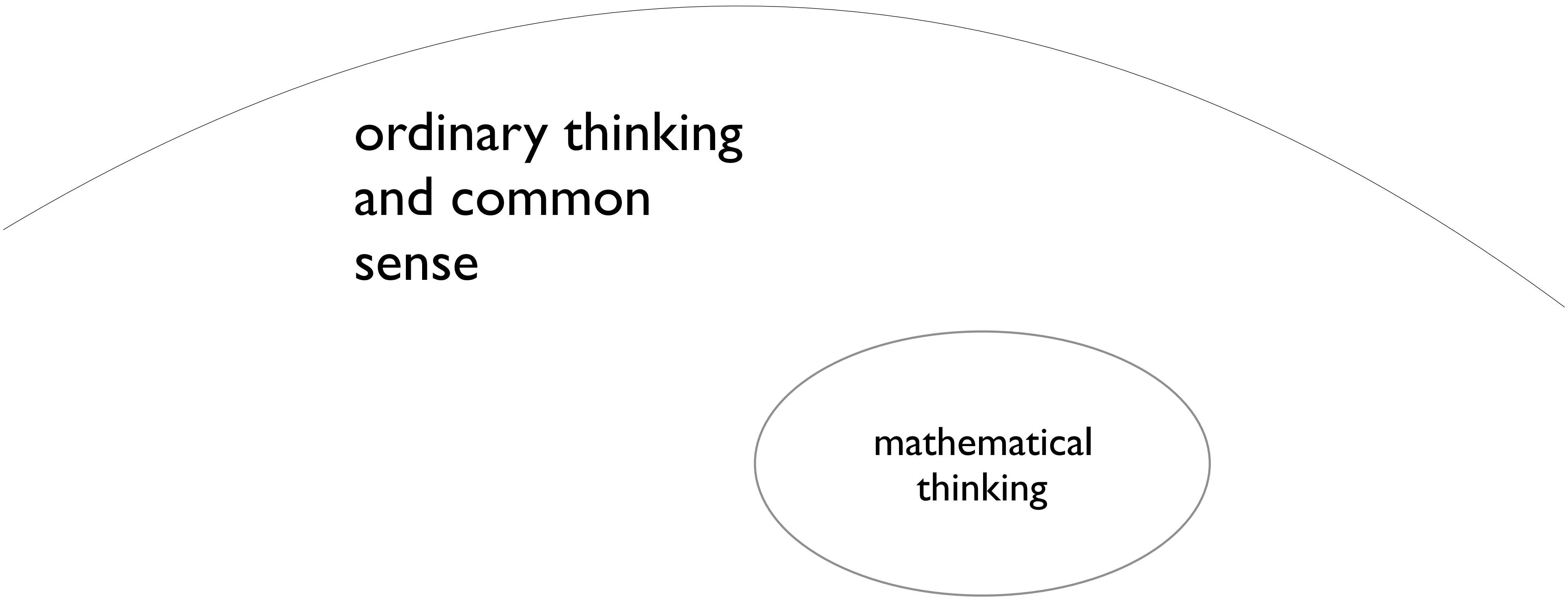
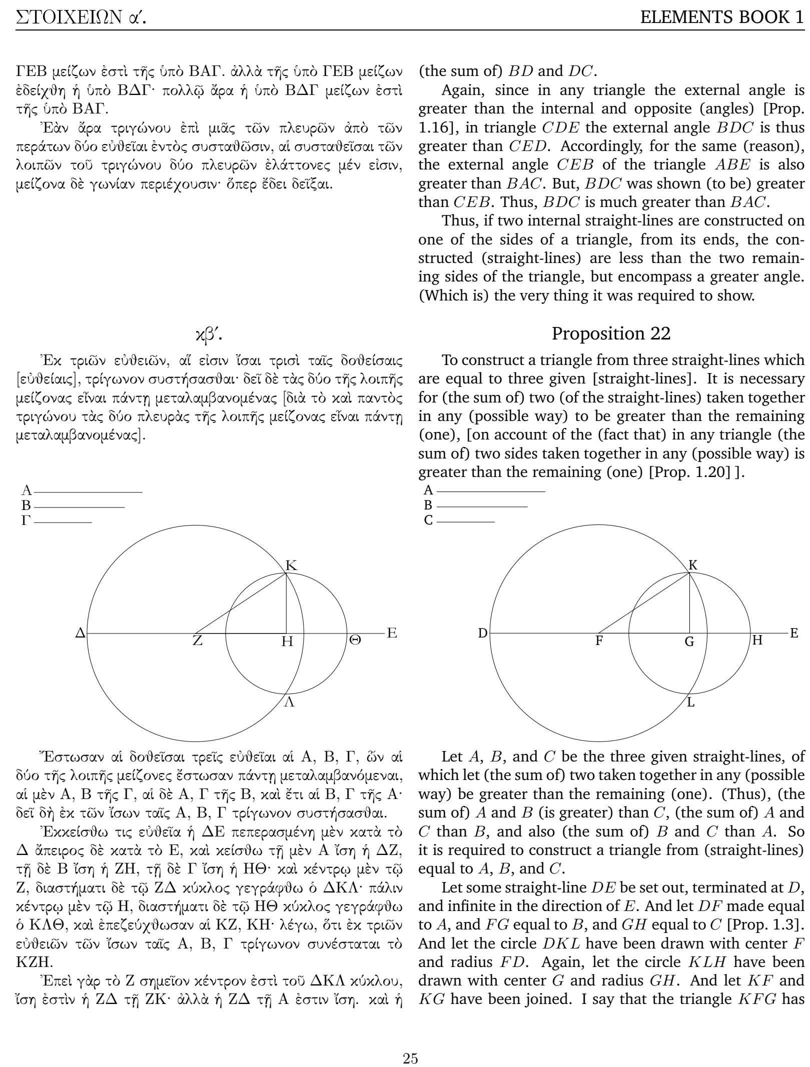
What is mathematical thinking, and why should we care?

Dag Wedelin, February 8, 2018

*“to think with the help of numbers,
shapes and other abstract patterns”*



Mathematical thinking
is a natural ability!



ΣΤΟΙΧΕΙΩΝ α΄. ELEMENTS BOOK 1

ΓΕΒ μείζων ἐστὶ τῆς ὑπὸ ΒΑΓ. ἀλλὰ τῆς ὑπὸ ΓΕΒ μείζων ἐδείχθη ἡ ὑπὸ ΒΔΓ· πολλῷ ἄρα ἡ ὑπὸ ΒΔΓ μείζων ἐστὶ τῆς ὑπὸ ΒΑΓ.

Ἐὰν ἄρα τριγώνου ἐπὶ μιᾶς τῶν πλευρῶν ἀπὸ τῶν περάτων δύο εὐθεῖαι ἐντὸς συσταθῶσιν, αἱ συσταθεῖσαι τῶν λοιπῶν τοῦ τριγώνου δύο πλευρῶν ἐλάττονες μέν εἰσιν, μείζονα δὲ γωνίαν περιέχουσιν· ὅπερ ἔδει δεῖξαι.

(the sum of) BD and DC.

Again, since in any triangle the external angle is greater than the internal and opposite (angles) [Prop. 1.16], in triangle CDE the external angle BDC is thus greater than CED. Accordingly, for the same (reason), the external angle CEB of the triangle ABE is also greater than BAC. But, BDC was shown (to be) greater than CEB. Thus, BDC is much greater than BAC.

Thus, if two internal straight-lines are constructed on one of the sides of a triangle, from its ends, the constructed (straight-lines) are less than the two remaining sides of the triangle, but encompass a greater angle. (Which is) the very thing it was required to show.

κβ΄.

Ἐκ τριῶν εὐθειῶν, αἵ εἰσιν ἴσαι τρισὶ ταῖς δοθείσαις [εὐθείαις], τρίγωνον συστήσασθαι· δεῖ δὲ τὰς δύο τῆς λοιπῆς μείζονας εἶναι πάντῃ μεταλαμβανομένας [διὰ τὸ καὶ παντὸς τριγώνου τὰς δύο πλευρὰς τῆς λοιπῆς μείζονας εἶναι πάντῃ μεταλαμβανομένας].

Proposition 22

To construct a triangle from three straight-lines which are equal to three given [straight-lines]. It is necessary for (the sum of) two (of the straight-lines) taken together in any (possible way) to be greater than the remaining (one), [on account of the (fact that) in any triangle (the sum of) two sides taken together in any (possible way) is greater than the remaining (one) [Prop. 1.20]].

Ἔστωσαν αἱ δοθεῖσαι τρεῖς εὐθεῖαι αἱ Α, Β, Γ, ὧν αἱ δύο τῆς λοιπῆς μείζονες ἔστωσαν πάντῃ μεταλαμβανόμεναι, αἱ μὲν Α, Β τῆς Γ, αἱ δὲ Α, Γ τῆς Β, καὶ ἔτι αἱ Β, Γ τῆς Α· δεῖ δὴ ἐκ τῶν ἴσων ταῖς Α, Β, Γ τρίγωνον συστήσασθαι.

Ἐκκείσθω τις εὐθεῖα ἡ ΔΕ πεπερασμένη μὲν κατὰ τὸ Δ ἄπειρος δὲ κατὰ τὸ Ε, καὶ κείσθω τῇ μὲν Α ἴση ἡ ΔΖ, τῇ δὲ Β ἴση ἡ ΖΗ, τῇ δὲ Γ ἴση ἡ ΗΘ· καὶ κέντρῳ μὲν τῷ Ζ, διαστήματι δὲ τῷ ΖΔ κύκλος γεγράφθω ὁ ΔΚΛ· πάλιν κέντρῳ μὲν τῷ Η, διαστήματι δὲ τῷ ΗΘ κύκλος γεγράφθω ὁ ΚΛΘ, καὶ ἐπεζεύχθωσαν αἱ ΚΖ, ΚΗ· λέγω, ὅτι ἐκ τριῶν εὐθειῶν τῶν ἴσων ταῖς Α, Β, Γ τρίγωνον συνέσταται τὸ ΚΖΗ.

