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Dag Wedelin[a b], Tom Adawi[b], Tabassum Jahan[b] & Sven Andersson[b]
[a] Department of Computer Science and Engineering, Chalmers University of Technology, SE-412 96 Gothenburg, Sweden
[b] Division of Engineering Education Research, Department of Applied IT, Chalmers University of Technology, SE-412 96 Gothenburg, Sweden
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Investigating and developing engineering students' mathematical modelling and problem-solving skills

Dag Wedelin[a,b*], Tom Adawi[b], Tabassum Jahan[b] and Sven Andersson[b]

[a]*Department of Computer Science and Engineering, Chalmers University of Technology, SE-412 96 Gothenburg, Sweden;* [b]*Division of Engineering Education Research, Department of Applied IT, Chalmers University of Technology, SE-412 96 Gothenburg, Sweden*

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How do engineering students approach mathematical modelling problems and how can they learn to deal with such problems? In the context of a course in mathematical modelling and problem solving, and using a qualitative case study approach, we found that the students had little prior experience of mathematical modelling. They were also inexperienced problem solvers, unaware of the importance of understanding the problem and exploring alternatives, and impeded by inappropriate beliefs, attitudes and expectations. Important impacts of the course belong to the metacognitive domain. The nature of the problems, the supervision and the follow-up lectures were emphasised as contributing to the impacts of the course, where students show major development. We discuss these empirical results in relation to a framework for mathematical thinking and the notion of cognitive apprenticeship. Based on the results, we argue that this kind of teaching should be considered in the education of all engineers.

Keywords: mathematical modelling; problem solving; metacognition; cognitive apprenticeship; qualitative case study

1. Introduction

Over the past two decades, scholars have investigated the nature of the mathematical thinking and problem-solving activities that engineers engage in at the workplace (Lesh, Hamilton, and Kaput 2007; Alpers 2010). Gainsburg (2006), for example, concluded that mathematical modelling constitutes a central and challenging aspect of everyday engineering work. Jonassen, Strobel, and Lee (2006) added that engineering workplace problems are often complex and *ill-structured*. It is therefore important that engineering students learn how to create mathematical representations of real-world problems – that is, *mathematical modelling* (Blum and Niss 1991; Blum and Borromeo Ferri 2009; Sole 2013) – and how to approach non-routine problems of different kinds – that is, *problem solving* (Martinez 1998; Mayer and Wittrock 2006; Bassok and Novick 2012). However, engineering education has been criticised for neglecting to provide students with adequate opportunities to develop these fundamental skills (Litzinger et al. 2011). For example, engineering students are seldom provided with opportunities to adapt or create mathematical models for a given situation (Zawojewski, Diefes-Dux, and Bowman 2008). More generally, classroom problems are often routine and well-structured (Jonassen, Strobel, and Lee

*Corresponding author. Email: dag@chalmers.se

2006; Mayer and Wittrock 2006). To bridge this gap between education and practice, the first author of this paper offers a course in mathematical modelling and problem solving to second-year engineering students at Chalmers University of Technology. The overall aim of the course is to enable the students to deal with real-world problems related to science and technology with the help of mathematics (Wedelin and Adawi 2014).

There has been relatively little research focusing on how engineering students approach ill-structured problems (or non-routine problems in general), or how to improve their ability to deal with such problems (Singer, Nielsen, and Schweingruber 2012). Mathematical modelling problems are often naturally ill-structured, and in this paper we investigate how students in the course approach mathematical modelling problems at the early stages of the course. We also examine what the students consider to be the most important impacts of the course and what aspects of the learning environment they believe contribute to these impacts. We discuss the results by drawing on a framework for *mathematical thinking* (Schoenfeld 1985, 1992), and on *cognitive apprenticeship* (Collins, Brown, and Newman 1989) as a framework for instructional design.

2. The course in mathematical modelling and problem solving

The course takes a broad view on mathematical modelling, and is based on a collection of about 30 small but reasonably realistic problems, which provides the context for learning in an *inquiry-based* (Prince and Felder 2006) setting. The relatively large number of problems provides a variation in applications, models and problem-solving approaches, and allows repeated feedback on the entire problem-solving process. The problems are calibrated to be challenging, and while some general introduction is offered, students do not in advance know any given model or method for solving the problems. Based on such problems, we have found it natural to teach mathematical modelling and problem solving together, which is also noted by other researchers, such as Lesh and Zawojewski (2007).

