inner product* of two vectors $\mathbf{a} = (a_1, a_2)$ and $\mathbf{x} = (x_1, x_2)$ is defined as $\mathbf{a}\ \mathbf{x} = a_1 x_1 + a_2 x_2$, while $\lambda\mathbf{x} = (\lambda x_1, \lambda x_2)$ is the product of a vector by an scalar λ and $|\mathbf{x}| = \sqrt{\mathbf{x}\ \mathbf{x}}$ is the *2-norm*. We denote by $\hat{\mathbf{x}}$ the vector $(x_2, -x_1)$ obtained from $\mathbf{x}$ by rotating clockwise by the angle $\pi/2$. Notice that $\mathbf{x}\ \hat{\mathbf{x}} = 0$.

The *distance* between two points $\mathbf{x}$ and $\mathbf{y}$ is defined to be $|\mathbf{x} - \mathbf{y}|$. For $\epsilon > 0$, the *ϵ-neighbourhood* of $\mathbf{x}$ is $B_\epsilon(\mathbf{x}) = \{\mathbf{y} \mid |\mathbf{x} - \mathbf{y}| < \epsilon\}$. The *interior* of $X \subseteq \mathbb{R}^2$, denoted by $\mathtt{int}(X)$, is the set of $\mathbf{x} \in X$ for which there exists $\epsilon > 0$ such that $B_\epsilon(\mathbf{x}) \subseteq X$.

For $\mathbf{x}_1, \ldots \mathbf{x}_n \in \mathbb{R}^2$ a *linear* combination is a vector $\mathbf{x} = \sum_{i=1}^{n} \lambda_i \mathbf{x}_i$ for some $\lambda_i \in \mathbb{R}$. A *positive combination* is a linear combination with $\lambda_i \geq 0$ for every i. The positive *hull* of a set $X \subseteq \mathbb{R}^2$ is the set of all positive combinations of points in X. Given a non-zero constant vector $\mathbf{a}$ and a constant b, a (closed) *half-space* is the set of all points $\mathbf{x}$ satisfying $\mathbf{a}\ \mathbf{x} \leq b$. A *convex closed polygonal set* P is the intersection of finitely many half-spaces. An *edge* e is a segment of line in $\mathbb{R}^2$.

An *angle* $\angle_{\mathbf{a}}^{\mathbf{b}}$ on the plane, defined by two non-zero vectors $\mathbf{a}, \mathbf{b}$ is the set of all positive linear combinations $\mathbf{x} = \alpha\ \mathbf{a} + \beta\ \mathbf{b}$, with $\alpha, \beta \geq 0$, and $\alpha + \beta > 0$. We can always assume that $\mathbf{b}$ is situated in the counter-clockwise direction from $\mathbf{a}$. That is, given $A = \{\mathbf{a}, \mathbf{b}\}$, with $\hat{\mathbf{a}}\ \mathbf{b} < 0$, $\angle_{\mathbf{a}}^{\mathbf{b}}$ is the positive hull of A.

Let S be a finite index set and $\mathbb{P} = \{P_s\}_{s \in S}$ be a finite set of convex closed polygonal sets, called *regions*, such that:

(1) For all $s \in S$, $\mathtt{int}(P_s) \neq \emptyset$;

(2) For all $s \neq r \in S$, $\mathtt{int}(P_s \cap P_r) = \emptyset$;

(3) $\bigcup_{s\in S} P_s = \mathbb{R}^2$.

Condition 1 states that regions are full dimensional. Condition 2 says that the intersection between two regions is either empty, an edge, or a point, whereas the third condition states that the regions covers the whole space. Thus, $\mathbb{P}$ is a *partition*. Condition 3 can be relaxed, and we consider then a partition of a subset of the plane.

Let $\mathbb{F} = \{\phi_s\}_{s\in S}$ be such that ϕ_s is the positive hull of two vectors $\mathbf{a}_s$ and $\mathbf{b}_s$ with $\hat{\mathbf{a}}_s\ \mathbf{b}_s < 0$ and $\mathbb{P}$ be a partition of the plane.

3. SPDI

A *polygonal differential inclusion system* [Schneider 2002] consists of a partition of the plane into convex polygonal regions, together with a constant differential inclusion associated with each region . More formally,

Definition 1. *A* polygonal differential inclusion system *(SPDI) is a pair* $\mathcal{H} = (\mathbb{P}, \mathbb{F})$. *Each region* P_s *has dynamics* $\dot{\mathbf{x}} \in \phi_s$ *for* $\mathbf{x} \in P_s$ *(given a generic region* P *we also use the notation* $\phi(P)$*).*

Let $E(P)$ be the set of edges of P. We say that $e \in E(P)$ is an *entry* of P if for all $\mathbf{x} \in e$ and for all $\mathbf{c} \in \phi(P)$, $\mathbf{x} + \mathbf{c}\epsilon \in P$ for some $\epsilon > 0$. We say that e is an *exit* of P if the same condition holds for some $\epsilon < 0$. We denote by $In(P) \subseteq E(P)$ the set of all entries of P and by $Out(P) \subseteq E(P)$ the set of all exits of P.

Assumption 1. All the edges in $E(P)$ are either entries or exits, that is, $E(P) = In(P) \cup Out(P)$.

