A new generation tree for permutations

Philippe Duchon & Romaric Duvignau

University of Bordeaux, LaBRI

What are we trying to do ?

We describe a new uniform generation tree for permutations with the specific property that, for most permutations, all of their descendants in the generation tree have the same number of fixed points. Our tree is optimal for the number of permutations having this property. Two probabilistic applications of such a tree are given: uniform derangement generation and Poisson variate generation.

Special Permutations $\sigma = (S, \delta)$ are permutations with δ being a critical derangement.

Uniform Generation Tree for Permutations

An infinite tree such that...

- Each node is a permutation
- Level n consists of $\mathcal{S}_n$, the set of permutations over $[n] = \{1, \dots, n\}$
- Each permutation of size n, has $n+1$ children

1. A description of a bijection ϕ_{n+1} between $\mathcal{S}_n \times [n+1]$ and $\mathcal{S}_{n+1}$, for each n
2. A generation algorithm of random uniform permutations

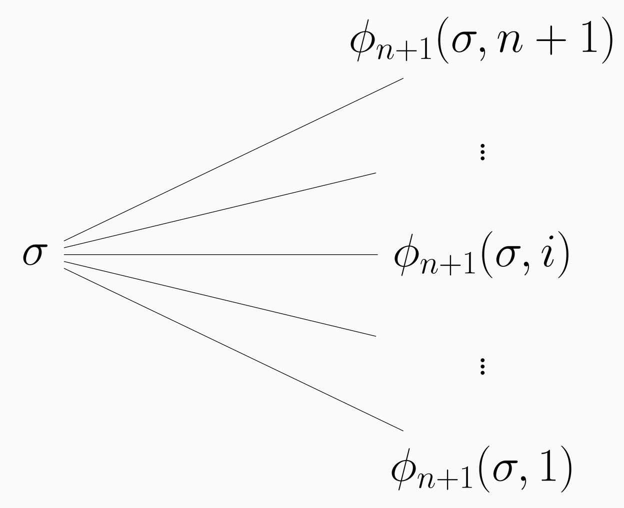
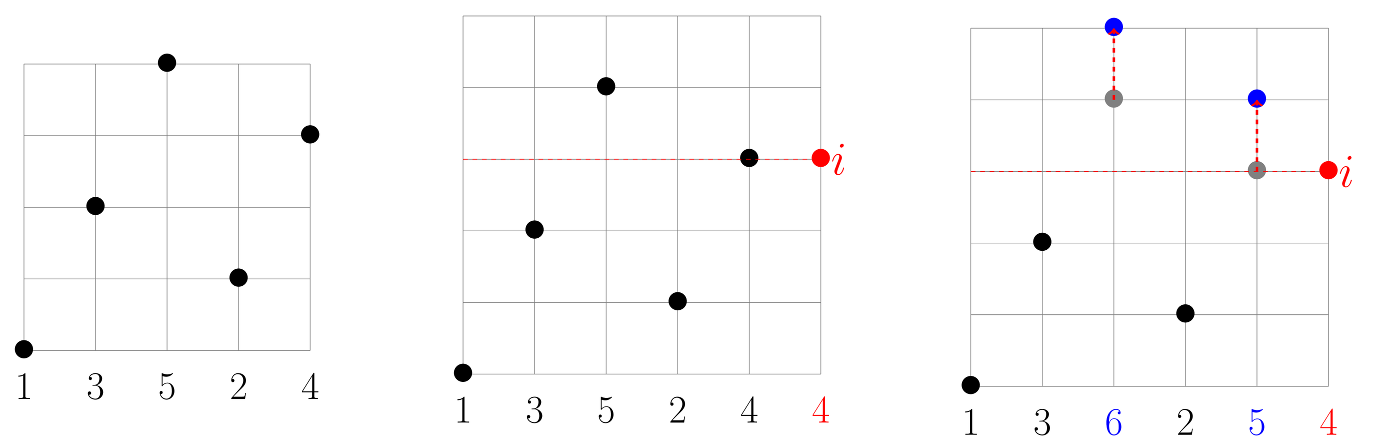


Fig. 1: A permutation σ of size n with its $n+1$ children.