The course is organised into six weekly modules, where a broad range of model types and applications are considered, see Figure 1. Each module focuses on applications in a certain modelling domain, and begins with an introductory lecture with general information about that week's module and related topics. The problems are solved in pairs, and the continuous interaction between students and teachers is emphasised. Extensive supervision is provided, mainly by asking general questions in a Socratic style to help the students to continue on their own, and in context reminding students of general problem-solving strategies. We take care not to give

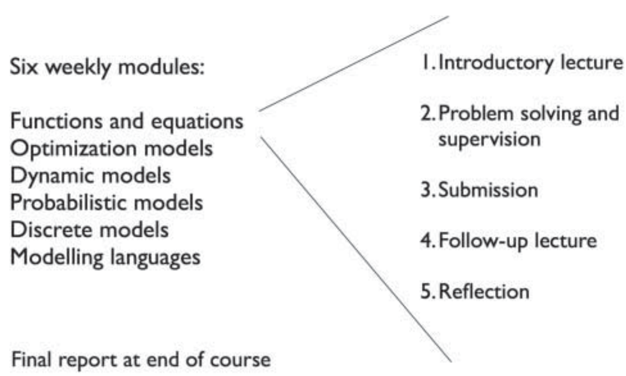

Figure 1. The structure of the course.

too much help, to encourage the students' own problem-solving skills and *develop their ability to explore*. As appropriate, however, we also share our general experience. The students are expected to do their best to solve each problem, but in order to provide a creative learning environment, they are not required to reach a final or correct solution, and it is perfectly acceptable to fail (see also Martinez 1998). In addition to the feedback students receive during supervision, each module also contains a follow-up lecture where collective feedback is provided. Different approaches are discussed as well as common mistakes and difficulties. The students are encouraged to compare their approaches to the presented ones, and to write their reflections as a part of the following module. We also extend the context of the problems solved.

Throughout the course, we encourage a continuous discussion and reflection on the modelling and problem-solving processes. For example, one way to describe the modelling process is in terms of the *modelling cycle*, consisting of the repeated application of the following steps: simplifying and mathematising the real problem, working mathematically, interpreting the mathematical solution in the real context and validating the solution (Edward and Hamson 1996; Borromeo Ferri 2006). For general problem solving, a well-known approach is that of Polya (1957), which contains four main phases: understanding, planning, executing and re-evaluating, together with different kinds of heuristic strategies. These and other characterisations provide a flexible framework and language for the problem-solving activities that the students actively engage in during the course, where the main aim is to encourage and help students to think by themselves in a productive way.

For a more detailed description of the course, see Wedelin and Adawi (2014).

3. The design of the study

3.1. *Methodology*

As the purpose of this study is to investigate how engineering students approach mathematical modelling problems, and how they develop the necessary skills and attitudes to effectively deal with such problems, a *qualitative case study* approach (Yin 1984; Merriam 1998, 2002; Case and Light 2011) was employed. *Qualitative research* attempts to understand how people *make meaning* of certain phenomena or situations, and these meanings are *mediated* by the researcher (Merriam 2002). The process is *inductive* and the product is *richly descriptive*. *Case studies* are, in addition, characterised by the analysis of a *bounded system* (Merriam 2002), and 'concentrate attention on the way particular groups of people confront specific problems, taking a holistic view of the situation' (Shaw 1978, 2). Yin (1984) argues that the case study has a distinct advantage for investigating 'how' and 'why' questions, and Merriam (1998, 33) suggests that the case study is a 'particularly suitable design if you are interested in process'.

3.2. *Data collection*

To investigate the students' ability to deal with mathematical modelling problems in the beginning of the course, we conducted a pilot group interview with five students, and then individual interviews with eight other students. The interviews were carried out during the third week of the course. The pilot interview focused on the students' early experiences in the course. The individual interviews focused mainly on how the students had dealt with one of two problems that they solved in the course; see Appendix 1 for the formulation of the problems.

The first problem (P1) is a curve-fitting problem and naturally lends itself to be solved by iteratively trying different functions to fit a series of given data points (time and distance). The problem requires the students to find suitable functions to try out, a way to actually do this, and to find some way to decide what is a good fit. No particular method for curve fitting, or the least squares method, is presented before attempting the problem.

The second problem (P2) is a facility location problem. This problem is about formulating a mathematical programming problem, which involves defining variables, an objective function and suitable constraints. It requires the students to interpret the problem sensibly, simplify some aspects of it and to find the variables of the optimisation problem. A brief introduction to such problems has been given, but the students are not used to this kind of modelling, so it is not a routine task.

During the interviews, the students were asked what specific steps they had taken in dealing with the problem, what challenges they experienced during the process, how they dealt with those challenges and what knowledge, skills or strategies they think are the most important for successfully dealing with the problem. The interviews were carried out in a *semi-structured* way (Cousin 2008), and probing questions, asking the students to explain or elaborate on something they said, were used to clarify answers. The interviews were audio-recorded and transcribed soon after the event.

To investigate what the students experienced as the main impacts of the course and their impressions of the course itself, we used the final reports that all groups submitted at the end of the course, so reports were collected from 103 students. In the final reports, students are encouraged to write freely about the course, for example, what they think is important as well as their own reflections. The reports therefore consist of a mix of facts about models, methods and specific problems, and the students' own reflections. We have in this paper focused on students' overall descriptions and reflections. These naturally fall into the three groups of *mathematical modelling*, *problem solving* and *the learning environment* (i.e. the course itself), and we sorted the extracted data into these three groups before engaging in the analysis.

This tapping into different methods for data collection, with different types of questions and tasks, tallies with common *triangulation techniques* employed in qualitative research to strengthen the validity of the findings (Krefting 1991).

