Chapter 35
Applied Mathematical Problem Solving: Principles for Designing Small Realistic Problems

Dag Wedelin and Tom Adawi

Abstract We discuss and propose principles for designing problems that let engineering students practice applied mathematical problem solving. The main idea is to simplify real-world problems to make them smaller, while retaining important characteristics such that the solution to the problem is still of practical or theoretical interest, and that the problem should invoke non-trivial modelling and problem solving activities. We formalize our analysis in three dimensions of learning, which provide a basis for reflection beyond just solving the problem. We further discuss the benefits of being able to consider a large and highly varied set of smaller problems for discerning problem solving patterns, and give examples of such problems. We finally discuss the relationship with other proposed ways of designing problems.

35.1 Introduction

If we wish to teach students to solve real-world problems, it is reasonable to assume that real-world problems will serve as good exercise problems. This is, for example, the starting point in both *project-based* and *problem-based learning* (Kolmos et al. 2009; Mills and Treagust 2003). However, since real-world problems – especially in engineering – are often large and complex, students may encounter only a few such problems during their studies, which may provide insufficient

D. Wedelin (✉)
Department of Computer Science and Engineering, Chalmers University of Technology, Gothenburg SE-412 96, Sweden
e-mail: dag@chalmers.se

T. Adawi
Division of Engineering Education Research, Chalmers University of Technology, Gothenburg SE-412 96, Sweden
e-mail: adawi@chalmers.se

G.A. Stillman et al. (eds.), *Mathematical Modelling in Education Research and Practice*, International Perspectives on the Teaching and Learning of Mathematical Modelling, DOI 10.1007/978-3-319-18272-8_35

variation to learn problem solving and to be able to effectively handle unknown future problems (Bowden and Marton 1997).

In this theoretical chapter, we therefore present, discuss and illustrate a set of design principles for smaller problems that preserve important characteristics of real-world problems. We motivate the design principles by drawing on the literature on problem solving and mathematical modelling, as well as our own previous work in the area. These principles extend – and in some aspects go against – related design principles that have been formulated for pure mathematical problem solving (Schoenfeld 1991; Taflin 2003).

The set of principles that we propose have been successfully implemented in a course in mathematical modelling and problem solving, developed by the first author and offered to second-year engineering students at Chalmers University of Technology (for a description of the course, see Wedelin and Adawi 2014). We therefore expect that the work described in this chapter is useful to anyone who wants to include similar problems in their courses, in order to develop their students' ability to apply mathematics in practice, and we expect the main principles to be applicable also in other domains.

35.2 Real-World Problems Have Solutions That Are of Interest

Why is a real-world problem a *real-world* problem? One answer is that the problem exists because someone is interested in its solution (for a similar characterization, see Jonassen 2011). This can be for different reasons, mainly: (1) The solution is directly needed in practice. (2) The solution contributes to the understanding of some topic. In the second case, the problem may be a theoretical problem whose solution is a useful result or insight within the context in which the problem is posed, for example for the engineer, applied mathematician or other specialist. Note that many school problems, designed just to practice the application of some method, will not fall under either category.

Principle 1 The solution to the problem should be of practical or theoretical interest.

For problems of this kind, it becomes natural not only to solve the problem itself, but also to consider the context in which it is posed, and how its solution can be interpreted and contribute to this context. The problem further acts as a reasonably truthful representative of what you can expect in practice, helping students to recognize the character of fruitful investigations, and what they typically lead to.

An obvious way to create a small problem with this property is to begin with a real-world problem, in either of the two categories above. For the first category, we may then *focus on and simplify a critical aspect of a known applied problem* that we may know about in general terms (such as predicting the weather from past data), a problem we may know from own experience, or for example a problem from a thesis project. The challenge is then to keep the *essence* of the real-world problem: the simplified problem should still be a reasonably truthful model of a real problem that is actually out there, and any solution and its derivation should be similar. This can be seen as a modelling exercise for the teacher. A meaningful textbook problem in some applied subject can possibly be used more or less as it is, provided that it is used in a way that also satisfies the considerations of Sect. 35.3.