Example 1. Consider the SPDI illustrated in Fig. 1. For each region R_i, $1 \le i \le 8$, there is a pair of vectors $(\mathbf{a}_i, \mathbf{b}_i)$, where:

$$
\begin{aligned}
&\mathbf{a}_1 = \mathbf{b}_1 = (1, 5),\\
&\mathbf{a}_2 = \mathbf{b}_2 = (-1, \tfrac{1}{2}),\\
&\mathbf{a}_3 = (-1, \tfrac{11}{60}) \text{ and } \mathbf{b}_3 = (-1, -\tfrac{1}{10}),\\
&\mathbf{a}_4 = \mathbf{b}_4 = (-1, -1),\\
&\mathbf{a}_5 = \mathbf{b}_5 = (0, -1),\\
&\mathbf{a}_6 = \mathbf{b}_6 = (1, -1),\\
&\mathbf{a}_7 = \mathbf{b}_7 = (1, 0),\\
&\mathbf{a}_8 = \mathbf{b}_8 = (1, 1).
\end{aligned}
$$

We define now the notion of *trajectory* which in the literature is sometimes called *run*.

Definition 2. *A* trajectory segment *of an SPDI is a continuous function* $\xi : [0, T] \to \mathbb{R}^2$ *which is smooth everywhere except in a discrete set of points, and such that for all* $t \in [0, T]$*, if* $\xi(t) \in P$ *and* $\dot{\xi}(t)$ *is defined then* $\dot{\xi}(t) \in \phi(P)$*. If* $T = \infty$*, a trajectory segment is called a* trajectory.

We are interested in making an abstraction in order to simplify part of our analysis. We will need to consider a discretisation of the trajectories, motivating the following definition. The *signature*, denoted $\mathsf{Sig}(\xi)$, is the ordered sequence of all edges traversed by the trajectory segment, that is, $e_1, e_2, \ldots$, where $\xi(t_i) \in e_i$ and $t_i < t_{i+1}$.

The following assumption avoids trajectories staying inside a region without leaving it.

Assumption 2. We will only consider trajectories with infinite signatures.

3.1 Successors and predecessors

For computing the invariance kernel of an SPDI we need to know how to compute the set of reachable points from and to a given interval, i.e. the *successor* and *predecessor* operators. In SPDIs, these operators have some nice mathematical properties inherited from a class of mathematical functions called *truncated affine multi-valued functions*. We define first the latter and then we give the definition of successors and predecessors.

3.1.1 Truncated affine multivalued functions

Truncated affine multi-valued functions (TAMFs) will be used to express *predecessors* and *successors*. Before defining such functions we need the notion of *affine (multivalued) functions*.

Definition 3. *A (positive)* affine *function $f : \mathbb{R} \to \mathbb{R}$ is such that $f(x) = ax+b$ with $a > 0$. An* affine multivalued *function $F : \mathbb{R} \to 2^{\mathbb{R}}$, denoted $F = \langle f_l, f_u \rangle$, is defined by $F(x) = \langle f_l(x), f_u(x) \rangle$ where f_l and f_u are affine and $\langle \cdot, \cdot \rangle$ denotes an interval.*

In what follows we will consider only well-formed intervals, i.e. $\langle l, u \rangle$ is an interval iff $l \leq u$. For notational convenience, we do not make explicit whether intervals are open, closed, left-open or right-open, unless required for comprehension. For an interval $I = \langle l, u \rangle$ we have that $F(\langle l, u \rangle) = \langle f_l(l), f_u(u) \rangle$.

Definition 4. *The* inverse *of F is defined by $F^{-1}(x) = \{y \mid x \in F(y)\}$. The* universal inverse *of F is defined by $\tilde{F}^{-1}(I) = I'$ iff I' is the greatest non-empty interval such that for all $x \in I'$, $F(x) \subseteq I$.*

Remark 1. It is not difficult to show that $F^{-1} = \langle f_u^{-1}, f_l^{-1} \rangle$ and similarly that $\tilde{F}^{-1} = \langle f_l^{-1}, f_u^{-1} \rangle$, provided that $\langle f_l^{-1}, f_u^{-1} \rangle \neq \emptyset$ (see Lemma 10 in the Appendix). Notice that if I is a singleton then $\tilde{F}^{-1}$ is defined only if $f_l = f_u$. The class of affine (multivalued) functions is closed under composition.

We define now a new class of functions which arises from the class of affine multivalued functions by truncating the argument and the image of such functions.

Definition 5. *A* truncated affine multivalued *function (TAMF)* $\mathcal{F} : \mathbb{R} \to 2^{\mathbb{R}}$ *is defined by an affine multivalued function F and intervals* $S \subseteq \mathbb{R}^+$ *and* $J \subseteq \mathbb{R}^+$ *as follows:* $\mathcal{F}(x) = F(x) \cap J$ *if* $x \in S$*, otherwise* $\mathcal{F}(x) = \emptyset$*.*

For convenience we write $\mathcal{F}(x) = F(\{x\} \cap S) \cap J$. For an interval I, $\mathcal{F}(I) = F(I \cap S) \cap J$ and $\mathcal{F}^{-1}(I) = F^{-1}(I \cap J) \cap S$. We say that $\mathcal{F}$ is *normalised* if $S = \mathsf{Dom}(\mathcal{F}) = \{x \mid F(x) \cap J \neq \emptyset\}$ (thus, $S \subseteq F^{-1}(J)$) and $J = \mathsf{Im}(\mathcal{F}) = \mathcal{F}(S)$. In what follows we only consider normalised TAMFs.

As for AMFs we define now the universal inverse of a TAMF.

Definition 6. *The* universal inverse *of* $\mathcal{F}$ *is defined by* $\tilde{\mathcal{F}}^{-1}(I) = I'$ *iff* I' *is the greatest non-empty interval such that for all* $x \in I'$*,* $F(x) \subseteq I$ *and* $F(x) = \mathcal{F}(x)$*.*

The following theorem states that TAMFs are closed under composition (see [Schneider 2002]).