3.3. *Data analysis*

The data were analysed using a *general inductive* approach for qualitative analysis described by Miles and Huberman (1994). The aim of this kind of analysis is to identify and describe *patterns* or *themes* that emerge from and cut across the data, and it is essentially a process of disassembling and reassembling data. The first step is to 'break up' the data into segments, or manageable pieces, and *tag* segments with short phrases intended to represent the idea that the segment conveys. The tags are then sorted and sifted in an iterative way, searching for patterns or themes. In Appendix 2, we provide a simplified example of how this analysis was carried out.

During the data analysis, intermediate findings were discussed among all authors of this paper to ensure a deeper reflexive analysis and to strengthen the validity of the findings (Krefting 1991). Moreover, two of the authors were actively involved in the course, which greatly assisted in interpreting the students' experiences.

The data are highly consistent between the different sources, as well as between students, and no students expressed views that are significantly different from those presented in this paper. We have therefore partly used representative student citations as a way to present our analysis, while at the same time giving direct impressions of what students actually say.

Section 4 describes the themes that emerged from the analysis of the interviews early in the course, and describes how students approach mathematical modelling problems at the early stages of the course. Section 5 describes what students perceive as the main impacts of the course, based on an analysis of the modelling and problem-solving sections of the final reports.

In Section 6, we draw on a framework for *mathematical thinking* (Schoenfeld 1985, 1992) to analyse student difficulties, and in Section 7 we draw on the notion of *cognitive apprenticeship* (Collins, Brown, and Newman 1989) to analyse student development. These two frameworks were not applied a priori, but were used to gain a deeper understanding of our results. In both these sections, the analysis is based on relevant parts of the data together with our understanding of the course itself.

4. Students' ability to solve modelling problems early in the course

The students did, as expected, experience difficulties related to mathematical modelling, such as how to best simplify a real-world problem and how to mathematise. However, the challenges that the students mostly talked about during the interviews concerned more general difficulties in dealing with the problems. In this section, we describe the three main challenges that the students experienced – *understanding the problem*, *exploring alternatives* and *having the right attitude* – and how they dealt with these challenges. We note that these issues have little to do with *mathematical modelling* as such, but are mainly related to *problem-solving* skills and attitudes.

4.1. *Understanding the problem*

It was evident from the interviews that, in the beginning of the course, most of the students were not aware of the importance of *understanding the problem*. It was only after failing to solve several of the problems that they started to realise how important it is to understand the problem well. Student S1, for example, said during the interview:

> The first week we really didn't spend much time on trying to understand the problem and what the problem was. We just dove in and that kind of backfired sometimes when we calculated something that wasn't what we were trying to do.

A clear example of how the students skipped important aspects of understanding the problem is when they, in problem P1, chose to base their solution on the logarithmic function. However, by looking at a plot of the data one can reasonably guess that the distance should go towards zero, when the time approaches zero. The suggested solution also ignores the physical context of the problem and the fact that with a logarithmic function the distance would take on negative values as time approaches zero. When confronted with this fact during the interview, S2 replied: 'We didn't actually reflect on what the data would represent.' He also offered the following reason for not taking the context into account: 'In previous mathematics courses, I believe it was less important to use the context to solve the problem.'

The students often returned to the importance of understanding the problem during the interviews. Several of the students considered this to be the most important aspect of problem solving. They noted that not understanding the problem well was a root cause for getting stuck at different stages of the problem-solving process. They thus emphasised the importance of continuously improving the understanding of the problem throughout the solution process.

At the same time, understanding the problem was seen as one of the most challenging aspects of problem solving. The students dealt with this challenge by using several strategies (on their own initiative or with some help from the supervisors). Some of these strategies are applicable to all types of mathematical problem solving, while others are more specific to the two problems, P1 and P2.

A strategy used by several students was to *reformulate the problem* in their own words. This helped them to identify the relevant information, important factors and what the goal of the problem was. According to the students, it was a good strategy that helped in all sorts of problem-solving, but it was useful in particular for problem P2, which was seen as a very complex problem containing a lot of information. Student S3, for example, said: 'By writing the problem formulation you spend more time trying to understand what you are trying to do in the question, and it is easier to solve it.' He also mentioned that he was then able to understand what information in the problem was relevant and easily identified the variables (the unknowns) and the constants of the problem. (Finding the variables is an important step in solving problem P2.)

The strategy of *splitting into parts* was used by some of the students and was said to have similar benefits as reformulating the problem. Student S4, for example, said: 'Dividing the problem too, is an important part. Not trying to solve the whole at once . . . it could be hard to look at the different parts if you are looking as a whole.'

Another strategy used by most of the students was *visualisation*. For example, most of the students plotted the data points in problem P1 as the first step, as they believed it was a good way of visualising the data. And by looking at the curve, they could start to guess a base function. Student S5 said that, by looking at the curve, he could recall $\sqrt{x}$ to have a similar shape, which is a crucial step in solving problem P1. Other ways of visualising the data were applied for problem P2 as well. Several students drew a transition matrix with binary numbers, indicating the sites reachable within eight minutes as '1' and not reachable within eight minutes as '0'. By having this transition matrix in 0s and 1s, they said, it was easier for them to see the nature of the problem and get a clearer idea about the constraints. In problem P2, visualising the data with the transition matrix enhanced their understanding of the problem but they needed to use other strategies to continuously improve their understanding and proceed towards a successful solution.