Ἐπεὶ γὰρ τὸ Ζ σημεῖον κέντρον ἐστὶ τοῦ ΔΚΛ κύκλου, ἴση ἐστὶν ἡ ΖΔ τῇ ΖΚ· ἀλλὰ ἡ ΖΔ τῇ Α ἐστιν ἴση. καὶ ἡ

Let A, B, and C be the three given straight-lines, of which let (the sum of) two taken together in any (possible way) be greater than the remaining (one). (Thus), (the sum of) A and B (is greater) than C, (the sum of) A and C than B, and also (the sum of) B and C than A. So it is required to construct a triangle from (straight-lines) equal to A, B, and C.

Let some straight-line DE be set out, terminated at D, and infinite in the direction of E. And let DF made equal to A, and FG equal to B, and GH equal to C [Prop. 1.3]. And let the circle DKL have been drawn with center F and radius FD. Again, let the circle KLH have been drawn with center G and radius GH. And let KF and KG have been joined. I say that the triangle KFG has

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Thinking creates knowledge!

What is needed to solve a problem?

very different balance for different problems

knowledge needed for solving a problem = knowledge from others + knowledge created by own thinking

because of the variation you often have to add something here!

For which problems and situations is mathematical thinking used?

“mathematics is used everywhere…”

(not so enlightening!)

Examples and problems

Rotary encoder

Lunch problem

Emergency care problem

Simple assignment

Translation problems

Consumer test ranking

Simple forecast

Arithmetic and geometric mean

Temperature control

Facility location

Balancing chemical reactions

Curve fitting

Map colouring

What is the revenue?

Shortest path

Achilles and the tortoise

Bridge problem

Beam on two supports

Predict weather

Medical test

Square root algorithm

Consumption problems

Reading everyday texts

When is optimality guaranteed?

Size of the world

Random text (and music)

Renewable energy system

Twelve balls problem

Bokeh

Homing

Sound intensity

Estimation

Throw ball

Project planning

Bouncing balls

Explain units

Whales and krill

Language recognition

Expert system

Interpreting quantitative information

Radioactive decay

Basic discrete structures

Medicine dose

Traffic simulation

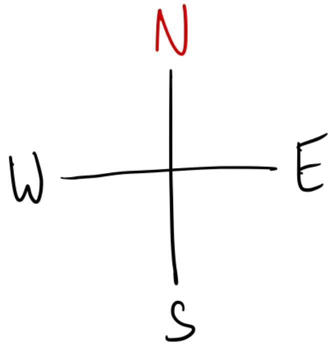
Dice simulation

Computer graphics

Prove algebraic laws

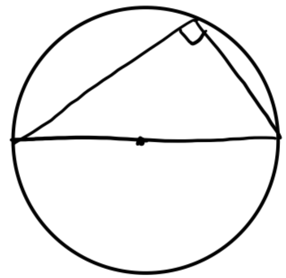
Data calibration

Sorting complexity



keeping track

Consumption problems
Translation problems
Explain units
Estimation
Interpreting quantitative information
Simple forecast
Homing
Reading everyday texts
Consumer test ranking
What is the revenue?
Data calibration
Basic discrete structures
…



investigating the abstract

Square root algorithm
Prove algebraic laws
Twelve balls problem
Arithmetic and geometric mean
Simple assignment
When is optimality guaranteed?
Lunch problem
Achilles and the tortoise
Dice simulation
Map colouring
Sorting complexity
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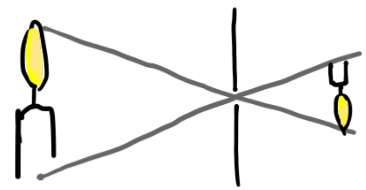
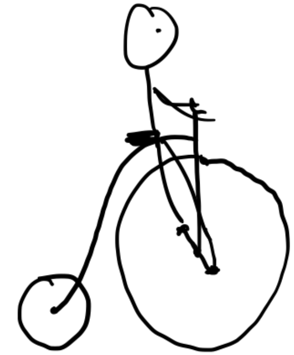


investigating the world

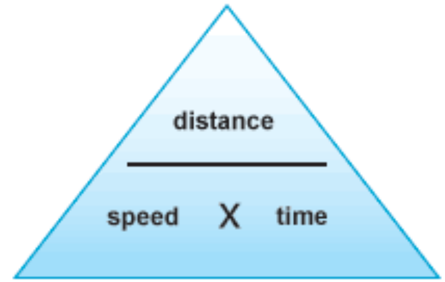
Beam on two supports
Sound intensity
Bouncing balls
Predict weather
Size of the world
Curve fitting
Bokeh
Bridge problem
Throw ball
Whales and krill
Traffic simulation
Radioactive decay
Medical test
Balancing chemical reactions
…

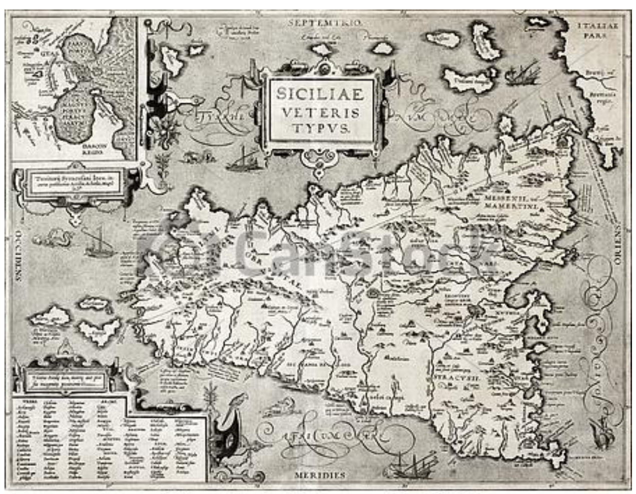


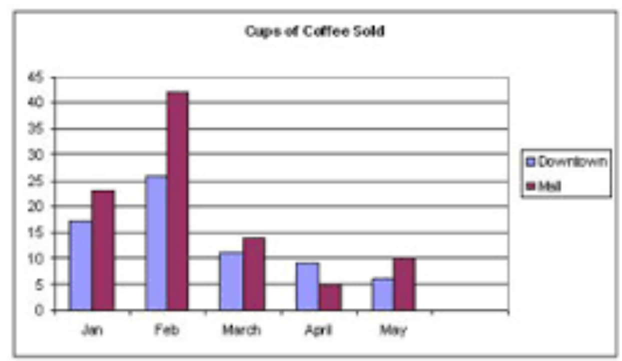
designing

Rotary encoder
Facility location
Medicine dose
Random text (and music)
Project planning
Expert system
Renewable energy system
Computer graphics
Emergency care problem
Shortest path
Temperature control
Language recognition
…

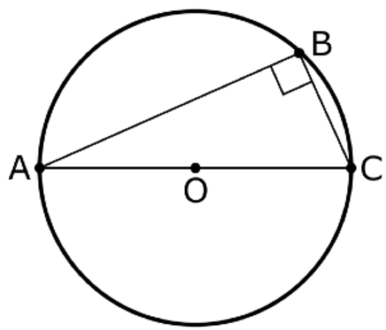
keeping track



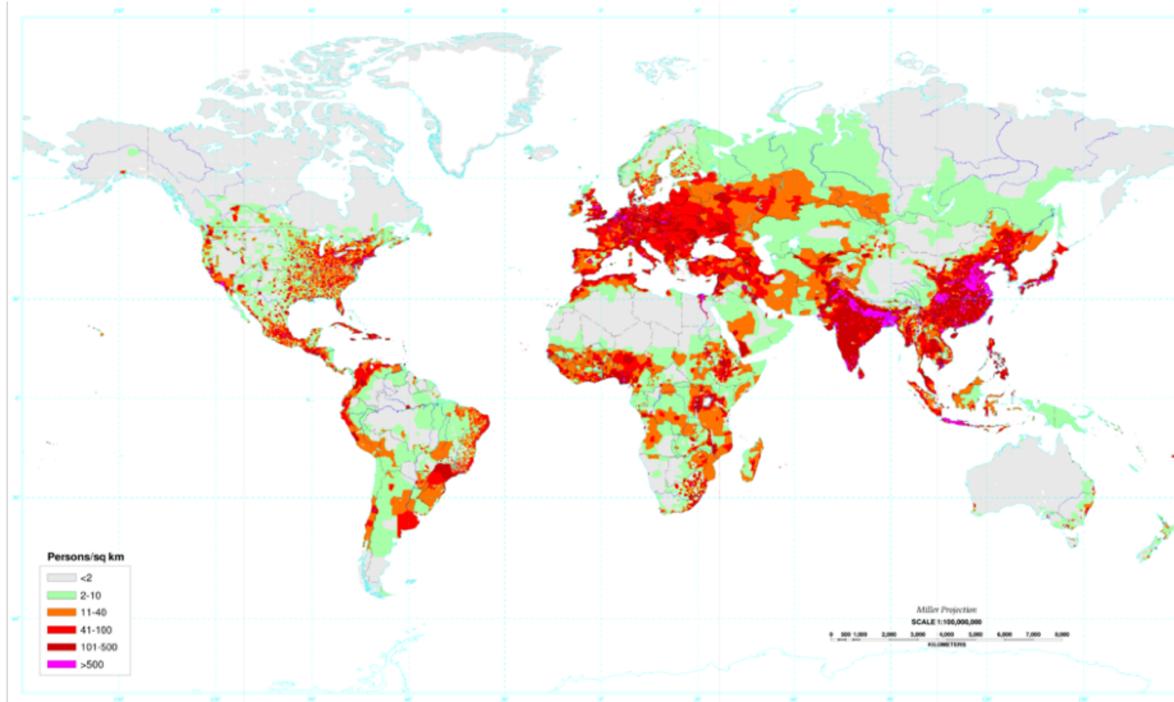




investigating the abstract



investigating the world



"Primitive" Weather Forecasting Equations

$p = \rho R T$ *Ideal Gas Law* (Equation of State)

$$\vec{a}_h = \sum \left(\vec{F}_h / m \right)$$ *Newton's Second Law of Motion*

$\Delta p = -\rho g \Delta z$

$(PGA)_v = g$

$$\vec{a}_v = \sum (\vec{F}_v / m) = (\vec{P}\vec{G}\vec{A})_v - \vec{g}$$

Hydrostatic Law (Obtained from the Equation of Vertical Motion)