For the second category, we may *focus on and simplify a critical aspect of a known historical research task*. This can be done by first thinking about a central concept or result in some area – and then considering a particular realistic situation where it is natural to use, explore or discover this concept or result. Or we may consider some highly simplified "model" problem if its solution illustrates some meaningful phenomenon or effect in the context in which it is posed; such problems are often used in practice as vehicles for improved understanding. We may situate the exploration directly in the theoretical context, such as defining a suitable concept, or performing some derivation or generalization. The result is then of interest as a part of the theory. Tasks of this kind are important since we want our students not just to be able to apply given knowledge, but to develop new specific or general knowledge as needed.

35.3 Real-World Problems Require Exploration

Real-world problems are often ill-defined (Mayer and Wittrock 2006; Jonassen 2011), and difficult to fully understand. They require finding a relevant point of view, using the context and making relevant assumptions to create a model of the real problem in terms of a simpler well-defined problem, which is actually solved, and an interpretation of any solution in the real situation. This is known as *modelling*, which can be seen as a key step in applied problem solving.

Then, in the modelling as well as in the theoretical analysis, real-world problems typically give rise to situations where the method of solving the problem – or some sub-problem – is not known, so investigating and exploring different ways to see and to solve the problem, becomes an integral part of the process in order to successfully proceed and not get stuck. This is traditionally known as *problem solving*. See, for example, Lesh and Zawojewski (2007), Schoenfeld (1992) and Jonassen (2011), for different perspectives on modelling and problem solving.

We can contrast these observations with the common practice in both mathematics and engineering classes (especially when mathematics is involved), to solve well-defined problems with a given method (Jonassen et al. 2006), which gives little room for exploration. Models are present everywhere in engineering, but are often perceived as truths, and many students remain unaware of the concept of a model (Wedelin et al. 2015). Additionally, with more or less given methods, there is little focus on developing problem solving skills.

So, we must ensure that the challenge in solving the small problem is similar to that of the real-world problem, and take care not to simplify too much. The problem then provides an opportunity to learn to explore, and to develop modelling and problem solving skills. Since the problem is not easily solved, it becomes natural to talk to others as a means of moving forward, which is useful in itself and for the supervision.

Principle 2 The problem should be challenging to understand in order to stimulate modelling skills and communication.

Principle 3 The problem should be challenging to solve in order to stimulate problem solving skills and communication.

Of course, the challenge is also a function of the students themselves, and how much theory the teacher provides in advance. A routine problem can become a challenging exploration simply by refraining to first introduce any theory for solving it, creating an opportunity to learn to explore. The scope of the challenge can also be controlled by formulating the question appropriately, possibly in a progression where most students will be able to find something, and where the full solution is within reach at least for some.

We note that a real problem sometimes requires learning and searching for existing theory, and this can certainly be sensible to practice. However, there is a risk that students then focus on the highly visible new theory and methods, and less on the invisible skills, creating an imbalance in the long run. In fact, many students believe that if they have difficulties in solving a problem they need to learn more theory, expecting that new given problems will always be solved with new given methods (Wedelin and Adawi 2014). If the learning objective is to especially improve students' own skills we may consider the following recommendation, although it has little to do with the realism of the problem itself:

Principle 4 The problem should not require extensive new theory to be learned before the problem is attempted.

35.4 Problems as Cases and Three Dimensions of Learning

A realistic problem designed along the lines we have discussed, can be seen as offering *learning opportunities in three dimensions*:

- *Familiarity with real-world problems*. A realistic problem and its solution (including any necessary derivation), acts as a representative example and contributes to a familiarity with real-world problems in the domain of interest.
- *Supporting knowledge*. The concepts and methods needed to solve the problem, and how they are used for this purpose (known in advance or created as a part of the solution process).

- *Processes and skills*. The particular way in which the solution (and its derivation) was found, among many different imaginable ways to approach the problem, and the modelling and problem solving techniques involved.

The first and last dimensions relate directly to the previous discussion, while the second is the conventional dimension relating to the knowledge required to solve a problem. We note that there is a rough relationship between these dimensions of learning and a framework for mathematical modelling competency by Blomhøj and Jensen (2007), although we are here concerned directly with properties of problems rather than competencies.