Some of the students also mentioned *discussing with peers* as a way to enhance their understanding of the problem. They noted that through discussion with their group partner, they could confirm, rectify or improve their understanding. Student S5 said: 'When you are able to explain to another person it means that you understood it yourself.'

4.2. *Exploring alternatives*

It was also clear that, at the beginning of the course, most of the students did not realise the importance of *exploring alternatives* in problem solving. The importance of this aspect of problem solving is something they realised, along with the importance of understanding the problem, only after failing to solve some of the problems. Student S7, for example, when discussing problem P1 during the interview, said: 'I don't think we would have got stuck like we did if we could think outside the box much earlier . . . and tried to see *different possibilities of getting forward*, and not getting to an answer directly.'

Several of the students also found exploring alternatives to be the most difficult aspect of problem solving. For example, student S8, who chose a logarithmic function as the base function in problem P1, said: 'The hardest thing was to give up the idea of a logarithmic function and try something else.' S2 said that the most difficult step in the problems was to 'look for a completely different function instead of just making it more complex'. The students often quickly decided on one way of approaching the problem and then went deep into details without exploring alternatives. It frequently turned out that they had persistently pursued an unsuccessful way to solve the problem. For example, when discussing problem P1 during the interview, student S5 said: 'When we had the quadratic function we tried to find the best quadratic function to fit the points, so we spent a lot of time doing the wrong thing.' Some of the other students never asked themselves if there could be another way of solving the problems.

Different types of unwarranted *beliefs* often prevented the students from exploring alternatives. For example, problem P2 seemed quite complicated to student S7, and she therefore expected it to have a complex solution. So she started to write quite complicated equations for the objective function since it seemed compatible with the apparent difficulty of the problem. When discussing the problem during the interview, she said:

> Well, what we were striving to find was a long complex formula, to kind of give us the answer. Because these are difficult problems, at least for us, to solve. I kind of think . . . well, I know it's not right; the best thing is always to find the easiest solution. But it's a hard problem so the solution must be very difficult too.

Since she expected a 'hard' problem to have a 'difficult' solution, she did not explore simpler alternatives for the solution. In the same spirit, several of the students used the words 'advanced' or 'complex' to characterise the function they looked for in problems P1 and P2.

It was, however, possible to identify a small number of strategies and resources that the students started to use in order to explore ways both to gain deeper understanding of the problem and to find a path towards a solution. These were visible in their solutions to the problems and discussed during the interviews. Most notable among these strategies were *asking suitable questions* and *seeing the problem from different angles*.

One way to see the problem differently is to *change the representation*. For example, student S7 intuitively chose $\sqrt{x}$ as the base function in problem P1. After trying for a while by adding parameters to $\sqrt{x}$ he did not notice much improvement and concluded that the best fit was $20\sqrt{x}+x$. Interestingly, during the interview it became clear that he did not think of the symbol $\sqrt{\ }$ as 'something one could change'. Student S8, in contrast, who also started with $\sqrt{x}$ as base function in problem P1, stated that by simply rewriting $\sqrt{x}$ as $x^{½}$, he was able to identify the possibility of 'playing with the exponent' and finally reached the best solution, which is $x^{2/3}$. So, in this case, changing the representation played an important role in exploring alternative ways for solving the problem.

4.3. *Having the right attitude*

As a further consequence of failing to solve the problems early on in the course, the students also realised the importance of having the right *attitudes* and *expectations*. Several students, when asked how their attitudes and expectations had changed, mentioned *approaching problems with a more open mind* and *expecting to try things out.* They also mentioned *courage*, *communication* and *daring to use common sense*. For example, student S3 said, 'You need to have some courage to dare to try different things and not just sit there and wait . . . and if it doesn't work you must be prepared to discard it and try another one.' *Patience* was also mentioned as a valuable asset in problem solving.

Several students mentioned the importance of *self-confidence*. This was the case for student S1: 'If you get stuck, it's important to do something, at least that leads to some understanding . . . when you don't understand the problem it's not fun anymore.' When asked why S1 did not do anything to overcome her difficulty whereas she could at least ask the supervisor for help, she said: 'because I felt like a loser'. She stated that she felt she was not 'smart enough' which affected her confidence to ask anyone for help or even try any other approach by herself.

5. Impacts of the course

Early in the course, students were mostly concerned with the problem-solving aspects we have described in the previous section. However, in the reports all students submitted at the end of the course, they reflected on both modelling and problem solving, and how these together contribute to their ability to solve real-world problems.

5.1. *Mathematical modelling*

Many students had little experience of *mathematical modelling* from previous courses, tending to see formulas as given, and with a vague notion of the concept of a model. They note that ‘ordinary math courses are abstract and do not address the earlier stages of real problem-solving’. They see modelling as extending mathematics from ‘a formulated problem to be solved’ to an important and useful tool for solving real-world problems. Students say that ‘the versatility of mathematical models is enormous’, and that modelling has ‘opened up the horizons for what problems we are able to solve’. They see links also to related areas of engineering, for example, that ‘modelling and programming have many similarities ... the course helps us to be better software engineers’.