$\Delta T = \Delta q / c_p + (1/\rho)\Delta p$ *First Law of Thermodynamics*

$(1/\rho)\Delta\rho/\Delta t = -DIV$

Conservation of Mass Applied to the Atmosphere (Equation of Continuity)

Zonal wind:

$$\frac{\partial u}{\partial t} = \eta v - \frac{\partial \Phi}{\partial x} - c_p\theta\frac{\partial \pi}{\partial x} - z\frac{\partial u}{\partial \sigma} - \frac{\partial(\frac{u^2+v^2}{2})}{\partial x}$$

Meridional wind:

$$\frac{\partial v}{\partial t} = -\eta\frac{u}{v} - \frac{\partial \Phi}{\partial y} - c_p\theta\frac{\partial \pi}{\partial y} - z\frac{\partial v}{\partial \sigma} - \frac{\partial(\frac{u^2+v^2}{2})}{\partial y}$$

Temperature:

$$\frac{\delta T}{\partial t} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}$$

Precipitable water:

$$\frac{\delta W}{\partial t} = u\frac{\partial W}{\partial x} + v\frac{\partial W}{\partial y} + w\frac{\partial W}{\partial z}$$

Pressure thickness:

$$\frac{\partial}{\partial t}\frac{\partial p}{\partial \sigma} = u\frac{\partial}{\partial x}x\frac{\partial p}{\partial \sigma} + v\frac{\partial}{\partial y}y\frac{\partial p}{\partial \sigma} + w\frac{\partial}{\partial z}z\frac{\partial p}{\partial \sigma}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + \omega\left(\frac{\partial T}{\partial p} + \frac{RT}{pc_p}\right) = \frac{J}{c_p} \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0 \qquad 0 = -\frac{\partial \phi}{\partial p} - \frac{RT}{p}$$

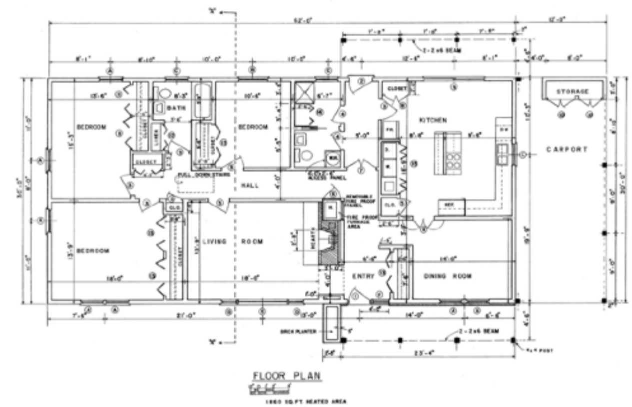
designing

the best materials

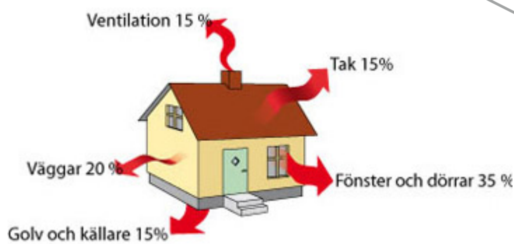
low energy

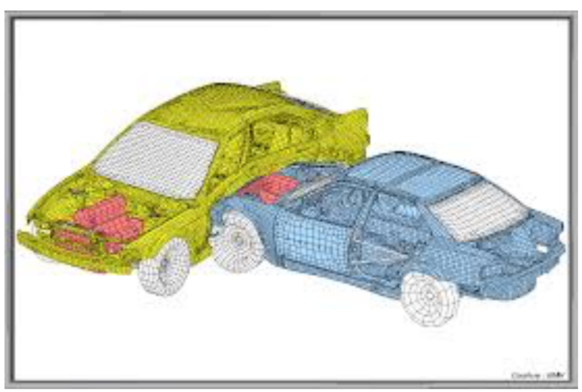
low cost

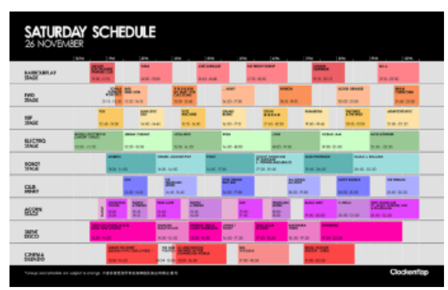
lots of light



iteration







design and creativity

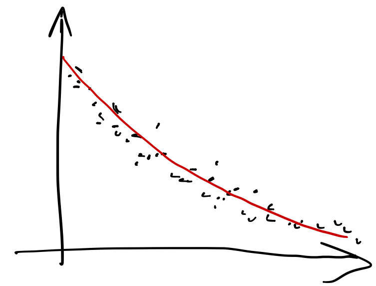
Understanding the nature of problems is important for your ability to solve them!

towards a theory of mathematical thinking

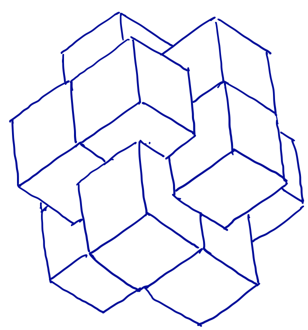
IF p THEN q

$$\equiv \neg p \vee q$$

mathematical reasoning



mathematical modelling



problem solving

mathematical reasoning

What is knowledge?

Nature of knowledge

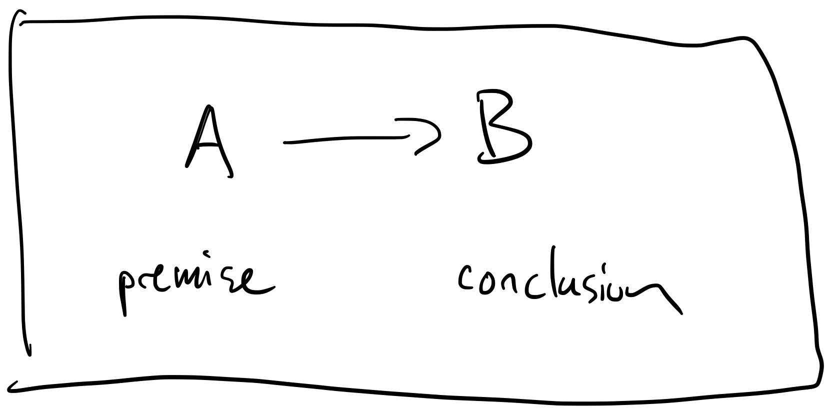
Awareness of what you know, what you believe and what you don't know

Understanding

Make every effort not to be wrong!

The scientific method

What is reasoning?



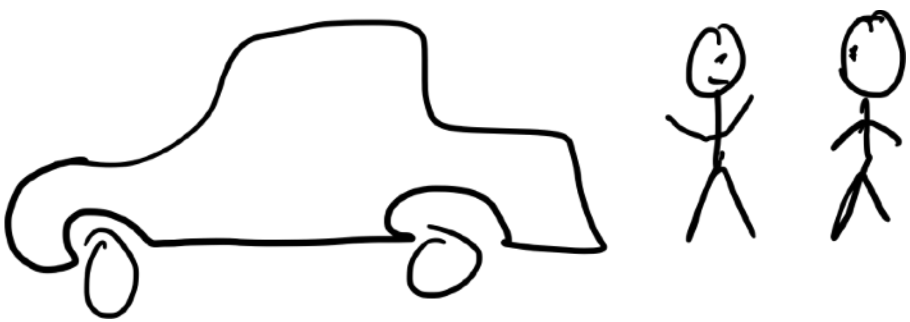
Reasoning concepts
(definition, conjecture, derivation, proof, calculation)

How reasoning connects statements

Nature of reasoning
(plausible/deductive, premise/conclusion, necessary/sufficient)

Reasoning errors

The importance of precision!

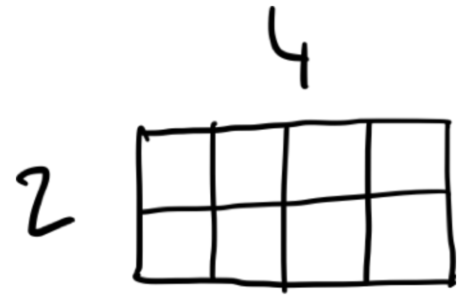


- How much fuel do we have?
- Quite a lot
- So how far can we go?
- Pretty far
- Will it be sufficient for our trip?
- ...

begin with clear definitions!

$$26 \times 31 = 806$$

$$31 \times 26 = 806$$

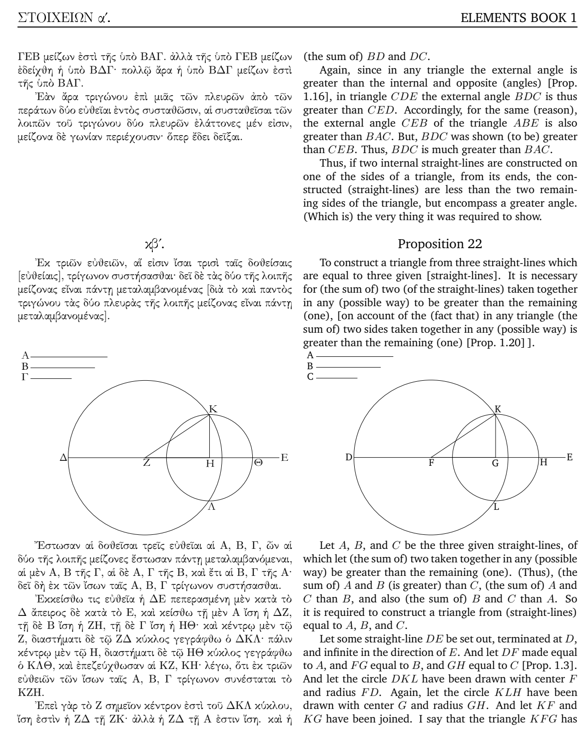


conjecture

proof

What kinds of statements is pure mathematics concerned with?