We note that while these dimensions are important aspects of a problem, an actual problem connects a particular real-world problem from which is has been created, particular knowledge needed to solve it, and some particular approach for solving it. It therefore contains more information than what can be seen in each dimension separately. The problem may also contain other potentially important aspects that we are unaware of.

So remembering the problem as a *case* makes it possible at a later time to discern relevant aspects of the problem, which may not have been of interest when the problem was encountered. This can be important for seeing problem solving patterns across problems, and to constructively relate to old problems when approaching new ones. The importance of cases is widely supported in the literature on applied problem solving (Jonassen 2011; Kolodner et al. 1996).

Even though the principles we have already suggested are likely to ensure that a problem cannot become too small or insignificant, we still – considering the abundance of very small and repetitive problems especially in mathematics – suggest the following principle as a safety precaution:

Principle 5 The problem should be easy to remember as a case.

35.5 Extending the Problem Solving Experience: Perspective and Variation

Given the three dimensions of learning, it makes sense for the students and the teacher to discuss and reflect on the problems especially from these three points of view. This includes the specifics of the problem in each dimension, as well as what the problem as a representative example can convey about each dimension in general. We may tell a story about the corresponding real-world problem, or show a large-scale version of a related problem. Some results, observations or methods can be explained to be generally important. When seeing the solution of the teacher, and alternative solutions, students can reflect on their own difficulties in solving the problem, and why they occurred. Overall aspects, such as strategies in modelling and problem solving, and other considerations, can be discussed based

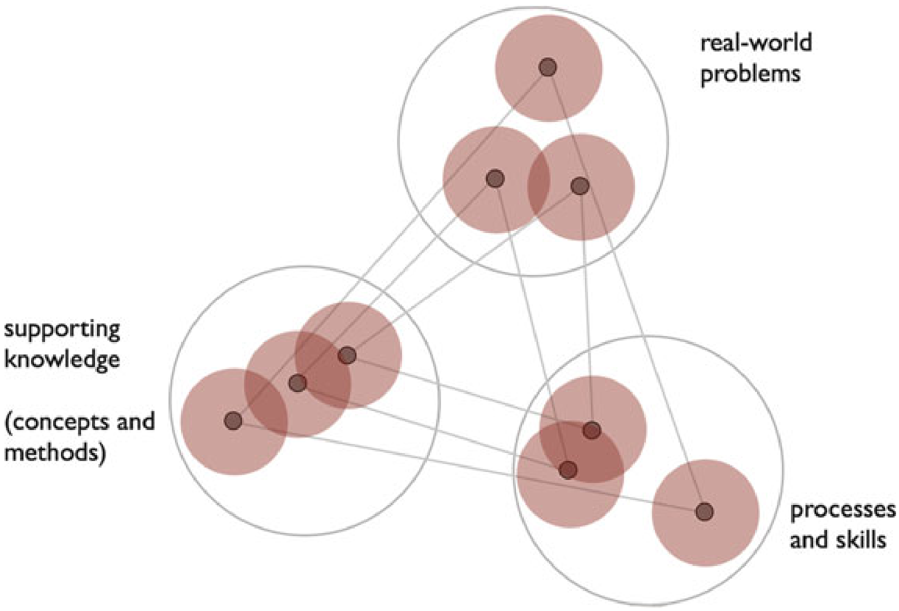

Fig. 35.1 Illustration of how several problems and their perspective span the three dimensions of learning (each *triangle* represents one problem)

on the students' own attempts to solve the problems. We refer to this entire reflection as a *perspective* on the problem.

In the perspective we include relevant *insights, attitudes, beliefs and expectations*. Such aspects are important to convey the views and ways of an experienced teacher, and are known to be important in problem solving (Schoenfeld 1992).

Principle 6 Provide a perspective on the problem in all three dimensions of learning.

We think of the perspective as an integral part of the problem, although it is not presented in the actual problem text, but provided to, and learned by, the students in other ways.