Modelling also had an impact on the students’ views on mathematics and mathematical thinking. They note that modelling is related to simplification, and that the value of a model can lie in its simplicity rather than in its precision. This is in sharp contrast with the students’ prior thinking, where ‘the goal was always to find the best solution rather than a good and simple one’. Moreover, the students are surprised to learn that even simple mathematics can be useful. Since it is natural to investigate simple models first in an iterative process, and models are not strictly about right and wrong, students also find that modelling makes mathematics less intimidating and more creative. They find that their own thinking is activated as it becomes necessary to analyse and interpret the problem before solving it, taking the context into account, rather than following given rules. They also mention how modelling helped them ‘realize that we have so much mathematical knowledge within us, and we trust our intuition much more’.

With respect to the relationship between modelling and problem solving, some students see modelling as the primary objective, and problem solving as the ability to solve such and other problems. However, many see themselves as ‘problem solvers’, with modelling as a particularly important aspect of problem solving. One student notes: ‘to solve a difficult problem we create a simplified version of it, a “model” of the original problem. So modelling is an important concept in all problem solving’.

5.2. *Problem solving*

What the students say about *problem solving* at the end of the course is highly consistent with what the interviewed students reported early in the course. They say that ‘before we were not aware and systematic’, and at the end of the course ‘the biggest change is a schema for working in a structured and careful way’.

Students emphasise how they now strive for a deeper understanding by seeing the problem from different angles. They recognise the iterative and exploratory nature of the problem-solving process, how asking questions is helpful for moving forward and how taking different roles individually or in discussion with others is helpful. They suggest numerous problem-solving schemes of their own design, and while the overall aspects of problem solving are in focus, they mention some heuristic strategies, of which *simplification* and *splitting in parts* are the most common.

Students explain how they need to be clear and careful, be self-aware and keep track of what they are doing, as well as how they need to evaluate and reflect. Failure is accepted as common and is no longer seen just as failure. Instead, ‘failure is learning’, and ‘the fear of not being on the right track’ is gone. An open mind, together with patience and the courage to try things out, is also mentioned here. Moreover, intuition and creativity are seen as important dimensions of problem solving.

Students have learned problem solving ‘like they have expected an engineer to be’. They see the value of the knowledge they have, they ‘think more and ask less’ and trust their intuition more. They report a significantly higher self-confidence, in solving problems and working with mathematics generally, and see problem solving as a 'driving force and a way to think'.

6. Describing student difficulties in terms of metacognition and beliefs

We find it illuminating to draw on a framework for *mathematical thinking* developed by Schoenfeld (1985, 1992) in discussing our results. (For related theories about problem solving in general, see e.g. Mayer and Wittrock 2006.) What Schoenfeld proposes is that four aspects of mathematical thinking are necessary and sufficient for understanding why individuals are successful or not when engaging in mathematical problem solving: (1) *resources* – the individual's mathematical knowledge base; (2) *heuristics* – the individual's problem-solving strategies; (3) *self-regulation* – the individual's monitoring and controlling of thinking processes and (4) *beliefs* – the individual's 'understandings and feelings that shape the ways that the individual conceptualizes and engages in mathematical behaviour'.

Our students were not primarily lacking in *resources*, i.e. their mathematical content knowledge – it is quite clear that they know more mathematics than they are able to use. They further appeared to have some idea of a few basic *heuristic strategies*, such as *finding similar problems*, *splitting the problem in parts* and *visualisation*, but demonstrated difficulties in actually seeing how to use them. Many students were however less familiar with strategies to simplify their thinking, such as *considering special or extreme cases*.

Our results mainly demonstrate that the students lacked effective *self-regulation* skills, which severely hindered them from solving the problems. The students describe how they were used to solving *clearly formulated problems in a linear way by following given patterns* ('the only way' as one student puts it). They were expected to select and follow a certain method – 'solving' rather than 'understanding' was emphasised, they point out. They had therefore not fully realised the importance of *understanding the problem* and *exploring alternatives*. As a consequence, they hastily decided on a solution path and persistently pursued that path, even if it often turned out to be a blind alley. The students seldom took a step back to judge whether the chosen path was effective and, if not, look for alternatives. In other words, the students did not engage in the kind of active *monitoring* and, as needed, subsequent *controlling* of their own thinking processes that successful problem solvers engage in. These twin processes, monitoring and controlling, comprise one central aspect of *metacognition* (Brown 1978; Flavell 1979). This aspect of metacognition is what Schoenfeld (1987, 190) calls *self-regulation*: 'How well do you keep track of what you are doing when you are solving problems, and how well (if at all) do you use the input from those observations to guide your problem solving actions?' The failed attempts in solving the problems led the students to become aware of the importance of self-regulation in mathematical problem solving.