Theorem 1. The composition of two TAMFs $\mathcal{F}_1(I) = F_1(I \cap S_1) \cap J_1$ and $\mathcal{F}_2(I) = F_2(I \cap S_2) \cap J_2$, is the TAMF $(\mathcal{F}_2 \circ \mathcal{F}_1)(I) = \mathcal{F}(I) = F(I \cap S) \cap J$, where $F = F_2 \circ F_1$, $S = S_1 \cap F_1^{-1}(J_1 \cap S_2)$ and $J = J_2 \cap F_2(J_1 \cap S_2)$. $\square$

3.1.2 Successors and Predecessors

We define now our main technical tools for computing invariance kernels, namely the *successor* and *predecessor* operators. We need first to give an effective representation of (rational) points and intervals on edges.

Given an SPDI, we fix a one-dimensional coordinate system on each edge to represent points laying on edges. For notational convenience, we indistinctly use letter e to denote the edge or its one-dimensional representation. Accordingly, we write $\mathbf{x} \in e$ or $x \in e$, to mean "point $\mathbf{x}$ in edge e with coordinate x in the one-dimensional coordinate system of e". The same convention is applied to sets of points of e represented as intervals (e.g., $\mathbf{x} \in I$ or $x \in I$, where $I \subseteq e$) and to trajectories (e.g., "ξ starting in x" or "ξ starting in $\mathbf{x}$"). Now, let $P \in \mathbb{P}$, $e \in In(P)$ and $e' \in Out(P)$. We are now in condition of defining the edge-to-edge successor of an SPDI.

Definition 7. *For* $I \subseteq e$*,* $\mathsf{Succ}_{ee'}(I)$ *is the set of all points in* e' *reachable from some point in I by a trajectory segment* $\xi : [0, t] \to \mathbb{R}^2$ *in P (i.e.,* $\xi(0) \in I \wedge \xi(t) \in e' \wedge \mathsf{Sig}(\xi) = ee'$*).*

Ege-to-edge successors are TAMFs [Schneider 2002], then $\mathsf{Succ}_{ee'}(I) = \mathcal{F}(I) = F(I \cap S) \cap J$ for some $S \subseteq e$ and $J \subseteq e'$. Notice that it is always possible to chose the positive direction on every edge in order to guarantee positive coefficients in the TAMF.

Example 2. Let us consider the SPDI of Example 1 and let $e_1, \ldots, e_8$ be as in Fig. 1 and $I = [l, u]$. We assume a one-dimensional coordinate system such that $e_i = S_i =$

$J_i = (0, 1)$. We have that:

$$F_{e_1e_2}(I) = \left[\frac{l}{2}, \frac{u}{2}\right] \qquad F_{e_2e_3}(I) = \left[l - \frac{1}{10}, u + \frac{11}{60}\right]$$

$$F_{e_ie_{i+1}}(I) = I \quad 3 \leq i \leq 7 \qquad F_{e_8e_1}(I) = \left[l + \frac{1}{5}, u + \frac{1}{5}\right]$$

with $\mathsf{Succ}_{e_ie_{i+1}}(I) = F_{e_ie_{i+1}}(I \cap S_i) \cap J_{i+1}$, for $1 \leq i \leq 7$, and $\mathsf{Succ}_{e_8e_1}(I) = F_{e_8e_1}(I \cap S_8) \cap J_1$.

By Theorem 1 we know that successors are closed by composition. Given a sequence $w = e_1, e_2, \ldots, e_n$, the successor of I along w defined as $\mathsf{Succ}_w(I) = \mathsf{Succ}_{e_{n-1}e_n} \circ \ldots \circ \mathsf{Succ}_{e_1e_2}(I)$ is a TAMF.

Example 3. Let $\sigma = e_1 \cdots e_8e_1$ be as before. In Example 2 we have defined the edge-to-edge AMFs $F_{e_1e_2}, \ldots, F_{e_7e_8}, F_{e_8e_1}$. We have that $\mathsf{Succ}_\sigma(I) = F(I \cap S) \cap J$, where F is obtained composing the above functions:

$$F(I) \quad = \quad \left[\frac{l}{2} + \frac{1}{10}, \frac{u}{2} + \frac{23}{60}\right] \tag{1}$$

$S = (0, 1)$ and $J = (\frac{1}{5}, \frac{53}{60})$ are computed using Theorem 1.

We can define in a similar way the edge-to-edge predecessor of an interval.

Definition 8. *For $I \subseteq e'$, $\mathsf{Pre}_{ee'}(I)$ is the set of points in e that can reach a point in I by a trajectory segment in P.*

We have that [Asarin *et al.* 2001]: $\mathsf{Pre}_{ee'} = \mathsf{Succ}_{ee'}^{-1}$ and $\mathsf{Pre}_\sigma = \mathsf{Succ}_\sigma^{-1}$.

Example 4. Continuing with our examples, let $\sigma = e_1 \ldots e_8e_1$ be as before and $I = [l, u]$. We have that $\mathsf{Pre}_{e_ie_{i+1}}(I) = F^{-1}_{e_ie_{i+1}}(I \cap J_{i+1}) \cap S_i$, for $1 \leq i \leq 7$, and $\mathsf{Pre}_{e_8e_1}(I) = F^{-1}_{e_8e_1}(I \cap J_1) \cap S_8$, where:

$$F^{-1}_{e_1e_2}(I) = [2l, 2u] \qquad F^{-1}_{e_2e_3}(I) = \left[l - \frac{11}{60}, u + \frac{1}{10}\right]$$

$$F^{-1}_{e_ie_{i+1}}(I) = I \quad 3 \leq i \leq 7 \qquad F^{-1}_{e_8e_1}(I) = \left[l - \frac{1}{5}, u - \frac{1}{5}\right]$$

Besides, $\mathsf{Pre}_\sigma(I) = F^{-1}(I \cap J) \cap S$, where $F^{-1}(I) = [2l - \frac{23}{30}, 2u - \frac{1}{5}]$.