Finally, smaller problems allow us to combine many different problems to create *variation* in all three dimensions of learning, as illustrated in Fig. 35.1. Variation is essential for being able to discern critical aspects of a problem (Bowden and Marton 1997; Marton and Trigwell 2000). Through the variation, students get a chance to experience differences and similarities between problems, including higher-order patterns, and we create a basis for conveying different forms of experience in the language of an expert. We note especially that modelling and problem solving strategies are difficult to fully formalize, and have meaning mainly if the students are able to experience patterns in their work across different problems.

Principle 7 Create a problem set with variation in all three dimensions of learning.

One way to do this is to more or less arbitrarily collect a number of problems, and then iteratively build a subset of these problems under the constraint that they should cover a number of aspects that we have defined in each dimension.

35.6 Examples of Mathematical Modelling Problems

In order to give an impression of how we implement the principles in practice – including the idea of variation – we briefly give five examples of mathematical modelling problems from our course. All of the problems place students in a mode of exploration, where they have to spend time to understand the problem and explore alternatives, and where important metacognitive aspects are trained. (For a detailed analysis of how the students deal with problems of this kind, see Wedelin et al. 2015.) A more exact problem formulation for these and other problems is available on the course homepage (Wedelin 2014); these formulations have been calibrated based on how students typically approach the problems.

Kepler Curve-Fitting Problem We ask the students to suggest a function for fitting a number of given points (time and distance). No method for curve fitting, or the least squares method, is given in advance. The table contains planet data, and the best solution to this problem is Kepler's third law, making it possible to extend the perspective to a historical context. The problem requires some informal judgement about what a "good" solution to this problem is, and invokes real problem solving where students need to explore different functions, discover the so called *modelling cycle*, and so on.

Telephone Network Problem A Swedish mobile phone operator wants to rent communication lines from the national fixed network operator to connect its base stations to their central switch. Given the character of these communication lines, and the prices, how can we decide which lines to rent for a low total cost? The problem is based on a Master's thesis, and many complicating details have been removed. The problem is given as a theoretical modelling exercise, in a progression, where students are first asked to solve an even simpler version of the problem, providing some insight into how varying the problem can influence the way you solve it. The problem can be modelled as a mathematical programming problem, but can also be solved heuristically with a modified spanning tree algorithm, which is well suited to the natural dynamicity of the problem.

Bridge Problem A simple road network including two cities, and some assumptions about how speed changes with traffic intensity, is given. What is the expected travel time between the cities, and how it might improve if a bridge is built? The problem requires additional assumptions about driver behaviour and a precise formulation of equilibrium conditions. It turns out that the travel time *increases* with the new bridge. This is known as Braess' paradox; it can happen in practice, and is an instance of a Nash equilibrium (See Wu et at., Chap. 9, Sect. 9.2).

Drug Dosage Problem How can we calculate the time interval and dosage for a drug? No specific details are given. Many assumptions are required, as well as a combined qualitative and quantitative understanding of the real-world problem and different models, and it is important to split the problem solving process in steps. The problem also shows how you can reach further than what most students expect without substantial subject knowledge, by making significant assumptions and seeing what they lead to, in a kind of thought experiment.

Project Planning Problem How can large projects consisting of many subprojects be represented mathematically and how can the total project time be estimated? We further ask how this time can be computed with the help of a shortest path algorithm. How can the model be extended if we have uncertainty in the estimated times? The well-known methods CPM (Critical Path Method) and PERT (Project Evaluation and Review Technique) have been the source of this problem. The problem includes modelling with graphs, the idea of modelling one problem (the longest path problem) in terms of another (the shortest path problem), and thinking about the modelling of uncertainty.

Insights and patterns that can be illustrated and discerned from this varied ensemble of problems include the usefulness of changing the representation, the importance of a qualitative understanding, how the problem solving can be split in simpler steps, the exploratory nature of problem solving, creating examples, considering extreme cases, the power of making assumptions and drawing their logical consequences to the limit, and so on.

35.7 Discussion of Related Work

Our work was inspired by similar work focusing on *rich problems* (for an overview, see Taflin 2003). Drawing, in particular, on work by Schoenfeld (1991), Taflin (2003) developed a list of design principles for rich problems in mathematics education. These recommendations for designing good problems are similar to ours in that the *problem solving* aspect is emphasized, and in that the problem should be challenging to solve. However, our recommendation that the solution to the problem should be of interest is for rich problems restricted to the *mathematical* domain. Moreover, our recommendation that the problem should be *challenging to understand* is contradicted for rich problems. This is because the notion of rich problems does not take *models and modelling* into account. Our problems offer significant learning opportunities in applied mathematical problem solving, in addition to purely mathematical problem solving and mathematical content knowledge.