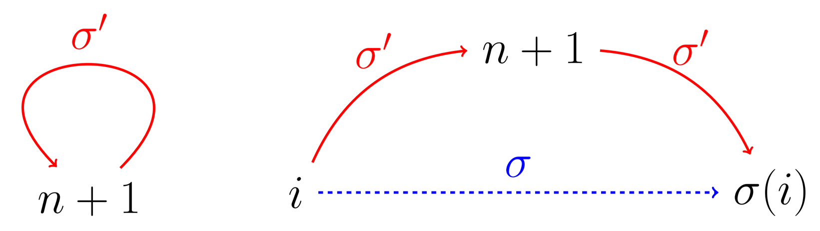
Example 1: Last value insertion

i becomes the last value, and all value greater than or equal to i are shifted by one.



Example 2: Cycle insertion

$n+1$ is *inserted* between i and $\sigma(i)$, in their cycle, or as a fixed point when $i = n+1$.



In the last value insertion generation tree, the number of fixed points may change dramatically between one permutation and its children. In the cycle insertion, it is potentially changed by only one, but it happens infinitely many times through an infinite random descent.

Our Generation Tree - Rules

Our tree is based on a non-standard way to represent a permutation $\sigma \in \mathcal{S}_n$ as **a pair** (S, δ), where S is the set of fixed points of σ and δ is the derangement of the non-fixed points of σ, normalized to fit in $[1..n-|S|]$. We note $\pi_S(i)$ the normalization of the non-fixed point i and $\gamma(\sigma)$ the greatest non-fixed point of σ (0 if σ is the identity).

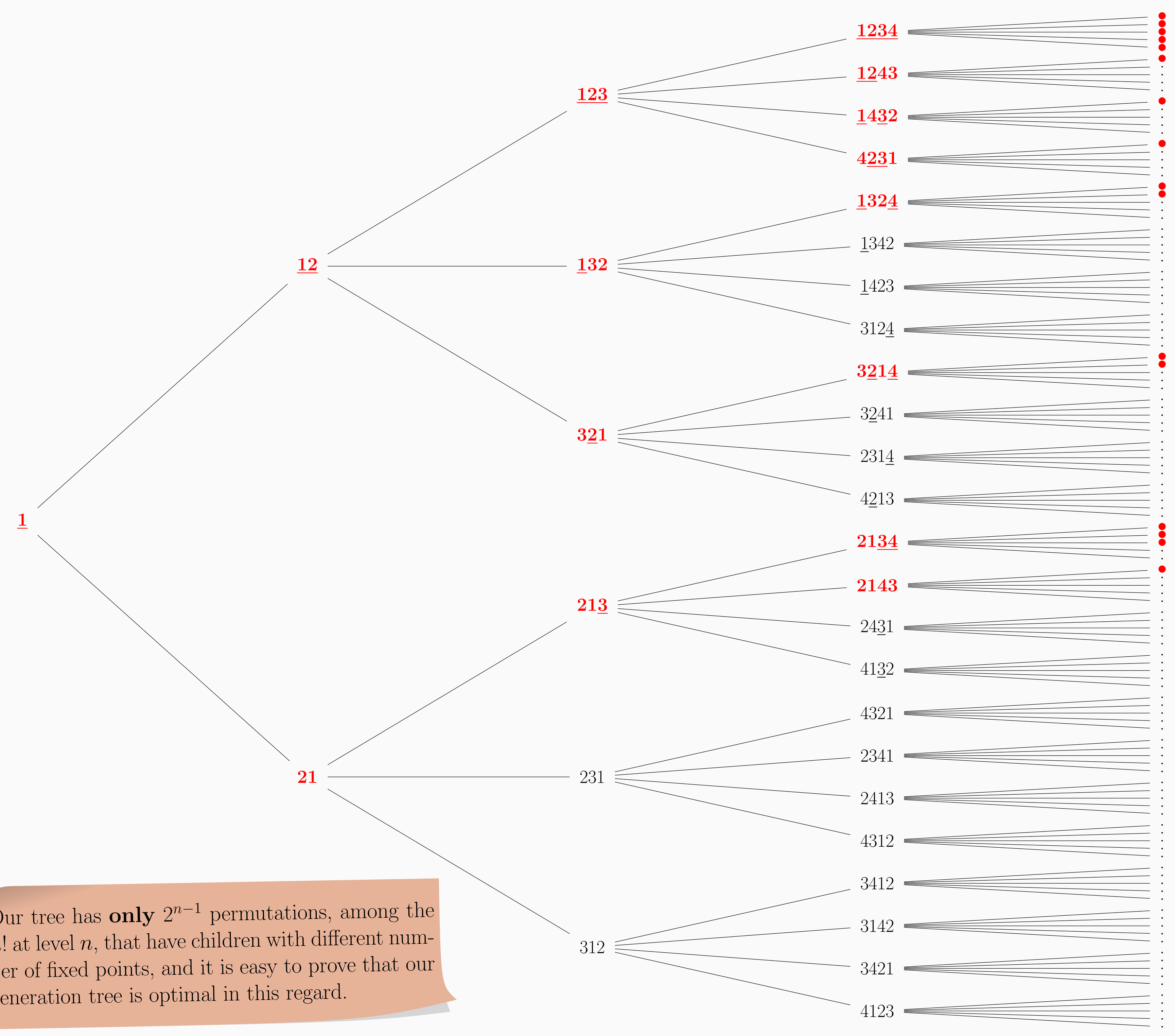
If $\sigma = (S, \delta)$ is a permutation of size n with k fixed points, then we construct σ', the i-th child of σ by the following rules:

1. If σ is special and $i = n+1$:
   $\sigma' = (S \cup \{n+1\}, \Delta_{n-k})$
2. If σ is special and $\gamma(\sigma) < i \leq n$:
   $\sigma' = (S \setminus \{i\}, \Delta_{n-k+2})$
3. If $i \in S$, σ is non-special or σ special and $i < \gamma(\sigma)$:
   $\sigma' = (S \setminus \{i\} \cup \{n+1\}, \tau_{n+1-k}(\delta, \pi_{S \setminus \{i\}}(i)))$
4. If $i \notin S$, σ is non-special or σ special and $i \neq n+1$:
   $\sigma' = (S, \tau_{n+1-k}(\delta, \pi_S(i)))$

Example

Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 5 & 3 & 4 & 6 & 2 & 7 \end{pmatrix} = (S, \delta)$.
Its set of fixed points is $S = \{1, 3, 4, 7\}$ and its normalized derangement is $\delta = (1\ 2\ 3)$ based on the derangement of its non-fixed points $(2\ 5\ 6)$.
Here $\pi_S(2) = 1$, $\pi_S(5) = 2$ and $\pi_S(6) = 3$.
Its 2-nd child is $\sigma' = (S, \delta')$ with $\delta' = \tau_4(\delta, 1) = (1\ 4\ 2\ 3)$. Its 3-rd child is $\sigma' = (S', \delta')$ with $S' = \{1, 4, 7, 8\}$, $\pi_{S'}(3) = 2$ and $\delta' = \tau_4(\delta, 2) = (1\ 2\ 4\ 3)$.

Our Generation Tree - First Levels



Our tree has **only** 2^{n-1} permutations, among the $n!$ at level n, that have children with different number of fixed points, and it is easy to prove that our generation tree is optimal in this regard.

Fig. 2: Our complete generation tree up to level 5. Each node at level n has $n+1$ children, depicted in order from bottom to top. Special permutations (red) are drawn in bold or bulleted. Fixed points are underlined.

Properties of our Tree

permutations σ with a different number of fixed points $\mathrm{fp}(\sigma)$ than their parent

- 2^{n-1} permutations of size n are **special**
- Being non-special is hereditary
- If σ is special, then
  - $\mathrm{fp}(\sigma) = \mathrm{fp}(\mathrm{parent}(\sigma)) \pm 1$
  - the last child of σ has $\mathrm{fp}(\sigma)+1$ fixed points, and the others have either $\mathrm{fp}(\sigma)$ or $\mathrm{fp}(\sigma)-1$ fixed points

Critical Derangements

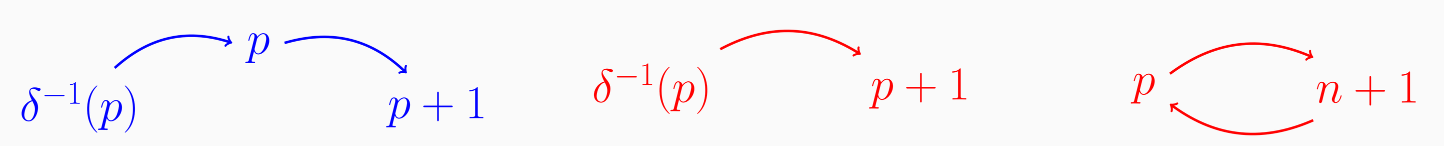
$\Delta_{2\ell}$ is the derangement of size 2ℓ into ℓ cycles of length 2, where each odd integer is paired with its successor, *i.e.* $\Delta_{2\ell} = (1\ 2)(3\ 4)\dots(2\ell-1\quad 2\ell)$.