The story of how people think to create mathematics is not told


ΣΤΟΙΧΕΙΩΝ α΄. — ELEMENTS BOOK 1

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Ἐὰν ἄρα τριγώνου ἐπὶ μιᾶς τῶν πλευρῶν ἀπὸ τῶν περάτων δύο εὐθεῖαι ἐντὸς συσταθῶσιν, αἱ συσταθεῖσαι τῶν λοιπῶν τοῦ τριγώνου δύο πλευρῶν ἐλάττονες μέν εἰσιν, μείζονα δὲ γωνίαν περιέχουσιν· ὅπερ ἔδει δεῖξαι.

(the sum of) BD and DC.

Again, since in any triangle the external angle is greater than the internal and opposite (angles) [Prop. 1.16], in triangle CDE the external angle BDC is thus greater than CED. Accordingly, for the same (reason), the external angle CEB of the triangle ABE is also greater than BAC. But, BDC was shown (to be) greater than CEB. Thus, BDC is much greater than BAC.

Thus, if two internal straight-lines are constructed on one of the sides of a triangle, from its ends, the constructed (straight-lines) are less than the two remaining sides of the triangle, but encompass a greater angle. (Which is) the very thing it was required to show.

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Ἔστωσαν αἱ δοθεῖσαι τρεῖς εὐθεῖαι αἱ Α, Β, Γ, ὧν αἱ δύο τῆς λοιπῆς μείζονες ἔστωσαν πάντῃ μεταλαμβανόμεναι, αἱ μὲν Α, Β τῆς Γ, αἱ δὲ Α, Γ τῆς Β, καὶ ἔτι αἱ Β, Γ τῆς Α· δεῖ δὴ ἐκ τῶν ἴσων ταῖς Α, Β, Γ τρίγωνον συστήσασθαι.

Ἐκκείσθω τις εὐθεῖα ἡ ΔΕ πεπερασμένη μὲν κατὰ τὸ Δ ἄπειρος δὲ κατὰ τὸ Ε, καὶ κείσθω τῇ μὲν Α ἴση ἡ ΔΖ, τῇ δὲ Β ἴση ἡ ΖΗ, τῇ δὲ Γ ἴση ἡ ΗΘ· καὶ κέντρῳ μὲν τῷ Ζ, διαστήματι δὲ τῷ ΖΔ κύκλος γεγράφθω ὁ ΔΚΛ· πάλιν κέντρῳ μὲν τῷ Η, διαστήματι δὲ τῷ ΗΘ κύκλος γεγράφθω ὁ ΚΛΘ, καὶ ἐπεζεύχθωσαν αἱ ΚΖ, ΚΗ· λέγω, ὅτι ἐκ τριῶν εὐθειῶν τῶν ἴσων ταῖς Α, Β, Γ τρίγωνον συνέσταται τὸ ΚΖΗ.

Ἐπεὶ γὰρ τὸ Ζ σημεῖον κέντρον ἐστὶ τοῦ ΔΚΛ κύκλου, ἴση ἐστὶν ἡ ΖΔ τῇ ΖΚ· ἀλλὰ ἡ ΖΔ τῇ Α ἐστιν ἴση. καὶ ἡ

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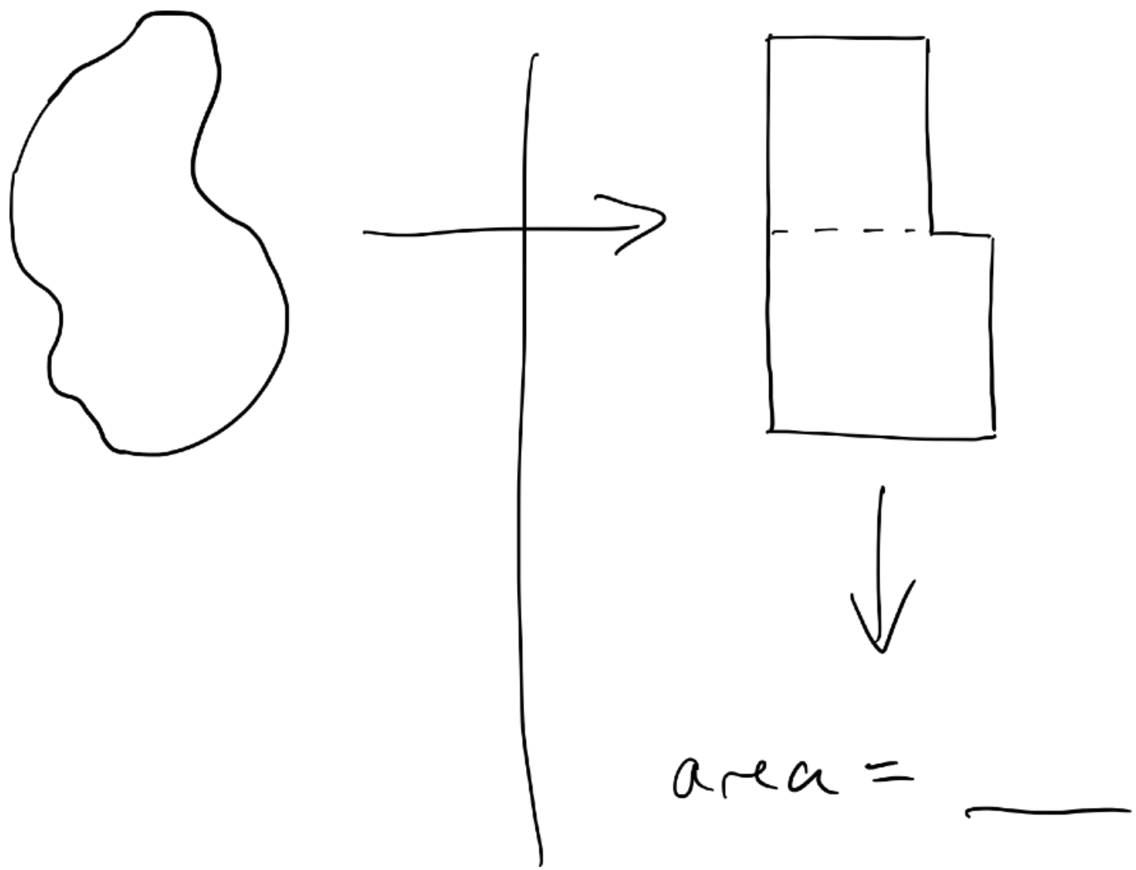
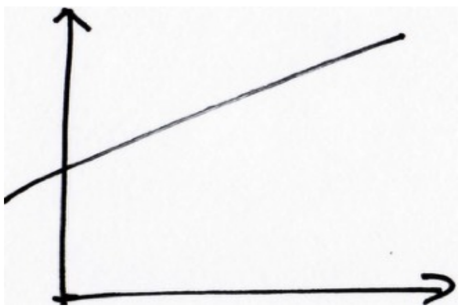
mathematical modelling

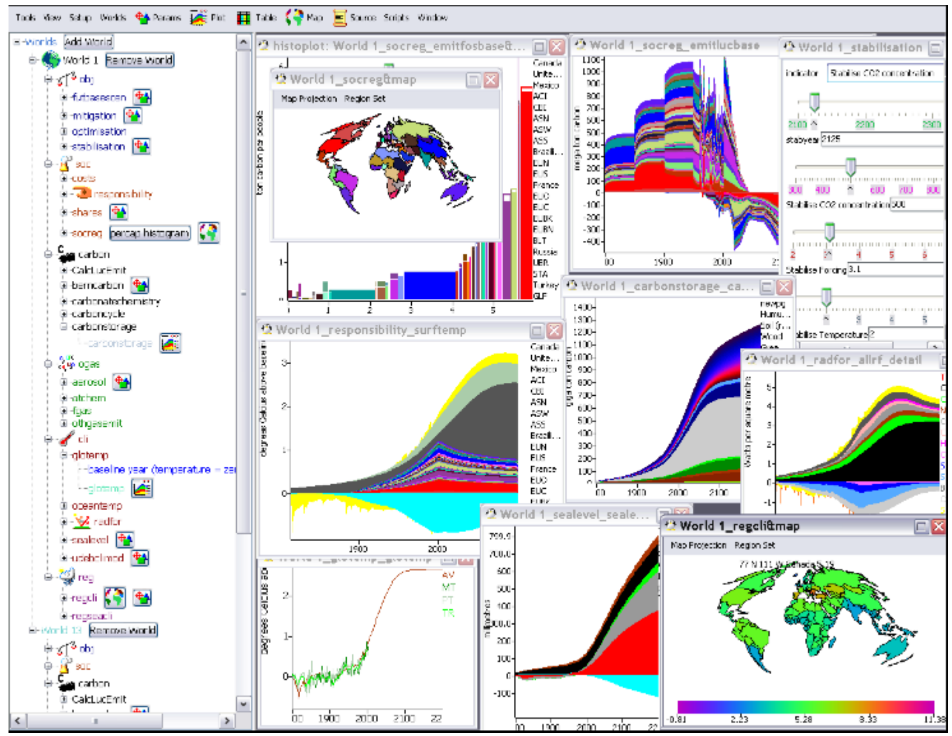
"a convenient way to represent reality so that we more easily can draw conclusions about it"

Why models?