Given two edges e and e' and an interval $I \subseteq e'$ we define the $\forall$-predecessor $\widetilde{\mathsf{Pre}}(I)$ in a similar way to $\mathsf{Pre}(I)$ using the universal inverse (see Definition 6) instead of just the inverse.

Definition 9. *For $I \subseteq e'$, $\widetilde{\mathsf{Pre}}_{ee'}(I)$ is the set of points in e such that any successor of such points are in I by a trajectory segment in P.*

We have that $\widetilde{\mathsf{Pre}}_{ee'} = \widetilde{\mathsf{Suc}}_{ee'}^{-1}$ and, given a signature $\sigma = e_1e_2 \ldots e_m$, $\widetilde{\mathsf{Pre}}_\sigma = \widetilde{\mathsf{Suc}}_\sigma^{-1}$.

3.2 Qualitative analysis of simple edge-cycles

We are interested in analysing the infinite behaviour of simple edge-cycles and we find then convenient to make a classification of such cycles. We need first some preliminary definitions.

Let $\sigma = e_1 \cdots e_k e_1$ be a simple edge-cycle, i.e., $e_i \neq e_j$ for all $1 \leq i \neq j \leq k$. Let $\mathsf{Succ}_\sigma(I) = F(I \cap S) \cap J$ with $F = \langle f_l, f_u \rangle$ (we suppose that this representation is normalised). We denote by $\mathcal{D}_\sigma$ the one-dimensional discrete-time dynamical system defined by Succ_σ, that is $x_{n+1} \in \mathsf{Succ}_\sigma(x_n)$.

Assumption 3. None of the two functions f_l, f_u is the identity.

Let l^* and u^* be the fix-points of f_l and f_u, respectively (the fix-point x^* is computed by solving a linear equation $f(x^*) = x^*$, which can be finite or infinite), and $S \cap J = \langle L, U \rangle$. By comparing the fix-points with the interval $\langle L, U \rangle$, we know (see [Schneider 2002]) that a simple cycle is of one of the following types:

STAY. The cycle is not abandoned neither by the leftmost nor the rightmost trajectory, that is, $L \leq l^* \leq u^* \leq U$.

DIE. The rightmost trajectory exits the cycle through the left (consequently the leftmost one also exits) or the leftmost trajectory exits the cycle through the right (consequently the rightmost one also exits), that is, $u^* < L \vee l^* > U$.

EXIT-BOTH. The leftmost trajectory exits the cycle through the left and the rightmost one through the right, that is, $l^* < L \wedge u^* > U$.

EXIT-LEFT. The leftmost trajectory exits the cycle (through the left) but the rightmost one stays inside, that is, $l^* < L \leq u^* \leq U$.

EXIT-RIGHT. The rightmost trajectory exits the cycle (through the right) but the leftmost one stays inside, that is, $L \leq l^* \leq U < u^*$.

Example 5. Let $\sigma = e_1 \cdots e_8 e_1$ be as before. We have that $S \cap J = \langle L, U \rangle = (\frac{1}{5}, \frac{53}{60})$. The fix-points of Equation (1) are such that $L = l^* = \frac{1}{5} < u^* = \frac{23}{30} < U$. Thus, σ is STAY.

The classification above gives us some information about the qualitative behaviour of trajectories. Any trajectory that enters a cycle of type DIE will eventually quit it after a finite number of turns. If the cycle is of type STAY, all trajectories that happen to enter it will keep turning inside it forever (provided they cycle at least once). In all other cases, some trajectories will turn for a while and then exit, and others will continue turning forever. This information is very useful for solving the reachability problem [Asarin *et al.* 2001] as well as for obtaining the invariance kernel.

Example 6. Consider again the cycle $\sigma = e_1 \cdots e_8 e_1$. Fig. 2 shows the reach set of the interval $[0.95, 1.0] \subset e_1$. Notice that the leftmost trajectory “converges to”

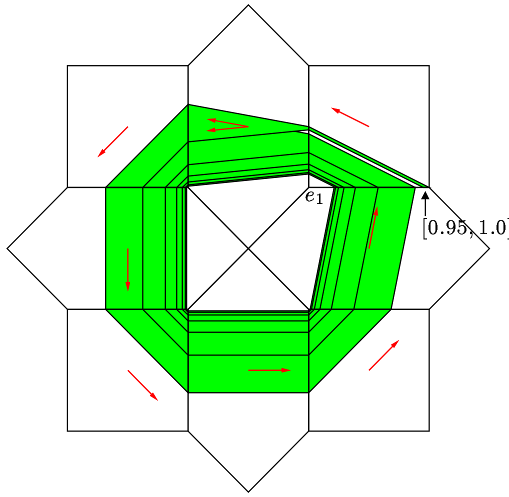

Fig. 2: Reachability analysis.

the limit $l^* = \frac{1}{5}$. Fig. 2 has been automatically generated by the SPeeDI toolbox [Asarin *et al.* 2002a] we have developed for reachability analysis of SPDIs based on the results given in [Asarin *et al.* 2001].

The above result does not allow us to directly answer other questions about the behaviour of the SPDI such as determine for a given point (or set of points) whether any trajectory (if it exists) starting in the point remains in the cycle forever. In order to do this, we need to further study the properties of the system around simple edge-cycles and in particular STAY cycles.

4. Invariance Kernel

In this section we present the main contributions of this paper, after defining *invariance kernels*. In a first part we prove some useful properties of STAY cycles, while in the last part we show how to effectively compute invariance kernels.

