What is mathematical thinking, and why should we care?

Dag Wedelin, February 8, 2018
ordinary thinking
and common
sense

“to think with the help of numbers,
shapes and other abstract patterns”
Mathematical thinking is a natural ability!
ΓΕΒ μέγεθος ἐστὶ τῆς ὑπὸ ΒΑΓ, ἀλλὰ ἐτοι ΒΕΒ μέγεθος ἐνικεῖται ἐν τῇ ὑπὸ ΒΑΓ· 
παραλλήλη δρόμος ἐν τῇ ὑπὸ ΒΑΓ μέγεθος ἐστὶ 
τῆς ὑπὸ ΒΑΓ·

Εἰς ὅρα ἄρα διεύρυνας ἅπαν τῶν εὐθειῶν ἀπὸ τῶν 
περίπου δῶν εὐθείας ἄντων συσταθῶσι, εἰς 
συσταθεῖσα τῶν 
κοινών τοῦ τριγώνου δῶν εὐθείας ἅπαν ἄρα 
μείζονα δὲ γωνίαν περιέχουσιν· ὅπερ ἔδει δεῖξαι

Ἐκ τριῶν εὐθειῶν, αἱ ἴσαι τὰς δοθείσαις

To construct a triangle from three straight-lines which are equal to three given (straight-lines). It is necessary for (the sum of) two (of the straight-lines) taken together in any (possible way) to be greater than the remaining (one), (on account of the fact that) in any triangle (the sum of) two sides taken together in any (possible way) is greater than the remaining (one) [Prop. 1.20].

Let A, B, and C be the three given straight-lines, of which let (the sum of) two taken together in any (possible way) be greater than the remaining (one). (Thus), (the sum of) A and B (is greater) than C, (the sum of) A and C than B, and also (the sum of) B and C than A. So it is required to construct a triangle from (straight-lines) equal to A, B, and C.

Let some straight-line DE be set out, terminated at D, and infinite in the direction of E. And let DF made equal to A, and FG equal to B, and GH equal to C [Prop. 1.3]. And let the circle DKL have been drawn with center F and radius FD. Again, let the circle KIH have been drawn with center K and radius G. And let KF and KG have been joined. I say that the triangle KFG has
What is needed to solve a problem?

knowledge needed for solving a problem = knowledge from others + knowledge created by own thinking

very different balance for different problems

because of the variation you often have to add something here!
For which problems and situations is mathematical thinking used?
“mathematics is used everywhere…”

(not so enlightening!)
Examples and problems

- Translation problems
- Interpreting quantitative information
- Homing
- Consumer test ranking
- Simple assignment
- What is the revenue?
- Data calibration
- Basic discrete structures
- Reading everyday texts
- Simple forecast
- Arithmetic and geometric mean
- Facility location
- Balancing chemical reactions
- Curve fitting
- What is the revenue?
- Shortest path
- Achilles and the tortoise
- Bridge problem
- Predict weather
- Consumption problems
- When is optimality guaranteed?
- Square root algorithm
- Size of the world
- Renewable energy system
- Map colouring
- Bokeh
- Homing
- Twelve balls problem
- Throw ball
- Whales and krill
- Project planning
- Expert system
- Interpreting quantitative information
- Language recognition
- Time control
- Random text (and music)
- Bokeh
- Emergency care problem
- Beam on two supports
- Medical test
- Sound intensity
- Arithmetic and geometric mean
- Temperature control
- Predict weather
- Bouncing balls
- Explain units
- Shortest path
- Renewable energy system
- Traffic simulation
- Dice simulation
- Radioactive decay
- Basic discrete structures
- Traffic simulation
- Prove algebraic laws
- Data calibration
- Computer graphics
- Sorting complexity
keeping track
Consumption problems
Translation problems
Explain units
Estimation
Interpreting quantitative information
Simple forecast
Homing
Reading everyday texts
Consumer test ranking
What is the revenue?
Data calibration
Basic discrete structures
...

investigating the abstract
Square root algorithm
Prove algebraic laws
Twelve balls problem
Arithmetic and geometric mean
Simple assignment
When is optimality guaranteed?
Lunch problem
Achilles and the tortoise
Dice simulation
Map colouring
Sorting complexity
...

investigating the world
Beam on two supports
Sound intensity
Bouncing balls
Predict weather
Size of the world
Curve fitting
Bokeh
Bridge problem
Throw ball
Whales and krill
Traffic simulation
Radioactive decay
Medical test
Balancing chemical reactions
...

designing
 Rotary encoder
Facility location
Medicine dose
Random text (and music)
Project planning
Expert system
Renewable energy system
Computer graphics
Emergency care problem
Shortest path
Temperature control
Language recognition
...
keeping track
investigating the abstract
investigating the world
designing

the best materials

low energy

low cost

lots of light

iteration

design and creativity
Understanding the nature of problems is important for your ability to solve them!
towards a theory of mathematical thinking
IF $\rho \text{ THEN } q$

$= \neg \rho \lor q$

Mathematical reasoning

Problem solving

Mathematical modelling
mathematical reasoning
What is knowledge?

Nature of knowledge

Awareness of what you know, what you believe and what you don’t know

Understanding

Make every effort not to be wrong!

The scientific method
What is reasoning?

Nature of reasoning
(plausible/deductive,
premise/conclusion,
necessary/sufficient)

Reasoning concepts
(definition, conjecture,
derivation, proof,
calculation)

How reasoning
connects statements

Nature of reasoning
(plausible/deductive,
premise/conclusion,
necessary/sufficient)

Reasoning
errors

A → B

premise conclusion

Note:
A doesn't nineteen
argument to be correct!

an not to be confused with causality!
The importance of precision!