Simple and complicated models for different purposes





The modelling cycle

simplify,
make precise,
select the modelling approach!

real situation or problem → mathematical model

mathematical model ↓ mathematical conclusions

mathematical conclusions → explanations and answers

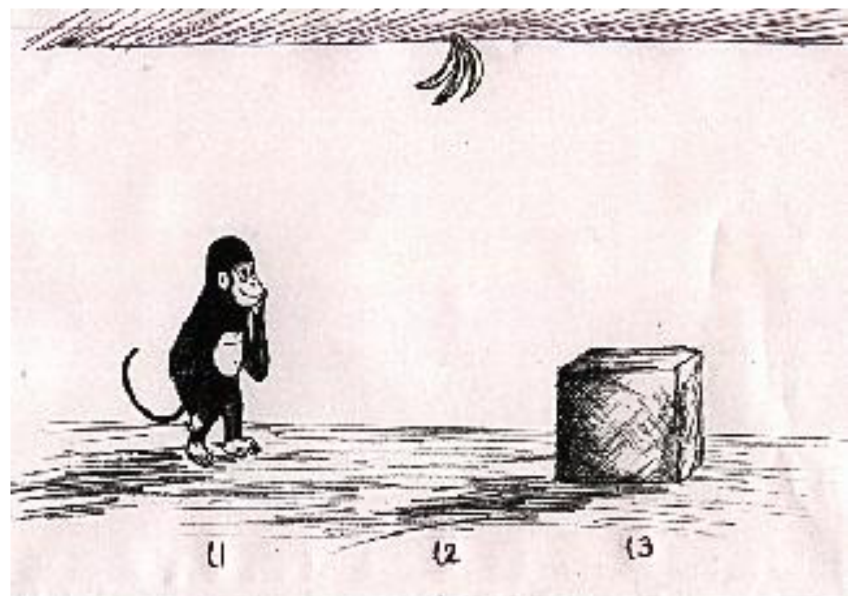
explanations and answers ↑ real situation or problem

reality | model

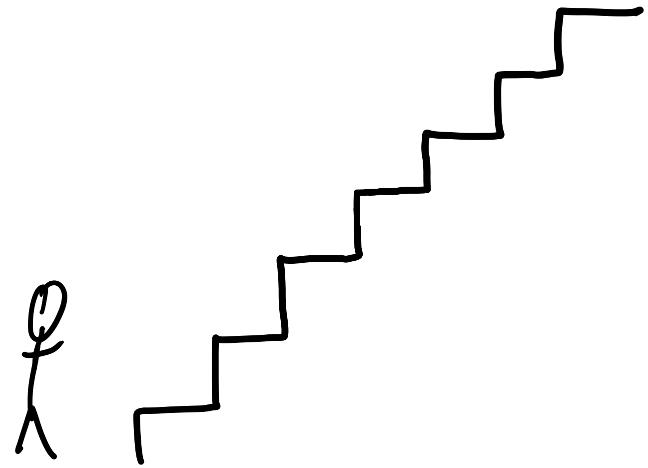
(actually a design problem)

problem solving

How can you handle your limited capacity?



A challenging task can be handled by working in small and efficient steps!



Typical workflow - easy problems

1. You easily understand the problem

2. You quickly see how to solve it

3. The problem is solved / implement the solution with no surprises

Typical workflow - intermediate problems

1. Understand the problem

2. Make a plan

3. Carry out the plan

4. Look back (check your result, reflect on the process, ...)

(Polya)

Typical workflow – more difficult problems

investigate for deeper understanding, define clearly

1. Understand the problem
2. Make a plan
3. Carry out the plan
4. Look back

explore different approaches, begin with something simple!

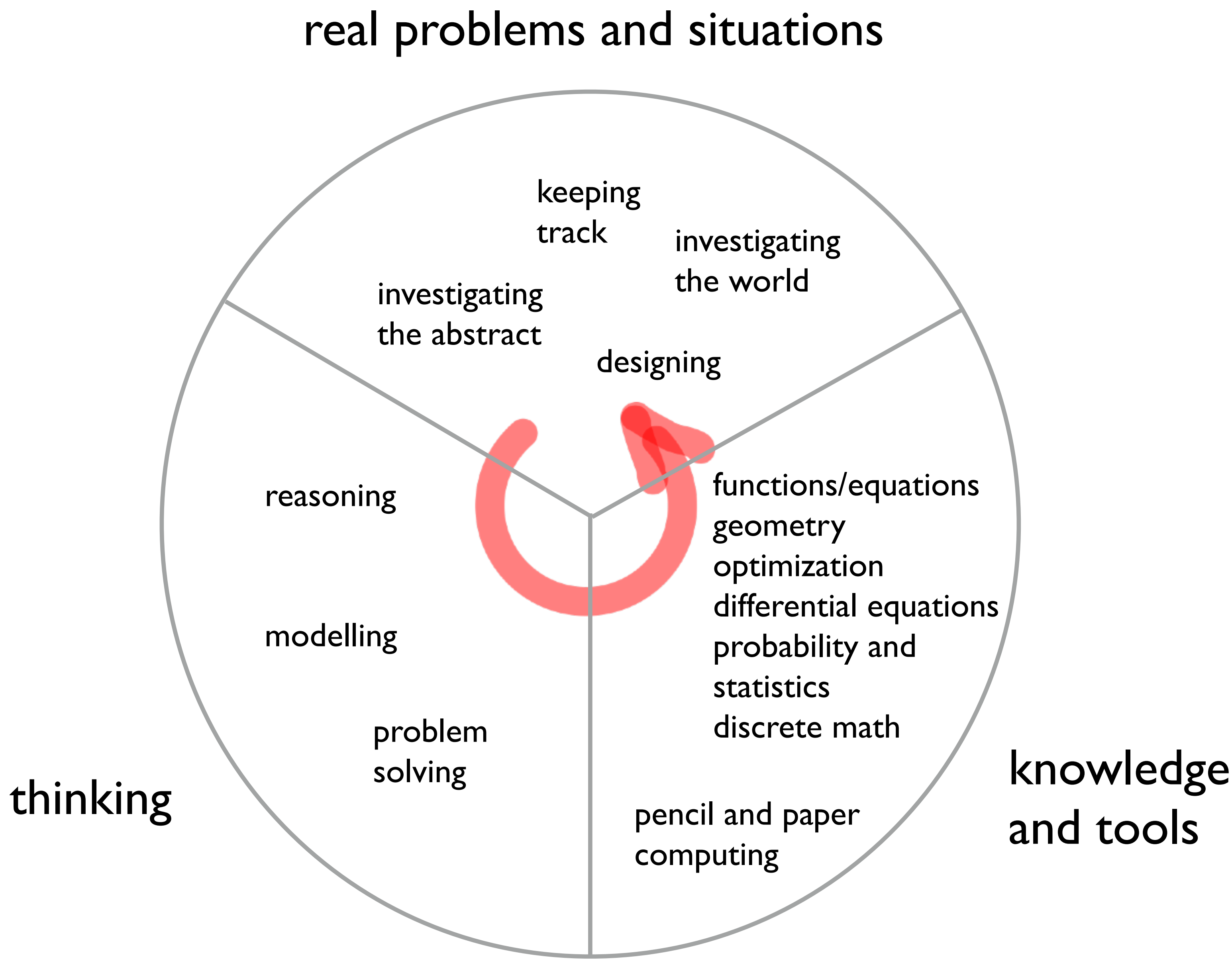
when you fail you learn and **go back**

Continuously reflect, go back and revise, manage your time

Problem solving is <u>not algorithmic</u> - even experts need to try things out!

You need to develop <u>experience and intuition</u> by solving quite a few realistic problems yourself!