The students also expressed different *beliefs* about mathematics, problem solving and themselves as learners. These beliefs, shaped by past experiences, often hindered them from solving the problems. One example is that a seemingly complicated problem must have a complex solution. Another is that there is a known method for everything, and that if you do not quickly see which method to apply, then you are unable to solve the problem. Yet another belief is that mathematics is about following rules and learning new theory, rather than own thinking. These and other beliefs shaped the attitudes the students had in the beginning, and illustrate how *beliefs can override self-regulation* in problem solving. Through failing to solve the problems, the students became more aware of their own beliefs and how these influenced their abilities to solve the problems. The students also report that they developed more productive beliefs and attitudes during the course.

Given their previous experience, it is not surprising that the students demonstrate significant weaknesses when they approach more open problems where *interpreting the problem and finding a method for solving it is a part of the task.* Modelling problems are typically of this character since, like many realistic problems, they are often not well-defined in their formulation, and it is difficult in advance to prescribe a detailed algorithm for their solution. These results are consistent with previous research on mathematical problem solving, demonstrating that self-regulation and beliefs are two aspects of mathematical thinking that strongly influence how successful students are when engaging in problem solving (Garofalo and Lester 1985; Schoenfeld 1987; Cardella and Atman 2005; Schaap, Vos, and Goedhart 2011). The results support the call for a broader view of mathematical competencies in line with Schoenfeld's aspects of mathematical thinking (Cardella 2008; Alpers 2013). Indeed, our study highlights that problem-solving abilities of this kind, sometimes seen as peculiar to a specialised domain of mathematical problem solving, are important for engineering students' ability to solve real-world problems.

7. Describing student development in terms of cognitive apprenticeship

It is not only the kind of well-structured problems that the students usually encounter that account for their lack of effective problem-solving skills, but also the way that traditional teaching is carried out. Even for more realistic problems, teachers seldom model the entire problem-solving process, including assumptions, alternative approaches and evaluation of results (Stice 1996). Consequently, what students get to see is the *product* but not the *process* behind the product.

In order to better understand how elements of the learning environment contribute to the impacts of the course, we will draw on *cognitive apprenticeship* (Collins, Brown, and Newman 1989) as a theoretical framework. Cognitive apprenticeship is an approach to instructional design that was proposed as a solution to the educational problems described earlier. The idea is to extend traditional apprenticeship to teach cognitive (rather than physical) skills and processes. It is based on teaching with *authentic tasks*, in a progression from simpler to more advanced tasks. The students *make their thinking visible* by practicing solving these tasks. The teacher *makes his or her thinking visible* by showing not only the result of a solved problem but also how the solution was found, including alternative approaches and other considerations, and by coaching the students by providing feedback during their attempts to solve the problems. In this way, student thinking will approach that of the expert teacher. All of this is well aligned with the *models and modelling perspective* on mathematical problem solving (Lesh and Doerr 2003), which specifically advocates the use of modelling problems for making mathematical thinking visible.

The problems in the course have been designed to be *reasonably realistic*, in that they have been adapted to the level of the students, while keeping important basic characteristics of real-world problems (see Wedelin and Adawi, forthcoming). Students are thereby exposed to the challenges that we wish them to master. In the reports, students often refer to specific problems, and it is clear that the case-by-case familiarity with the collection of problems is in itself an important part of what students learn (see also Jonassen, Strobel, and Lee 2006). The problems are experienced as different from the ones the students are used to:

> They are somewhere between the theoretical calculations and real-world applications, which make them interesting. It feels worth putting in the effort, because it gives something back beyond just finishing the task. Working with them becomes very pleasant and it raises our interest in a much deeper way

and 'not presenting a method first is more stimulating'. They have been adapted to be *challenging* so that they are on the border of what students are able to handle: 'the problems are so difficult that they encourage problem solving. They require that you see them from different perspectives, and different approaches need to be tested', and 'to be a good problem solver you must solve many relatively difficult problems. To simulate this, many problems are vaguely formulated requiring thinking from the start'. Students point to the value of solving a moderate number of 'high complexity problems, as compared to most mathematics courses that provide hundreds of problems but of low complexity, and which usually can be solved using almost automatic procedures that put the mind to sleep'.

Turning to the culture of the course, students are *allowed* to think and solve problems in their own way: before 'smart solutions have just been presented to us', and there was 'often just one right way to solve the problem and several wrong, and answers could only be right or wrong. Here we are allowed to think as we like'. Otherwise, student thinking can never be genuinely visible and be developed. There is a 'focus on process rather than facts', and there is 'no completely new knowledge but you look at problems differently'. Students' own thinking is also encouraged in the assessment, where failure in solving problems is anticipated and accepted (see also Martinez 1998). However, students are also *expected* to think on their own, and this comes at a certain price. Some students acknowledge that they at first did not see the purpose of the course and were sceptical. They mention how they 'felt utterly useless in the beginning', and the experience was 'scary because we have to let go of given rules'. They say that 'working like this is completely new to us', and that it is 'totally new that we could be allowed to test and try things out'.

During the supervision sessions as well as in the introductory and follow-up lectures of each weekly module, the teacher strives to make the entire problem-solving process visible, by using all available means. Examples are presented in an interactive way, and important decision points in the problem-solving process are highlighted. Different ways to characterise modelling and problem solving are discussed, including attitudes and expectations. In the supervision, due to the difficulty of the problems, students have to communicate with each other and with the teacher. The teacher gives expert advice focusing on the process rather than the solution, trying to make students self-aware and realise how one, in principle, may think to move forward.