The kind of problems we propose bear a strong resemblance to a class of problems known as *model-eliciting activities* (MEAs), which is increasingly being used in engineering education (Diefes-Dux et al. 2008; Hamilton et al. 2008). MEAs are *scaled-down*, real-world problems that require students to develop or adapt a mathematical model for a given situation. With roots in mathematics education research, MEAs were originally developed to serve as instruments for investigating student and teacher thinking at school level (Lesh et al. 2000). A difference is that we start by looking at properties of real-world problems, rather than first considering the kinds of problems that most effectively reveal student thinking. Moreover, we emphasize what we have called a *perspective* for the problems, and the use of many small and *varied* problems in order to help students to discern higher order patterns, such as problem solving strategies. In other words, we take advantage of the opportunities that come with using many small problems. On a more general level, MEAs appear to focus on most phases/ roles in a project, whereas our focus is more specifically on the kind of work undertaken by an applied mathematician involved in the project.

Regarding the notion of *authenticity* (Vos 2011, 2015), we consider our problems to be authentic with respect to the real-world problem characteristics that we intentionally retain, and in some way in how we work with the problems. It is, however, not our goal to be as authentic as possible – the simplification itself is clearly for educational purposes, in order to create smaller problems to highlight important aspects of interest, and to exploit variation. And our point of view is that of a specialist engaged in a particular role in several different projects, rather than a person engaging in all aspects of a single project. We note that definitions of authenticity do not generally consider the *difficulty* of the problems, since real-world problems can be both easy and difficult.

35.8 Conclusions

The proposed way of creating smaller problems from real-world problems underlines aspects of real-world problems that we consider to be especially important for developing applied problem solving skills. The smaller problems have the potential to help students to develop in all of the *three dimensions of learning*, and they provide the students with *a case library* to draw on in future courses and in the workplace. We have also emphasized the importance of a *perspective* for the problems – a reflection on the problems in all three dimensions of learning – as an important complementary part of the problem. A prepared perspective also makes it easier for other teachers to fully understand and use the problems.

Importantly, the use of smaller problems enables *repeated and continuous feedback on the entire problem solving process*. Moreover, working with a set of varied problems opens up for *reflections on higher order patterns*, for example related to problem solving, and allows the teacher to talk about and convey his or her general experience in a meaningful way. Due to their limited size the problems are relatively easy to supervise.

The challenging nature of the problems is, in our experience, very effective for making students' own thinking visible, enabling more pointed feedback for developing complex skills and constructive attitudes. The problems encourage the students to engage in creative thinking, and convey the message that students are expected to do more than just applying given or known methods, including developing new theoretical concepts, models and methods as needed and that they are able to do so.

How do students in our course experience this approach to designing and using problems then? Most students find it extremely motivating, yet quite frustrating. At the beginning of the course they struggle, in particular, to develop effective problem solving skills, and they are hampered by unsuitable attitudes and expectations. However, the response of the students has been exceptionally positive, and after taking the course most students express and demonstrate a fundamental change in their abilities to "think mathematically", in their understanding of the nature of mathematics and its role in their future profession (Wedelin et al. 2015).

Compared to full real-world problems, the small realistic problems we propose have a bias towards being more condensed and simplified, with potential discoveries just around the corner, and with less time-consuming (and possibly boring) work. However, we have found these smaller problems very useful, and see them as a stepping-stone towards a full ability to handle real and larger projects. A course like the one we give therefore acts as an intermediate step between traditional engineering courses and full-scale projects of the kind our students will meet in their profession.

References

Blomhøj, M., & Jensen, T. H. (2007). What's all the fuss about competencies? In W. Blum, P. Galbraith, H. W. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics education: The 14th ICMI study* (pp. 45–56). New York: Springer.

Bowden, J., & Marton, F. (1997). *The university of learning*. London: Kogan Page.