summary of how we think and work mathematically

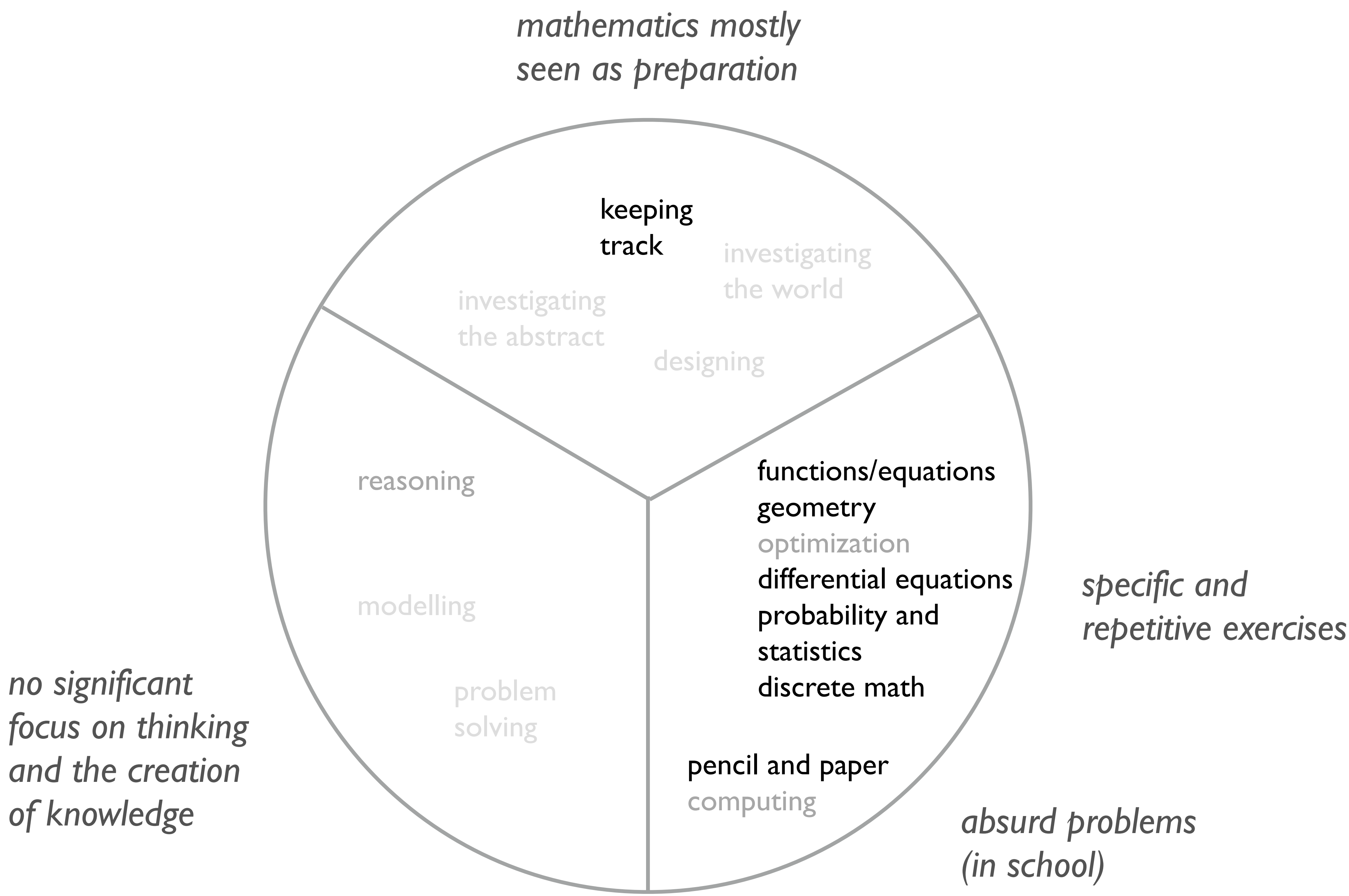

real problems and situations
keeping track
investigating the world
investigating the abstract
designing
reasoning
modelling
problem solving
thinking
functions/equations
geometry
optimization
differential equations
probability and statistics
discrete math
pencil and paper
computing
knowledge and tools

How does this relate to mathematics education?

Most mathematics education

knowledge needed for solving a problem **=** **knowledge from others** **+** knowledge created by own thinking

the emphasis is here!


mathematics mostly
seen as preparation
keeping
track
investigating
the world
investigating
the abstract
designing
reasoning
functions/equations
geometry
optimization
differential equations
probability and
statistics
discrete math
specific and
repetitive exercises
modelling
no significant
focus on thinking
and the creation
of knowledge
problem
solving
pencil and paper
computing
absurd problems
(in school)

The result is that students are often not able to use to use the mathematics they already know!

(this is also why I started to engage in this!)

In the beginning of our modelling and problem solving course

(2016 reports - mostly software engineering students end of year 2)

"We always thought that there were ready-made formulas for everything."

"The distinct difference between reality and mathematics was something we had never reflected over."

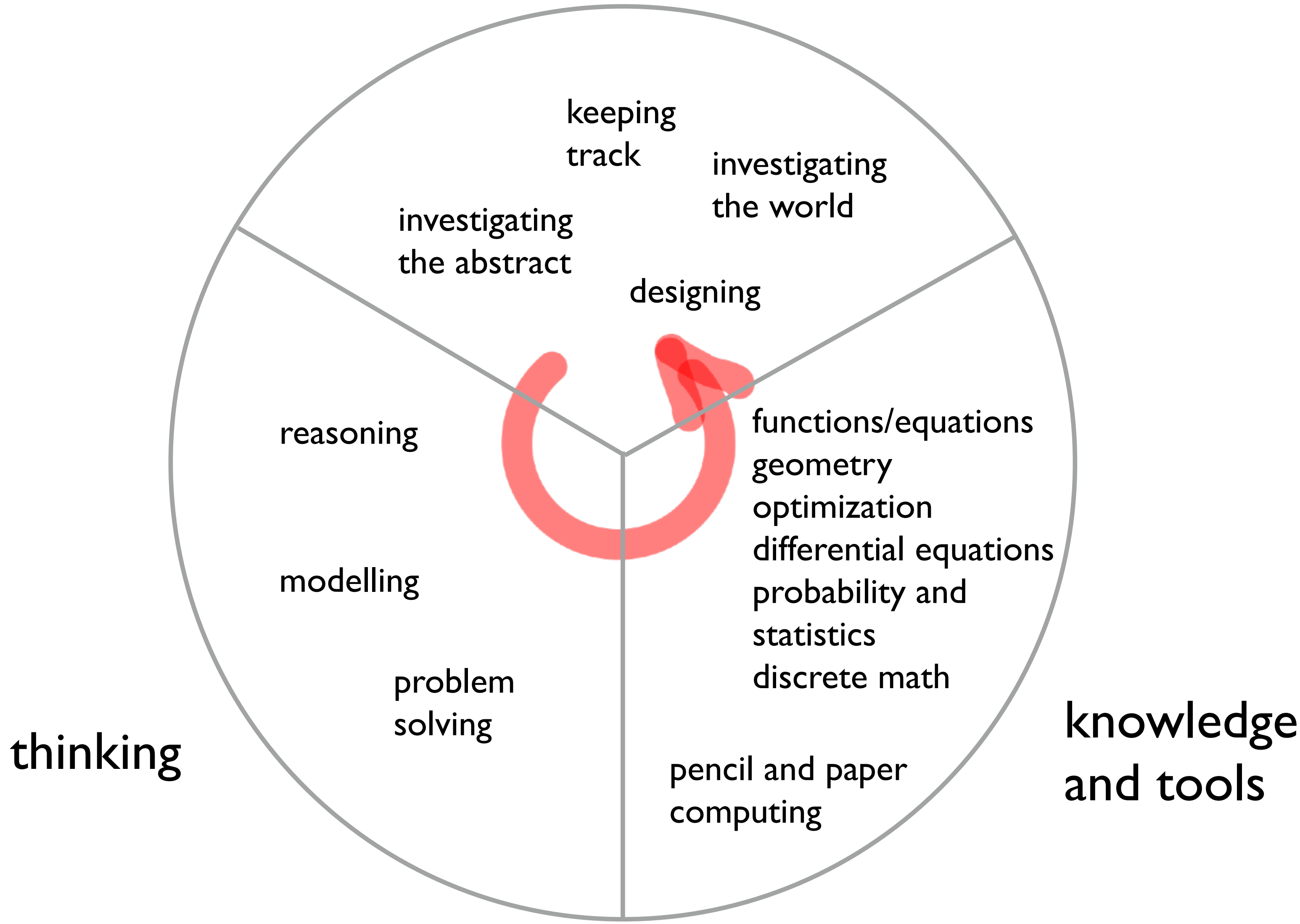
"If George Polya had seen us, he would probably say that we went against everything he ever said."

"Math was so much more than just doing calculations."