Students emphasise that the *Socratic questioning* was very helpful: 'Just by being asked by someone else, you realize a lot of things yourself.' However, it was 'at first annoying not to get clear answers', and

> it's worth noticing that the tutoring differs a lot from other courses in the sense that they will not really help solve the problem but instead tries to make you understand what you might be doing wrong and help you discover another possible direction to explore, not spoiling the problem itself.

It is appreciated that reflection on the answer is encouraged: 'you repeatedly ask us to improve ... allowing us to go further and deeper'. Students perceive a progression in the supervision: 'first more direct help, later in the course more discussion. The breakthrough was to go from waiting for somebody to tell you something, to a discussion on possible approaches'.

In the follow-up lectures, preferred solutions to the weekly problems are shown, and alternative ways to solve the problems as well as common problem-solving difficulties are discussed. Students are encouraged to reflect on their own solutions by comparing them with the presented ones. This inductive approach, where standard methods are mostly discussed afterwards, is seen to be more motivating.

These are typical elements in a cognitive apprenticeship environment. The supervision and other guidance are considered by the students to be very important for their learning, and 'learning has been very efficient'. Students say that the deeply integrated supervision 'has been the most important factor for our learning', and that 'practicing in this controlled environment has been enormously rewarding'. Another student remarks: 'Can you really learn problem solving, learn to see patterns? In my own experience with this course, yes. It is not a talent, it is a way to think that you can learn.'

8. Summary and conclusions

Using a qualitative case study approach, we have investigated engineering students' ability to deal with mathematical modelling problems. Our results demonstrate that the students had little prior experience of mathematical modelling and – more generally – of solving realistic problems. The students therefore experienced significant difficulties when attempting to solve problems that would otherwise be well within their reach in terms of mathematical knowledge (see also Soon, Lioe, and McInnes 2011). These difficulties were reflected, in particular, in the lack of *problem-solving skills*, as many students were unaware of the importance of *understanding the problem* and *exploring alternatives* in the problem-solving process. In this way, the students lacked effective *self-regulation* or *metacognitive* skills (Flavell 1979). Moreover, different types of *beliefs, attitudes and expectations* – mainly shaped by past experiences of solving problems in the context of mathematics courses – hindered the students from solving the problems.

After attending the course on mathematical modelling and problem solving, the demonstrated and self-reported learning outcomes, as well as the general response from the students, have been overwhelmingly positive and remarkably consistent. With respect to mathematics, students feel that they have been empowered, confirming that the course 'fills the gap between reality and pure math', and that 'now we have all we need to apply math as an engineer'. Important impacts of the course belong to the metacognitive domain, where students report that they have gained 'enormous self-insight'. They are 'more confident', 'are now much better problem solvers' and have learned 'a new way to think, beyond given rules'. We may here add that the students gave the current iteration of the course a score of 4.8 out of 5, in the standard evaluation of courses at Chalmers.

The nature of the problems in tandem with the Socratic supervision and the follow-up lectures were the components of the learning environment that the students believed most strongly contributed to their development. These components are related to the two hallmarks of a *cognitive apprenticeship* environment: *authentic problems* and *making thinking visible* (Collins, Brown, and Holum 1991).

We note, however, that this study is based on a course that has been refined over several years, carefully addressing challenges in the design of the problems and in the cognitive apprenticeship environment, as well as in communicating the culture of the course. We further note that the strong positive impression of the course depends on the fact that the course offers a new kind of learning environment to the students. In traditional mathematics and engineering courses, an explicit treatment of mathematical modelling is often overlooked (Zawojewski, Diefes-Dux, and Bowman 2008), making it difficult for students to apply the mathematics they know to real-world problems; the present course is here in line with calls for a broader view of mathematical competencies (Cardella 2008; Alpers 2013). More generally, traditional course problems typically differ substantially from engineering workplace problems (Jonassen, Strobel, and Lee 2006), and traditional courses often encourage students to follow the teacher (Gattie et al. 2011). In contrast, the set of small but reasonably realistic problems incrementally develops the students' ability to deal with *non-routine* and *ill-structured* problems, and the course is poles apart in how it encourages students to take control of their own thinking. This helps students to not see themselves as *followers* – as users of existing methods, trying to satisfy the expectations of others. We therefore see the course as an intermediate step between traditional engineering education and an ability to solve larger and more complex engineering problems.

Based on the findings in this paper, and the consistent response from the students over several years, we argue that a course in mathematical modelling and problem solving, or similar teaching, should be considered in the education of all engineers. This is supported by the students, who conclude that 'the course should be compulsory for all engineers', and that 'this kind of teaching should be given to everyone, at all stages'. Moreover, our findings offer insights into such curriculum development – how to implement cognitive apprenticeship to create a safe and supportive learning environment, in order to help students to become independent real-world problem solvers.

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About the authors

Dag Wedelin is Associate Professor at the Department of Computer Science and Engineering at the Chalmers University of Technology. He is also affiliated with the Division of Engineering Education Research (EER). His research interests in computer science and mathematics include algorithms, AI, optimisation and mathematical modelling. Within EER, he is interested in the teaching and learning of applied mathematics and more generally complex skills such as problem-solving.