In general, an *invariant set* is a set of points such that for any point in the set it exists an infinite trajectory starting in such point and every such trajectory remains in the set forever and the *invariance kernel* is the largest of such sets.

In particular, for SPDI, given a cyclic signature, an *invariant set* is a set of points which keep rotating in the cycle forever and the *invariance kernel* is the largest of such sets. We show that this kernel is a non-convex polygon (often with a hole in the middle) and we give a non-iterative algorithm for computing the coordinates of its vertexes and edges.

In what follows, let $K \subset \mathbb{R}^2$. We recall the definition of *viable* trajectory [Aubin 1991].

DEFINITION 10. *A trajectory ξ is* viable *in K if $\xi(t) \in K$ for all $t \geq 0$. K is a* viability domain *if for every* $\mathbf{x} \in K$*, there exists at least one trajectory* ξ*, with* $\xi(0) = \mathbf{x}$*, which is viable in* K*.*

We give now the formal definition of invariance kernel.

DEFINITION 11. *We say that a set* K *is* invariant *iff for any* $x \in K$ *there exists at least one trajectory starting in it and every trajectory starting in* x *is viable in* K*. Given a set* K*, its largest invariant subset is called the* invariance kernel *of* K *and is denoted by* $\mathsf{Inv}(K)$*.*

4.1 Properties of STAY cycles

In this section we state some properties of STAY cycles. Recall that $\mathcal{D}_\sigma$ denotes the one-dimensional discrete-time dynamical system defined by Succ_σ, that is $x_{n+1} \in \mathsf{Succ}_\sigma(x_n)$. The concepts above can be defined for $\mathcal{D}_\sigma$, by setting that a trajectory $x_0 x_1 \ldots$ of $\mathcal{D}_\sigma$ is viable in an interval $I \subseteq \mathbb{R}$, if $x_i \in I$ for all $i \geq 0$. Similarly we say that an interval I in an edge e is invariant if *any* trajectory starting on $x_0 \in I$ is viable in I.

In what follows we provide a characterisation of one-dimensional discrete-time invariant but before we need the following notational convention. Remember that $\mathsf{Succ}(I) = F(I \cap S) \cap J$. Henceforth, whenever the signature σ is understood from the context we will use indistinctly the notation Succ_σ and $\mathcal{F}$ for the successor function, given that successors are TAMFs. Similarly for the predecessor function Pre_σ and $\mathcal{F}^{-1}$.

LEMMA 1. *For* $\mathcal{D}_\sigma$ *and* σ *a STAY cycle, the following is valid. If* I *is such that* $F(I) \subseteq I$ *and* $F(I) = \mathcal{F}(I)$ *then* I *is invariant. On the other hand if* I *is invariant then* $F(I) = \mathcal{F}(I)$*.*

PROOF. Suppose that $F(I) = \mathcal{F}(I)$ and $F(I) \subseteq I$, then $\mathcal{F}(I) \subseteq I$, thus by definition of STAY and monotonicity of $\mathcal{F}$, we know that for all n, $\mathcal{F}^n(I) \subseteq I$. Hence I is invariant. Let suppose now that I is invariant, then for any trajectory starting on $x_0 \in I$, $x_0 x_1 \ldots$ is in I and trivially $F(I) = \mathcal{F}(I)$. □

The following lemma states that, for STAY cycles, the image of the non-truncated successor function applied to $S \cap J$ is in $S \cap J$.

LEMMA 2. *For STAY cycles,* $F(S \cap J) \subseteq S \cap J$*.*

PROOF. Direct from the definition of STAY cycles and the fact that the fix-points are global attractors. □

We show now that for the kind of cycles under consideration, applying the truncated or the non-truncated successor function gives the same result, given that the argument is the interval $S \cap J$.

LEMMA 3. *For STAY cycles,* $\mathcal{F}(S \cap J) = F(S \cap J)$*.*

Proof. Suppose that $x \in \mathcal{F}(S \cap J)$, then $x \in F(S \cap J) \cap J$ by definition of $\mathcal{F}$, thus $x \in F(S \cap J)$. Let's prove now that $F(S \cap J) \subseteq \mathcal{F}(S \cap J)$. Let $x \in F(S \cap J)$, then $x \in S \cap J$ by Lemma 2. Hence $x \in F(S \cap J) \ \wedge \ x \in S \cap J$, i.e. $x \in F(S \cap J) \cap (S \cap J)$ or what is the same, $x \in (F(S \cap J) \cap J) \ \wedge \ x \in S$. Thus $x \in \mathcal{F}(S \cap J)$. □

The following result shows that the image of the successor function of a STAY cycle is included in its domain. That means that if we apply a successor to an interval $I \subseteq J$, we are sure that the result will be in its domain, thus we will be able to apply the function once more and again.

Lemma 4. *For STAY cycles, $J \subseteq S$.*

Proof. By normalisation, $J = \mathcal{F}(S)$ and by definition of STAY $\mathcal{F}(S) \subseteq S$, thus $J \subseteq S$. □

We give now a bound on the greatest interval having the property of equating truncated and non-truncated successors.

Lemma 5. *For STAY cycles, the greatest interval I_K such that $\mathcal{F}(I_K) = F(I_K)$ is bounded by $S \cap J$ and S, i.e. $S \cap J \subseteq I_K \subseteq S$.*

Proof. That $S \cap J \subseteq I_K$ follows immediately from Lemma 3 and the hypothesis. We prove now that $I_K \subseteq S$. By hypothesis and by definition of $\mathcal{F}$, $F(I_K) = F(I_K \cap S) \cap J$, then $F(I_K) \subseteq F(I_K \cap S)$, but by monotonicity of F this is only possible if $I_K \subseteq I_K \cap S$ that implies that $I_K \subseteq S$. □

Next lemma gives an upper bound for the universal inverse of a non-truncated successor when applied to its image J.