…

realistic problems and situations

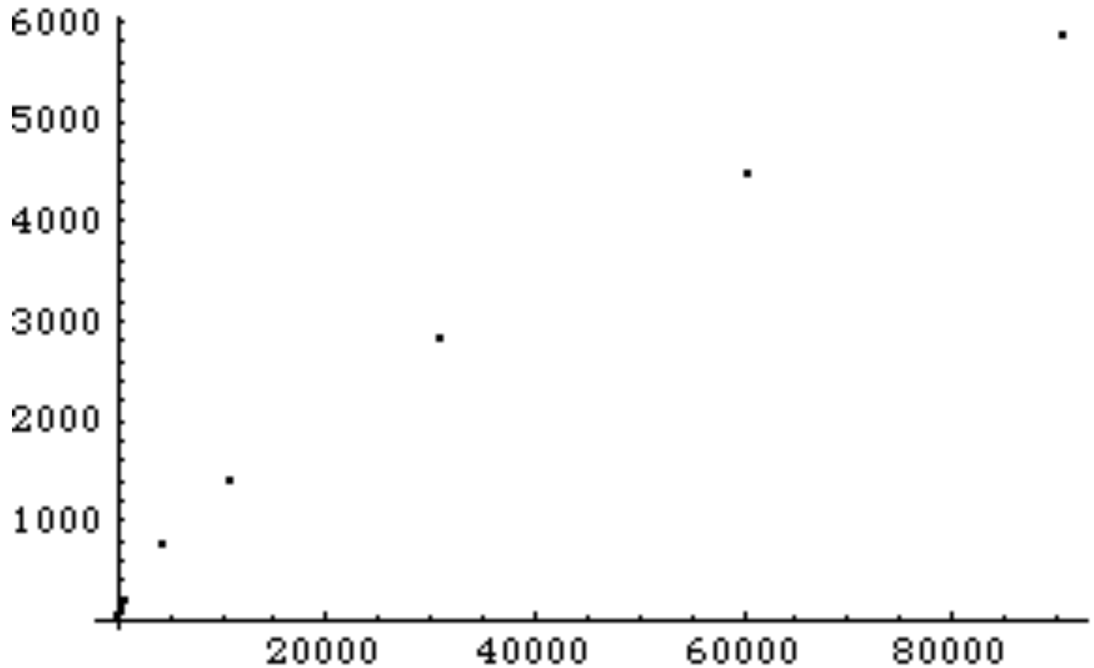
for eight weeks…

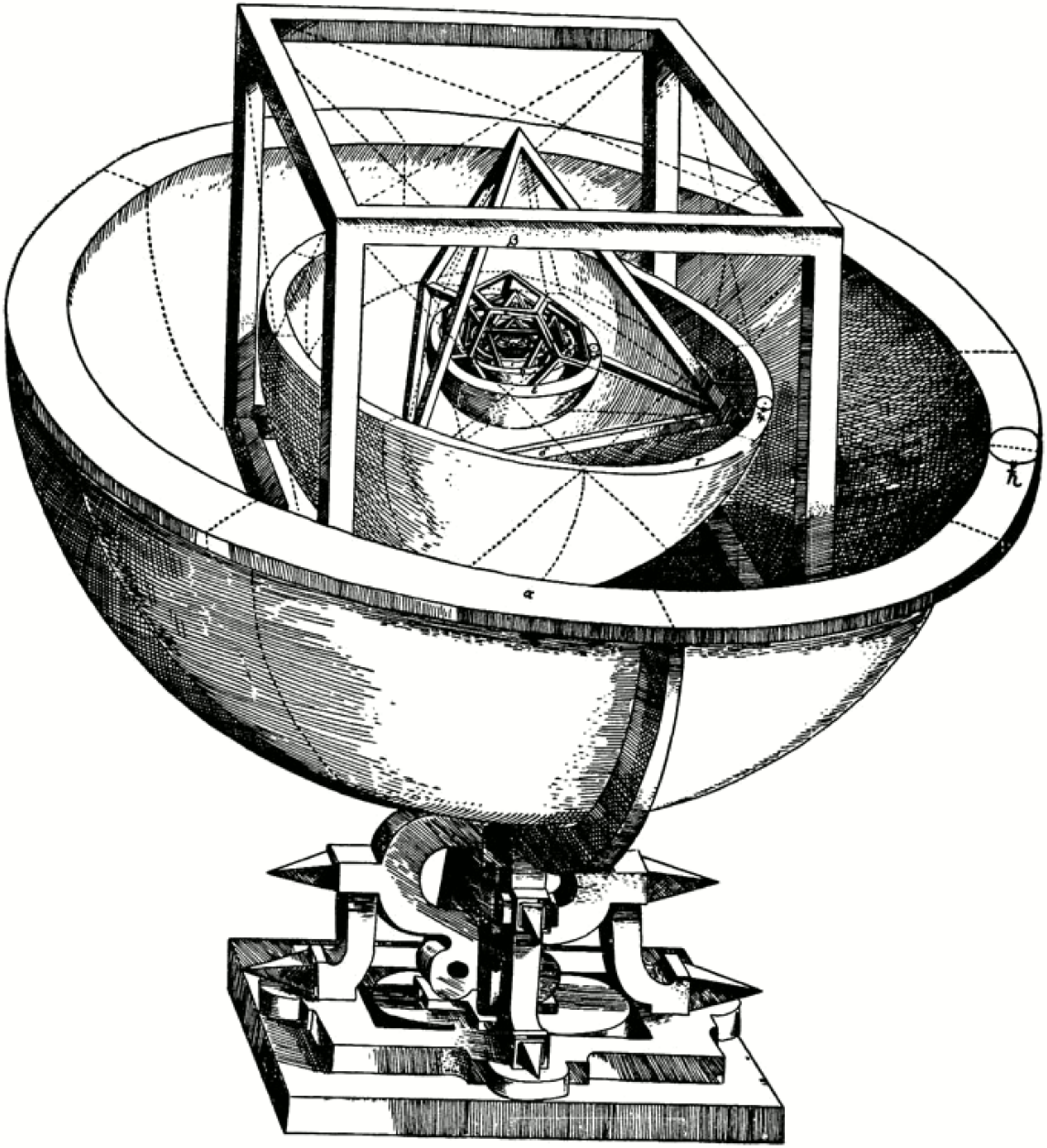


Example: finding a function for a physical relationship

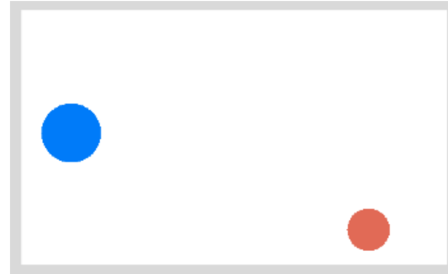
T (time)	D (distance)
88.0	57.9
224.7	108.2
365.3	149.6
687.0	228.07
4332	778.434
10760	1428.74
30684	2839.08
60188	4490.8
90467	5879.13

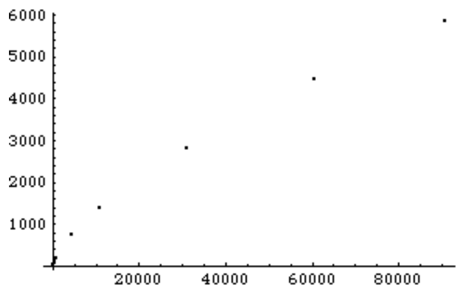
Inquiry-based learning example



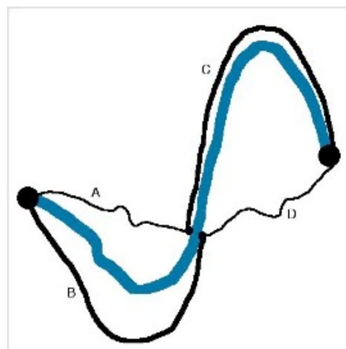


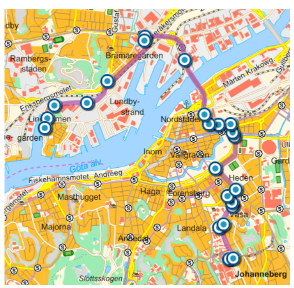
A <u>*varied*</u> set of *realistic* and *challenging* problems



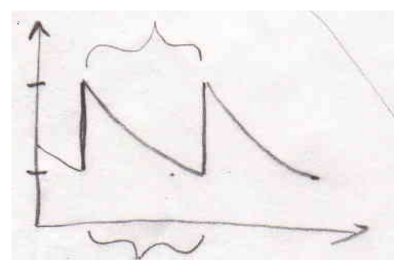


1	3	5	1
4	5	3	2
7	4	6	9
8	4	7	3





We'll use the first played with great deal with your facility spend waiting 1000 times…



Students build a *case library* of experiences!

Patterns across problems become visible!

At the end of the course (2016 reports)

"We have learned a new way of thinking."

"A unique property of the course is the balance between reasoning and the use of established techniques."

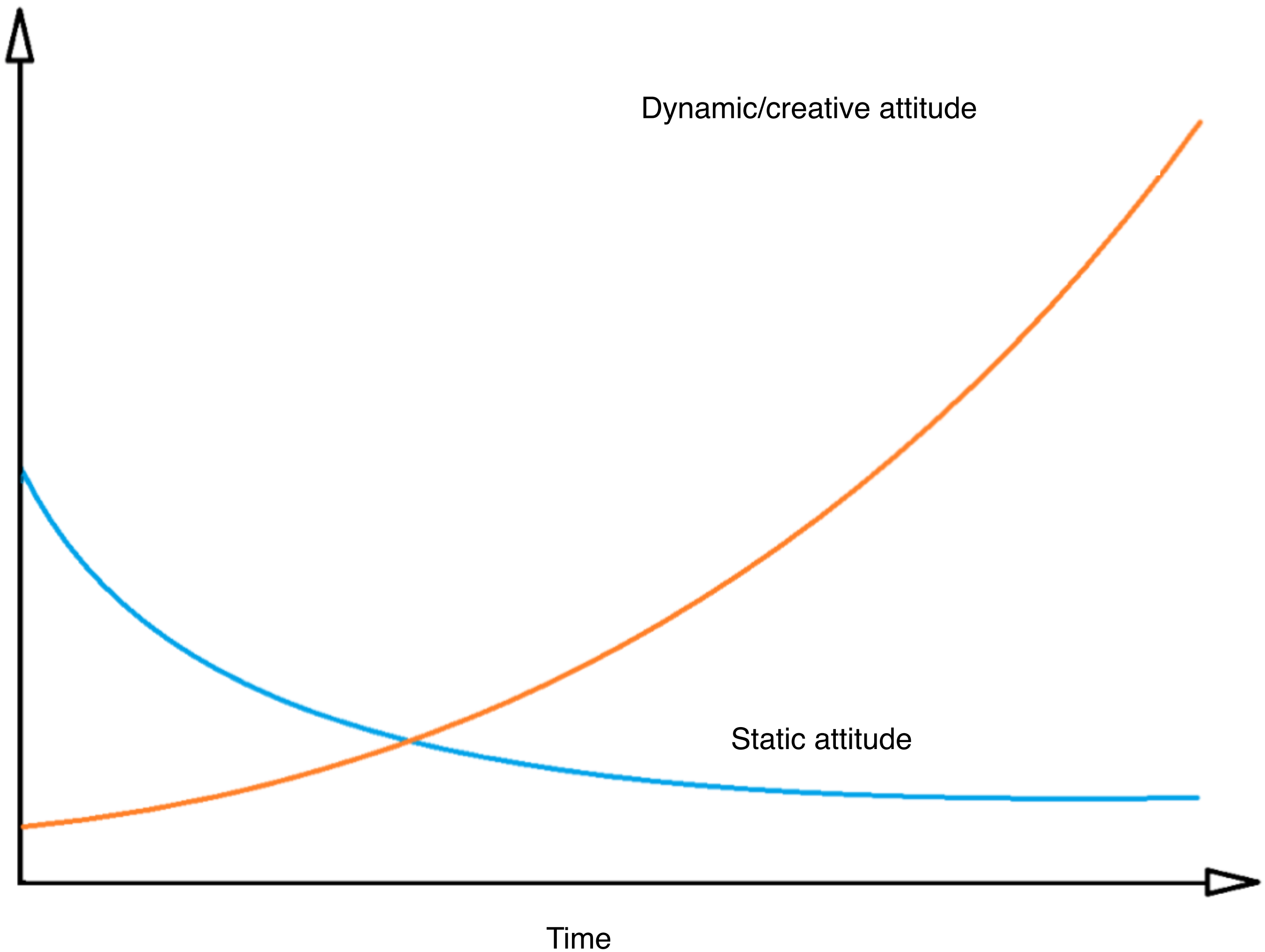
"The first course that made us feel as engineers."

"We had not realized how good this course is for us as future mathematics teachers."

"My rate of development has been enormous"

"Imagine if we had been given more of this earlier in our education."

…



(student group 2016)

Cooperation with Chalmers EER

Wedelin D., Adawi T. (2014). Teaching mathematical modelling and problem solving - a cognitive apprenticeship approach to mathematics and engineering education. International Journal of Engineering Pedagogy 4(5).

Wedelin D., Adawi. T, Jahan T., Andersson S. (2015). Investigating and developing students' mathematical modelling and problem solving skills. European Journal of Engineering Education.