Tom Adawi is Professor in Engineering Education Research (EER) and is heading the Division of EER at the Chalmers University of Technology. He is the Chair of the Nordic Network of Engineering Education Research (NNEER) and his research interests include students' understanding of threshold concepts in science and engineering, the interplay between and the development of conceptual/mathematical understanding, the scholarship of teaching and learning, as well as theoretical and methodological issues in EER.

Tabassum Jahan is a Ph.D. student at the Division of EER at the Chalmers University of Technology. Her research area is mathematical modelling and problem solving, with a focus on improving teaching and learning in engineering education.

Sven Andersson is Associate Professor in Theory of Science and with the Division of EER at the Chalmers University of Technology. He is responsible for several staff development courses within the Diploma of Higher Education.

Appendix 1. The two problems used in the interviews early in the course

Problem P1. For two related physical variables the following relationship has been measured, see Table A1.

Table A1. Time and distance table.

T (time)	D (distance)
88.0	57.9
224.7	108.2
365.3	149.6
687.0	228.07
4332	778.434
10760	1428.74
30684	2839.08
60188	4490.8
90467	5879.13

Suggest a mathematical equation describing the relation between these two quantities. Feel free to use your creativity. Any method that leads to a good result is fine.

Problem P2. A city wishes to make a long-term study to decide where to best locate emergency care. The city has been partitioned into regions, and it has been decided that an emergency care site can acceptably service regions of the city which are within a driving distance of eight minutes. The goal is to choose a set of stations at minimum cost. There are seven regions to cover, and six potential sites have been identified. Distances in minutes between regions and potential sites are shown in Table A2, and the costs for building at the different sites are shown in Table A3.

Table A2. Driving times from regions to sites.

Site #	1	2	3	4	5	6
Region 1	15	3	12	5	17	20
Region 2	12	9	13	16	3	4
Region 3	13	16	9	4	7	11
Region 4	3	22	12	5	16	18
Region 5	4	7	6	22	5	14
Region 6	8	10	5	16	13	5
Region 7	13	10	5	6	13	21

Table A3. Cost of the sites.

	Cost
Site 1	710,000
Site 2	610,000
Site 3	650,000
Site 4	910,000
Site 5	720,000
Site 6	570,000

(a) Model this problem mathematically by defining variables, constraints and an objective function. To get started, you can simply begin to define some variables, write some equations and see what you get along the way. (It is best if you can make the constraints and the objective function linear, since then the problem becomes easier to solve mathematically.)
(b) Now try to solve your model by using Mathematica function NMinimize. Try to solve it as a linear programming problem without using any special options of the NMinimize function or constrains written by you to say that the variables are integer or binary. Describe any difficulties you run into. What conclusions can you draw from the solution you obtain? Then, use any method you can think of to solve the problem and give the answer.

Appendix 2. An example of the data analysis process

Table A4. An example of the data analysis process, illustrating how the data was tagged and sorted into themes. We note that the tags are here stripped of much of the context, which is also used for arriving at the themes. This context is typically accounted for in the main text of the paper, and a more detailed description of how we arrived at these particular themes is found in Section 4.

<table>
<tr><th>Quotes</th><th>Tags</th><th>Patterns and themes</th></tr>
<tr><td>‘Sometimes it helps to divide the big problem into parts and deal with one part at a time and be aware of it’</td><td>Split into parts

Control and self-awareness</td><td rowspan="3">Formulate the problem in your own words (reformulate the problem)

Simplify

Not understanding the problem

Split into parts

Drawing, sketching (visualisation)

Discuss with group mate

Look at examples

…

Understanding the problem</td></tr>
<tr><td>‘Have self-confidence to evaluate your work, you will lose if you don’t dare to try different things’</td><td>Self-confidence

Dare to try things out</td></tr>
<tr><td>‘An important thing is to systematically test different ideas, constantly being active, and always trying different methods’</td><td>Systematically test ideas

Persistence in trying

Try different methods</td></tr>
<tr><td>‘Sometimes I write the problem in my own words. Simple in my own words. It clears my mind. Also drawing graphs and sketching makes things more clear’</td><td>Formulate the problem in your own words

Simplify

Drawing graphs and sketching clarifies issues</td><td rowspan="2">Systematically test ideas, try different methods and try things out

Try doing something

Look at examples

…

Exploring alternatives</td></tr>
<tr><td>‘You think you are on the right path, you work on it and after a while you realize what you thought to be the problem is not the problem at all’</td><td>Not understanding the problem</td></tr>
<tr><td>‘What helps in this process is trusting in oneself that a solution exists. If you try doing something long enough, you will find it. You can discuss with your group mate and clarify confusions, look at example problems’</td><td>Trust yourself

Persistence gives success

Try doing something

Discuss with group mate

Look at examples</td><td rowspan="2">Self-confidence

Controland self-awareness

Dare to try things out

Persistence gives success, persistence in trying

Trust yourself

…

Having the right attitude</td></tr>
<tr><td>…</td><td>…</td></tr>
</table>