Lemma 6. *For STAY cycles, $\tilde{F}^{-1}(J) \subseteq S$.*

Proof. By Lemma 4, $J \subseteq S$ and by Lemma 2, $F(J) \subseteq J$. On the other hand $\mathcal{F}(S) = J$ and then $J \subseteq F(S)$ (since $\mathcal{F}(H) \subseteq F(H)$ for any H). Applying $\tilde{F}^{-1}$ to both sides we obtain $\tilde{F}^{-1}(J) \subseteq \tilde{F}^{-1}(F(S))$ and by Lemma 12 (see Appendix), $\tilde{F}^{-1}(F(S)) = S$ and then $\tilde{F}^{-1}(J) \subseteq S$. □

In what follows we show that the image of the successor function, namely J, is a fix-point of the composition function of the truncated successor and its universal inverse.

Lemma 7. *For STAY cycles, $\mathcal{F}(\tilde{\mathcal{F}}^{-1}(J)) = J$.*

Proof.

$$
\begin{aligned}
J &= \mathcal{F}(\tilde{\mathcal{F}}^{-1}(J)) & \\
&= \mathcal{F}(\tilde{F}^{-1}(J) \cap S) & \text{(by definition of } \tilde{\mathcal{F}}^{-1}\text{)}\\
&= F(\tilde{F}^{-1}(J) \cap S) \cap J & \text{(by definition of } \mathcal{F}\text{)}\\
&= F(\tilde{F}^{-1}(J)) \cap J & \text{(by Lemma 6)}\\
&= J \cap J & \text{(by Lemma 12 – see Appendix)}\\
&= J &
\end{aligned}
$$

□

The last lemma of this section shows that J is also a fix-point of the composition function of the non-truncated successor and the universal inverse of the truncated successor.

LEMMA 8. *For STAY cycles,* $F(\tilde{\mathcal{F}}^{-1}(J)) = \mathcal{F}(\tilde{\mathcal{F}}^{-1}(J))$.

PROOF. That $\mathcal{F}(\tilde{\mathcal{F}}^{-1}(J)) \subseteq F(\tilde{\mathcal{F}}^{-1}(J))$ follows directly from the definition of F and $\mathcal{F}$. We prove now that $F(\tilde{\mathcal{F}}^{-1}(J)) \subseteq \mathcal{F}(\tilde{\mathcal{F}}^{-1}(J))$. Let $x \in F(\tilde{\mathcal{F}}^{-1}(J))$, then by Lemma 11 (see Appendix), $x \in F(\tilde{F}^{-1}(J) \cap S)$. But by Lemma 6, $\tilde{F}^{-1}(J) \subseteq S$ and then $\tilde{F}^{-1}(J) \cap S \neq \emptyset$. We have then that $x \in F(\tilde{F}^{-1}(J)) \cap F(S)$ and by Lemma 12 (see Appendix) $x \in J \ \wedge \ x \in F(S)$. On the other hand, from $x \in F(\tilde{F}^{-1}(J) \cap S)$ and $x \in J$, $x \in F(\tilde{\mathcal{F}}^{-1}(J) \cap S) \cap J$ and by definition of $\mathcal{F}$, $x \in \mathcal{F}(\tilde{\mathcal{F}}^{-1}(J))$. □

Besides being important properties of STAY cycles, all the results shown in this section are crucial for computing invariance kernel, as shown in next section.

4.2 Invariance Kernel Computation

Before showing how to compute an invariance kernel for a cycle $\sigma = e_1 \ldots e_n e_1$, we will show how to compute it for the one-dimensional discrete-time system $\mathcal{D}_\sigma$, i.e. $\mathsf{Inv}(e_1)$.

THEOREM 2. *For* $\mathcal{D}_\sigma$, *if* $\sigma = e_1 \ldots e_n e_1$ *is STAY then* $\mathsf{Inv}(e_1) = \widetilde{\mathsf{Pre}}_\sigma(J)$, *otherwise* $\mathsf{Inv}(e_1) = \emptyset$.

PROOF. That $\mathsf{Inv}(e_1) = \emptyset$ for any type of cycle but STAY follows directly from the definition of each type of cycle.

Let's consider a STAY cycle with signature σ. Let $I_K = \tilde{\mathcal{F}}^{-1}(J) = \widetilde{\mathsf{Pre}}_\sigma(J)$. We know that $F(\tilde{\mathcal{F}}^{-1}(J)) = \mathcal{F}(\tilde{\mathcal{F}}^{-1}(J)) = J$ (see Lemmas 7 and 8). By Lemmas 2, 3 and 4, we have that $\mathcal{F}(J) \subseteq J$, so $J \subseteq \tilde{\mathcal{F}}^{-1}(J)$ and then $F(\tilde{\mathcal{F}}^{-1}(J)) \subseteq \tilde{\mathcal{F}}^{-1}(J)$. We have then, by Lemma 1, that I_K is invariant. We prove now that I_K is indeed the greatest invariant. Let suppose that there exists an invariant $H \subseteq S$ strictly greater than I_K. By assumption we have that $I_K = \tilde{\mathcal{F}}^{-1}(J) \subset H$, then by monotonicity of $\mathcal{F}$, $\mathcal{F}(\tilde{\mathcal{F}}^{-1}(J)) \subset \mathcal{F}(H)$ and by Lemma 7 we have that $J \subset \mathcal{F}(H)$, but this contradicts the monotonicity of $\mathcal{F}$ since $J = \mathcal{F}(S) \subset \mathcal{F}(H)$ and then $S \subset H$ which contradicts the hypothesis that $H \subseteq S$. Hence, $\mathsf{Inv}(e_1) = \widetilde{\mathsf{Pre}}_\sigma(J)$. □