Wedelin D., Adawi T. (2015). Applied mathematical problem solving - principles for designing small realistic problems. In Stillman, Blum, Biembengut (eds.), Mathematical Modelling in Education Research and Practice, Springer.

…

what about computers and programming?

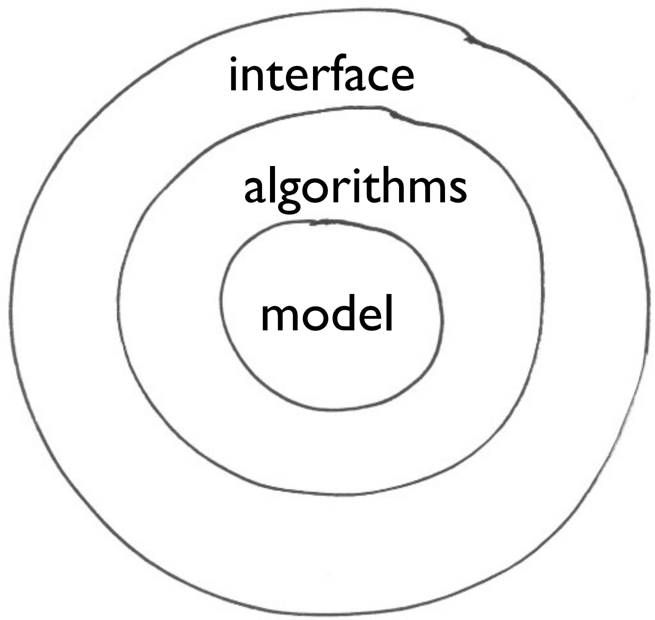
Programming enables a new range of creative problem solving tasks

Can reduce the use of artificial repetitive exercises?

Requires an attitude of being careful

Tools like Mathematica are also important

Models, algorithms and software



Vienna Symphonic Library

Bösendorfer mic'd for sampling

Example of changing underlying models: electronic pianos

1960's: simple waveform and decay synthesis

1980's: sampling synthesis

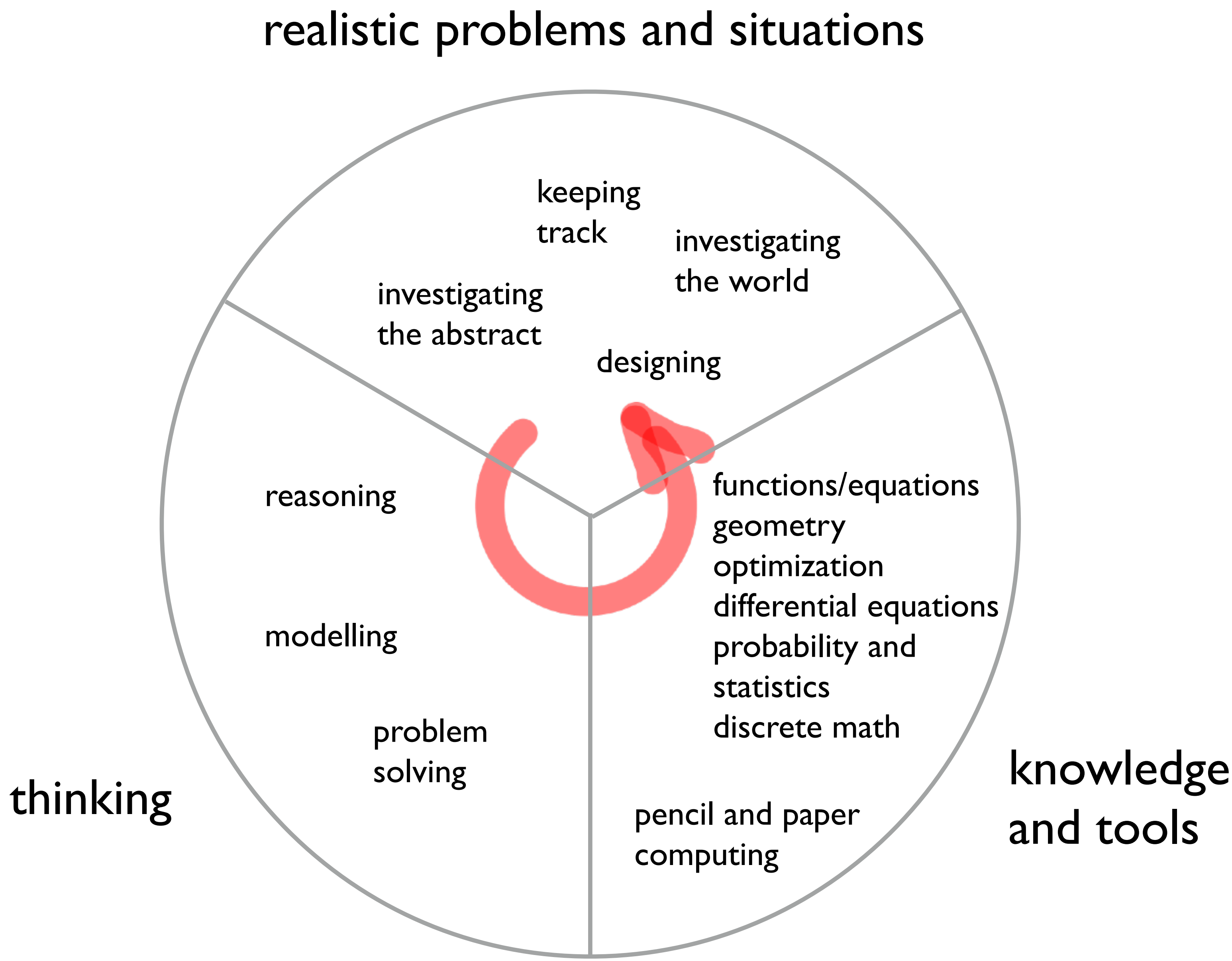
2000- : physical modelling

Bösendorfer mic'd for sampling

summary

knowledge needed for solving a problem = knowledge from others + knowledge created by own thinking

you often need to add something yourself!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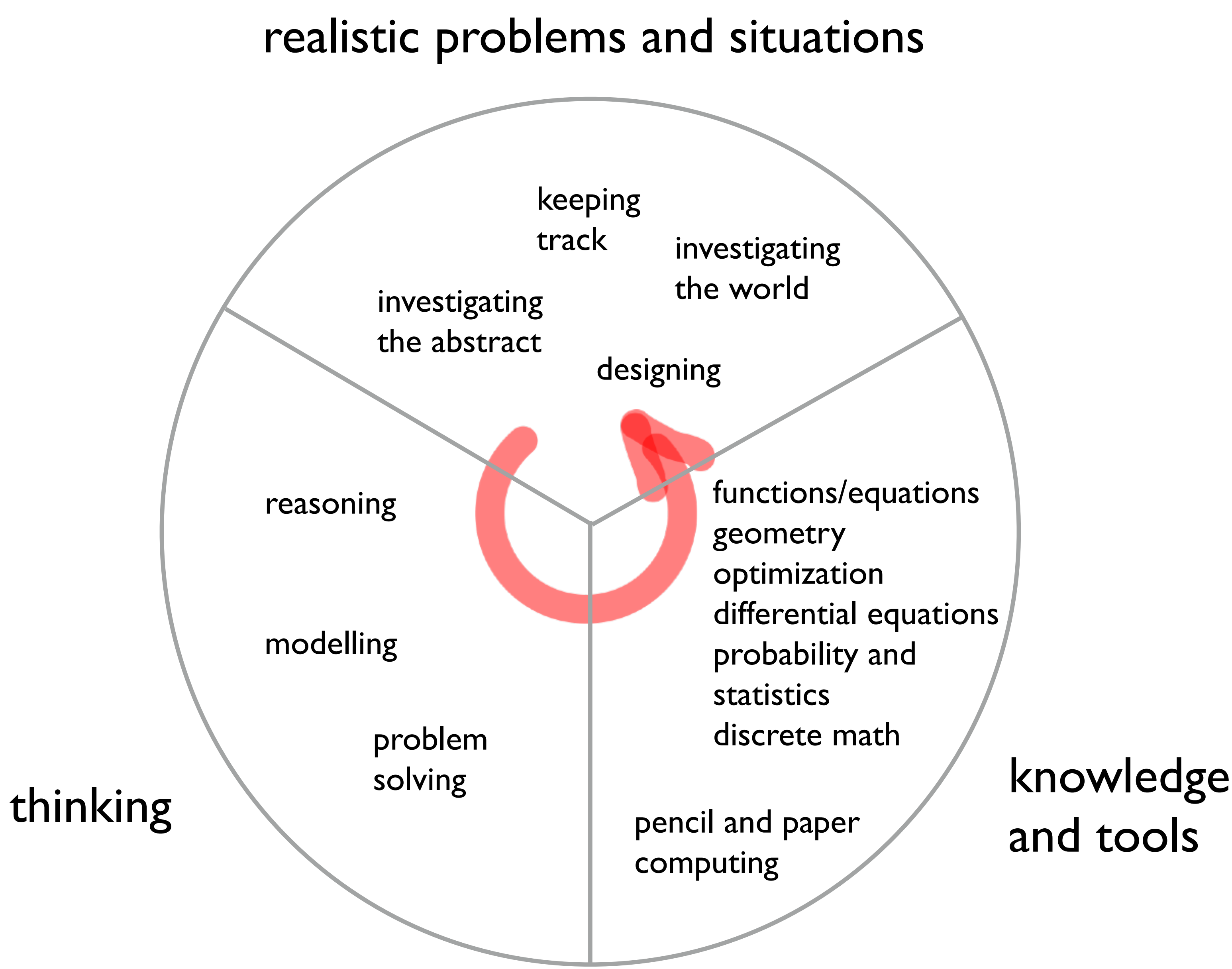
discrete math
pencil and paper
computing
knowledge and tools


What's on your plate kids?
Salad & Veggies
Keep it Colourful
Protein
Carbohydrates
Nestlé Healthy Kids
AIS
Healthy Eating is as easy as 1,2,3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END