τ does not produce $\Delta_{2\ell}$, and it is not defined on $(\Delta_{2\ell}, 2\ell+1)$

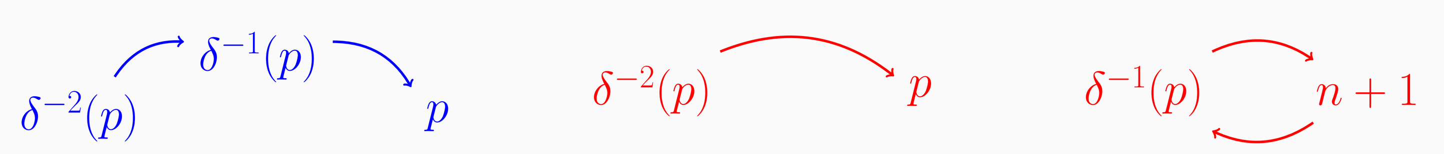
Bijection on Derangements

We need a bijection $\tau_{n+1} : \mathcal{D}_n \times [n+1] \setminus (\Delta_n, n+1) \to D_{n+1} \setminus \{\Delta_{n+1}\}$ for derangements (permutations with no fixed points) of size n in order to produce derangements of size $n+1$. We use directly for that purpose Rakotondrajao's bijection [3] over derangements that proves combinatorially the recurrence $d_n = nd_{n-1} + (-1)^n$, where d_n stands for $|\mathcal{D}_n|$.

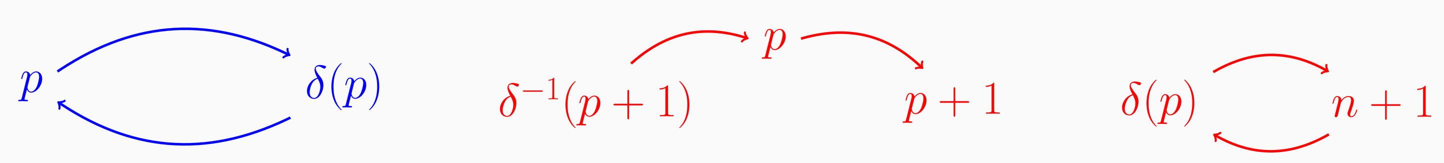
- $\tau_{n+1}(\delta, i)$ corresponds to the Cycle insertion whenever $i < n+1$
- Otherwise $i = n+1$ and $\tau_{n+1}(\delta, i)$ is defined as follows: let p be the greatest integer such that $\delta|_{[p-1]} = \Delta_{p-1}$ *i.e.* $\delta = (1\ 2)(3\ 4)\dots(p-2\ p-1)(p\ \delta(p)\ \dots)\dots$
  - If $\delta(p) = p+1$ (then $\delta(p+1) \neq p$):



  - If $\delta(p) \neq p+1$ and p is in a cycle of length > 2:



  - If $\delta(p) \neq p+1$ and p is in a cycle of length 2:



Probabilistic Applications

Derangement Generation

1. Perform a random descent in the tree until reaching a non-special permutation σ or level n, whichever comes first.
2. If the reached permutation is a derangement, continue the descent until reaching level n and return the derangement; otherwise, repeat Step 1.

This algorithm uses in expectation $n + O(1)$ calls to `Random()` – equivalent to $n\log_2(n) + o(n\log(n))$ random bits assuming perfect sampler [1] – which is asymptotically optimal, improving in this regard some recent techniques [2].

Poisson Variate

Sample from the Poisson distribution $p_k = e^{-\lambda}\lambda^k/k!$ with parameter 1, i.e. return $k \geq 0$ with probability $1/(ek!)$:

- Perform a random descent in the generation tree, until a non-special permutation is reached; output its number of fixed points.

This algorithm uses, in expectation, between 6.89 and 6.9 random bits.

Algorithm Poisson(1)

```
n ← 1, g ← 0, k ← 1
loop
    i ← Random(n + 1)
    if i = n + 1 then
        k ← k + 1
    else if i > g then
        k ← k − 1, g ← n + 1
    else
        return k
    end if
    n ← n + 1
end loop
```


[1] Donald E Knuth and Andrew C Yao. The complexity of nonuniform random number generation. In *Proceedings of Symposium on Algorithms and Complexity*, pages 357–428. Academic Press, 1976.

[2] Conrado Martínez, Alois Panholzer, and Helmut Prodinger. Generating random derangements. pages 234–240. SIAM, 2008.

[3] Fanja Rakotondrajao. K-fixed-points permutations. In *Integers: Electronic Journal of Combinatorial Number Theory*, volume 7, 2007.