- How much fuel do we have?
- Quite a lot
- So how far can we go?
- Pretty far
- Will it be sufficient for our trip?
- ...

begin with clear definitions!
26 \times 31 = 806
31 \times 26 = 806
What kinds of statements is pure mathematics concerned with?

The story of how people think to create mathematics is not told.
mathematical modelling
Why models?

“a convenient way to represent reality so that we more easily can draw conclusions about it”
Simple and complicated models for different purposes
The modelling cycle

real situation or problem

mathematical model

explanations and answers

mathematical conclusions

simplify, make precise, select the modelling approach!

reality model

(at least a design problem)
problem solving
How can you handle your limited capacity?
A challenging task can be handled by working in small and efficient steps!
Typical workflow - easy problems

1. You easily understand the problem

2. You quickly see how to solve it

3. The problem is solved / implement the solution with no surprises
Typical workflow - intermediate problems

1. Understand the problem
2. Make a plan
3. Carry out the plan
4. Look back (check your result, reflect on the process, ...)

(Polya)
Typical workflow – more difficult problems

1. Understand the problem
2. Make a plan
3. Carry out the plan
4. Look back

Investigate for deeper understanding, define clearly
Explore different approaches, begin with something simple!
When you fail you learn and go back

Continuously reflect, go back and revise, manage your time
Problem solving is **not algorithmic** - even experts need to try things out!

You need to develop *experience and intuition* by solving quite a few realistic problems yourself!
summary of how we think and work mathematically
real problems and situations

thinking

- problem solving
- modelling
- reasoning

knowledge and tools

- keeping track
- investigating the world
- investigating the abstract

functions/equations
- geometry
- optimization
- differential equations
- probability and statistics
- discrete math

pencil and paper computing
How does this relate to mathematics education?
Most mathematics education

knowledge needed for solving a problem = knowledge from others + knowledge created by own thinking

the emphasis is here!
mathematics mostly seen as preparation

- functions/equations
- geometry
- optimization
- differential equations
- probability and statistics
- discrete math

- pencil and paper computing
- designing
- investigating the world
- investigating the abstract
- reasoning
- modelling
- problem solving
- keeping track

no significant focus on thinking and the creation of knowledge

specific and repetitive exercises

absurd problems (in school)
The result is that students are often not able to use the mathematics they already know!

(this is also why I started to engage in this!)
In the beginning of our modelling and problem solving course
(2016 reports - mostly software engineering students end of year 2)

“We always thought that there were ready-made formulas for everything.”

“The distinct difference between reality and mathematics was something we had never reflected over.”

“If George Polya had seen us, he would probably say that we went against everything he ever said.”

“Math was so much more than just doing calculations.”

...
thinking

knowledge and tools

realistic problems and situations

for eight weeks...

keeping track

investigating the world

designing

functions/equations
geometry
optimization
differential equations
probability and statistics
discrete math

pencil and paper computing

reasoning

modelling

problem solving

investigating the abstract

problem solving
Example: finding a function for a physical relationship

<table>
<thead>
<tr>
<th>T (time)</th>
<th>D (distance)</th>
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<tbody>
<tr>
<td>88.0</td>
<td>57.9</td>
</tr>
<tr>
<td>224.7</td>
<td>108.2</td>
</tr>
<tr>
<td>365.3</td>
<td>149.6</td>
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<td>4490.8</td>
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<tr>
<td>90467</td>
<td>5879.13</td>
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</tbody>
</table>
Inquiry-based learning example
A varied set of realistic and challenging problems

We'll use the first played with great deal with your facility spend waiting 1000 times…

Patterns across problems become visible!

Students build a case library of experiences!
At the end of the course (2016 reports)

“We have learned a new way of thinking.”

“A unique property of the course is the balance between reasoning and the use of established techniques.”

“The first course that made us feel as engineers.”

“We had not realized how good this course is for us as future mathematics teachers.”

“My rate of development has been enormous”

“Imagine if we had been given more of this earlier in our education.”
Dynamic/creative attitude

Static attitude

Time

(student group 2016)
Cooperation with Chalmers EER


…
what about computers and programming?
Programming enables a new range of creative problem solving tasks

Can reduce the use of artificial repetitive exercises?

Requires an attitude of being careful

Tools like Mathematica are also important
Models, algorithms and software
Bösendorfer mic'd for sampling
Example of changing underlying models: electronic pianos

1960’s: simple waveform and decay synthesis

1980’s: sampling synthesis

2000- : physical modelling
summary
knowledge needed for solving a problem = knowledge from others + knowledge created by own thinking

you often need to add something yourself!
realistic problems and situations

thinking

knowledge and tools

keeping track

functions/equations

reasoning

designing

geometry

investigating the world

optimization

investigating the abstract

differential equations

problem solving

discrete math

pencil and paper computing

computing
What's on your plate kids?

Healthy Eating is as easy as 1,2,3

Salad & Veggies
Keep It Colourful

Protein

Carbohydrates
realistic problems and situations

- investigating the abstract
- investigating the world
- designing
- functions/equations
- geometry
- optimization
- differential equations
- probability and statistics
- discrete math
- pencil and paper
- computing

thinking

- problem solving
- modelling
- reasoning

knowledge and tools

keeping track

- keeping track
- problem solving
- modelling
- reasoning
- thinking
END