The invariance kernel for the continuous-time system can be now found by propagating $\widetilde{\mathsf{Pre}}(J)$ from e_1 using the following operator. The *extended* ∀*-predecessor* of an output edge e of a region R is the set of points in R such that every trajectory segment starting in such point reaches e without traversing any other edge. More formally,

DEFINITION 12. *Let R be a region and e be an edge in* Out(R). *The e-*extended ∀-predecessor *of I,* $\widetilde{\widetilde{\mathsf{Pre}}}_e(I)$ *is defined as:*

$$\widetilde{\widetilde{\mathsf{Pre}}}_e(I) = \{\mathbf{x} \mid \forall\xi \,.\, (\xi(0) = \mathbf{x} \Rightarrow \exists t \geq 0 \,.\, (\xi(t) \in I \wedge \mathsf{Sig}(\xi[0,t]) = e))\}.$$

The above notion can be extended to cyclic signatures (and so to edge-signatures) as follows. Let $\sigma = e_1 \ldots e_k$ be a cyclic signature. For $I \subseteq e_1$, the *σ-extended $\forall$-predecessor* of I, $\widetilde{\overline{\mathsf{Pre}}}_\sigma(I)$ is the set of all $\mathbf{x} \in \mathbb{R}^2$ for which any trajectory segment ξ starting in $\mathbf{x}$, reaches some point in I, such that $\mathsf{Sig}(\xi)$ is a suffix of $e_2 \ldots e_k e_1$.

It is easy to see that $\widetilde{\overline{\mathsf{Pre}}}_\sigma(I)$ is a polygonal subset of the plane which can be calculated using the following procedure. First compute $\widetilde{\overline{\mathsf{Pre}}}_{e_i}(I)$ for all $1 \leq i \leq k$ and then apply this operation k times:

$$\widetilde{\overline{\mathsf{Pre}}}_\sigma(I) = \bigcup_{i=1}^{k} \widetilde{\overline{\mathsf{Pre}}}_{e_i}(I_i)$$

with

$$\begin{aligned} I_1 &= I, \\ I_k &= \widetilde{\mathsf{Pre}}_{e_k e_1}(I_1) \text{ and} \\ I_i &= \widetilde{\mathsf{Pre}}_{e_i e_{i+1}}(I_{i+1}), \text{ for } 2 \leq i \leq k-1. \end{aligned}$$

Now, let us define the following set:

$$K_\sigma \quad = \quad \bigcup_{i=1}^{k} (\mathtt{int}(P_i) \cup e_i) \tag{2}$$

where P_i is such that $e_{i-1} \in \mathit{In}(P_i)$, $e_i \in \mathit{Out}(P_i)$ and $\mathtt{int}(P_i)$ is the interior of P_i. We can now compute the invariance kernel of K_σ.

THEOREM 3. *If $\sigma = e_1 \ldots e_n e_1$ is STAY then* $\mathsf{Inv}(K_\sigma) = \widetilde{\overline{\mathsf{Pre}}}_\sigma(\widetilde{\mathsf{Pre}}_\sigma(J))$*, otherwise* $\mathsf{Inv}(K_\sigma) = \emptyset$.

PROOF. Trivially $\mathsf{Inv}(K_\sigma) = \emptyset$ for any type of cycle but STAY. That $\mathsf{Inv}(K_\sigma) = \widetilde{\overline{\mathsf{Pre}}}_\sigma(\widetilde{\mathsf{Pre}}_\sigma(J))$ for STAY cycles, follows directly from Theorem 2 and definition of $\widetilde{\overline{\mathsf{Pre}}}$. □

EXAMPLE 7. Let $\sigma = e_1 \ldots e_8 e_1$ on the SPDI considered before. Fig. 3 depicts: (a) K_σ, and (b) $\widetilde{\overline{\mathsf{Pre}}}_\sigma(\widetilde{\mathsf{Pre}}_\sigma(J))$.

We have at this point all the elements for providing an algorithm to compute all the invariance kernels of simple edge-cycles of a given SPDI.

for each simple cycle σ
  if σ is STAY
    then $\longleftarrow \widetilde{\overline{\mathsf{Pre}}}_\sigma(\widetilde{\mathsf{Pre}}_\sigma(J))$
    else $\longleftarrow \emptyset$

Soundness of the algorithm follows directly from Theorem 3.

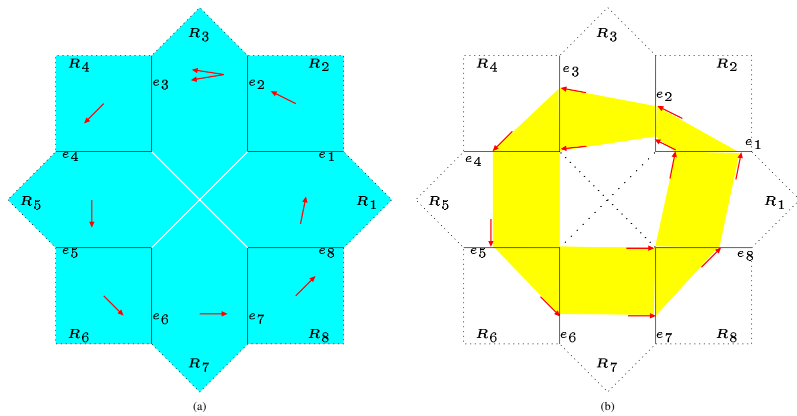

Fig. 3: Invariance kernel.