Diefes-Dux, H. A., Hjalmarson, M. A., Miller, T. K., & Lesh, R. (2008). Model-eliciting activities for engineering education. In J. S. Zawojewski, H. A. Diefes-Dux, & K. J. Bowman (Eds.), *Models and modeling in engineering education* (pp. 17–35). Rotterdam: Sense.

Hamilton, E., Lesh, R., Lester, F., & Brilleslyper, M. (2008). Model-eliciting activities (MEAs) as a bridge between engineering education research and mathematics education research. *Advances in Engineering Education, 1*(2), 1–25.

Jonassen, D. (2011). *Learning to solve problems*. New York: Routledge.

Jonassen, D., Strobel, J., & Lee, C. B. (2006). Everyday problem solving in engineering: Lessons for engineering educators. *Journal of Engineering Education, 95*(2), 139–151.

Kolmos, A., de Graaff, E., & Du, X. (2009). Diversity of PBL – PBL learning principles and models. In X. Du, E. de Graaff, & A. Kolmos (Eds.), *Research on PBL practice in engineering education* (pp. 9–21). Rotterdam: Sense.

Kolodner, J. L., Hmelo, C. E., & Narayanan, N. H. (1996). Problem-based learning meets case-based reasoning. *Proceedings of the International Conference on Learning Sciences* (pp. 188–195). Evanston: International Society of the Learning Sciences.

Lesh, R., & Zawojewski, J. (2007). Problem solving and modeling. In F. Lester (Ed.), *The second handbook of research on mathematics teaching and learning* (pp. 763–804). Charlotte: Information Age Publishing.

Lesh, R., Hoover, M., Hole, B., Kelly, A., & Post, T. (2000). Principles for developing thought-revealing activities for students and teachers. In A. Kelly & R. Lesh (Eds.), *Research design in mathematics and science education* (pp. 591–646). Mahwah: Erlbaum.

Marton, F., & Trigwell, K. (2000). Variatio est mater studiorum. *Higher Education Research and Development, 19*(3), 381–395.

Mayer, R., & Wittrock, M. (2006). Problem solving. In P. A. Alexander & P. H. Winne (Eds.), *Handbook of educational psychology* (pp. 287–303). Mahwah: Erlbaum.

Mills, J. E., & Treagust, D. F. (2003). Engineering education: Is problem-based or project-based learning the answer? *Australasian Journal of Engineering Education, 3*, 2–16.

Schoenfeld, A. H. (1991). What's all the fuss about problem-solving? *Zentralblatt für Didaktik der Mathematik, 23*(1), 4–8.

Schoenfeld, A. (1992). Learning to think mathematically: Problem solving, metacognition and sense making. In D. Grouws (Ed.), *Handbook of research in mathematics teaching and learning* (pp. 334–370). New York: Macmillan.

Taflin, E. (2003). *Problem solving – An analysis of rich mathematical problems* (in Swedish). Licenciate thesis, Department of Mathematics, Umeå University.

Vos, P. (2011). What is "authentic" in the teaching and learning of mathematical modelling? In G. Kaiser, W. Blum, R. Borromeo-Ferri, & G. Stillman (Eds.), *Trends in teaching and learning of mathematical modelling* (pp. 713–722). New York: Springer.

Vos, P. (2015). Authenticity in extra-curricular mathematics activities: Researching authenticity as a social construct. In G. A. Stillman, W. Blum, & M. S. Biembengut (Eds.), *Mathematical modelling in education research and practice: Cultural, social and cognitive influences* (pp. 105–113). Cham: Springer.

Wedelin, D. (2014). Course home page at www.cse.chalmers.se/edu/course/DAT026 (or via the home page of Dag Wedelin).

Wedelin, D., & Adawi, T. (2014). Teaching mathematical modelling and problem solving: A cognitive apprenticeship approach to mathematics and engineering education. *International Journal of Engineering Pedagogy, 4*(5), 49–55.

Wedelin, D., Adawi, T., Jahan, T., & Andersson, S. (2015). Investigating and developing engineering students' mathematical modelling and problem solving skills. Accepted for publication in *European Journal of Engineering Education*. doi: 10.1080/03043797.2014.987648