5. Conclusion

The main contribution of this paper is an automatic procedure to obtain an important object of the phase portrait of polygonal differential inclusions systems (SPDIs), namely invariance kernels. We have characterised where these kernels "live", namely in STAY cycles, and we have also proved many useful properties of such cycles. Two remarkable features of our algorithm is that it depends only on the computation of the predecessor and successor operators and that the computation is *exact*.

Related Work The use of differential inclusions as a tool for modeling uncertainties has been known for more than 70 years (Marchaud [1934]; Zaremba [1936]; see also the book by Aubin and Cellina [1984] for a more complete treatment of the subject). One of the reasons for a renewed interest in differential inclusions was the development of *viability theory* [Aubin 1991]. Invariance, as well as *viability* and *controllability* kernels may be defined under the framework of the viability theory. Despite of the similarity on the definition of viability and invariance kernels, they differ on the quantification over trajectories: For viability kernels we demand the *existence* of at least one trajectory remaining in the set forever while for invariance kernels we ask so for *every* trajectory.

Some results concerning the computation of viability kernels for hybrid systems using viability theory were given by Aubin *et al.* [2001], Aubin and Saint-Pierre [2003] and Saint-Pierre [2002], which is for sure a non exhaustive list. In particular, Aubin *et al.* [2001] give a characterisation of invariance kernels for impulsive differential inclusions. Works based on viability theory differ from ours mainly in that they usually compute an approximation of the kernels (using numerical methods) while in our case we compute them *exactly*. This is possible since we consider a specific class of hybrid systems in the plane, while the above-mentioned works are more general.

As already mentioned in the introduction, besides their inherent importance as phase portrait objects, invariance kernels are used to prove termination of a breadth-first search reachability algorithm for SPDIs, as shown in [Pace and Schneider 2003] (see also [Pace 2003] for more details about the algorithm).

Future Direction Invariance kernels provide an insight about the qualitative behaviour of SPDIs, characterising the "sinks" on simple edge-cycles. Since SPDIs could be used for approximating non-linear differential equations, one interesting application of such kernels is their use for finding the limit cycles of such equations.

Another relevant question is whether it would be possible to extend the result presented in this paper for higher dimensional hybrid systems. We believe it is possible to compute the invariance kernels of SPDIs defined over two dimensional manifolds, even though the reachability problem for such systems is an open question [Asarin and Schneider 2002].

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Appendix A. Some Properties of Successors and Predecessors

Remember that $\mathsf{Succ}(I) = F(I \cap S) \cap J$. As the successor functions Succ are TAMFs, we will use in what follows the notation $\mathcal{F}$ instead of Succ. Similarly for the predecessor function.

The following lemma gives a characterisation of $\tilde{F}^{-1}$:

Lemma 9. *If I' is non empty, then $\tilde{F}^{-1}(I) = I'$ iff $F(I') = I$.*

Proof. That $F(I') = I$ implies $\tilde{F}^{-1}(I) = I'$ follows directly from the definition of $\tilde{F}^{-1}$.

Assume now that $\tilde{F}^{-1}(I) = I'$, then by definition I' is the greatest non-empty interval such that for all $x \in I'$, $F(x) \subseteq I$. Hence $F(I') \subseteq I$. We prove now that $I \subseteq \tilde{F}^{-1}(I')$. Let $x \in I$, then it exists $y \in \tilde{F}^{-1}(x)$, i.e. $x \in F(y)$. Thus, $x \in F(I')$ and then $I \subseteq \tilde{F}^{-1}(I')$. We have then proved that $\tilde{F}^{-1}(I) = I'$ implies $F(I') = I$. □

It is not difficult to show that $\tilde{F}^{-1} = \langle f_l^{-1}, f_u^{-1} \rangle$:

Lemma 10. *Given an AMF $F = \langle f_l, f_u \rangle$, then $\tilde{F}^{-1} = \langle f_l^{-1}, f_u^{-1} \rangle$.*

Proof. Let $I = \langle l_I, u_I \rangle$ and suppose that $I' = \langle l_{I'}, u_{I'} \rangle$ is a non-empty interval.

$$
\begin{aligned}
\tilde{F}^{-1}(I) = I' \quad & \Longleftrightarrow F(I') = I && \text{(by Lemma 9)} \\
& \Longleftrightarrow \langle f_l(l_{I'}), f_u(u_{I'}) \rangle = \langle l_I, u_I \rangle \rangle \\
& \Longleftrightarrow f_l(l_{I'}) = l_I \;\wedge\; f_u(u_{I'}) = u_I \\
& \Longleftrightarrow l_{I'} = f_l^{-1}(l_I) \;\wedge\; u_{I'} = f_u^{-1}(u_I) \\
& \Longleftrightarrow \langle l_{I'}, u_{I'} \rangle = \langle f_l^{-1}(l_I) = f_u^{-1}(u_I) \rangle \\
& \Longleftrightarrow I' = \langle f_l^{-1}(l_I) = f_u^{-1}(u_I) \rangle \\
& \Longleftrightarrow \tilde{F}^{-1}(I) = \langle f_l^{-1}(l_I) = f_u^{-1}(u_I) \rangle
\end{aligned}
$$

□

The following Lemma relate $\tilde{F}^{-1}$ with $\tilde{\mathcal{F}}^{-1}$:

Lemma 11. *Given a TAMF $\mathcal{F}(I) = F(I \cap S) \cap J$, then $\tilde{\mathcal{F}}^{-1}(I) = \tilde{F}^{-1}(I \cap J) \cap S$.*

Proof. Similar to Lemma 1 of [Schneider 2002, p. 41]. □

Lemma 12. *If $\tilde{F}^{-1}(I) \neq \emptyset$, then $\tilde{F}^{-1}(F(I)) = F(\tilde{F}^{-1}(I)) = I$.*

Proof. Follows directly from the definition of $\tilde{F}^{-1}$. □

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