A cubical type theory

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Type theory and univalence

Extension of type theory in which Voevodsky's univalence axiom is *provable*

This is an example of a *presheaf model extension* of type theory

Type theory and univalence

The idea is to allow elements and types to depend on "names"

$$u(i_1, \ldots, i_n)$$

Purely *formal* objects which represent elements of the unit interval $[0, 1]$

Type theory and univalence

At any point we can do a "re-parametrisation"

$$i_1 = f_1(j_1, \ldots, j_m)$$

$$\ldots$$

$$i_n = f_n(j_1, \ldots, j_m)$$

using the operations $max(r, s), min(r, s), 1 - r$ and constants $0, 1$

Structure of *de Morgan algebra*

Dependent type theory

Γ, Δ	$::=$	$() \mid \Gamma, x : A$	Contexts
t, u, A, B	$::=$	$x \mid \lambda x : A.\ t \mid t\ u \mid (x : A) \to B$	Π-types
	$\mid$	$(t, u) \mid t.1 \mid t.2 \mid (x : A) \times B$	Σ-types
	$\mid$	$0 \mid \mathsf{s}\ u \mid \mathsf{natrec}\ t\ u \mid \mathsf{N}$	Natural numbers

We write $A \to B$ for the non-dependent function space and $A \times B$ for the type of non-dependent pairs

Terms and types are considered up to α-equivalence of bound variables

Dependent type theory

$$\frac{\Gamma \vdash t = u : A \qquad \Gamma \vdash A = B}{\Gamma \vdash t = u : B}$$

$$\frac{\Gamma, x : A \vdash t : B \qquad \Gamma \vdash u : A}{\Gamma \vdash (\lambda x : A.\ t)\ u = t(x/u) : B(x/u)}$$

$$\frac{\Gamma, x : A \vdash t\ x = u\ x : B}{\Gamma \vdash t = u : (x : A) \to B}$$

Dependent type theory

$$\frac{\Gamma \vdash t : A \qquad \Gamma \vdash u : B(x/t)}{\Gamma \vdash (t,u).1 = t : A \qquad \Gamma \vdash (t,u).2 = u : B(x/t)}$$

$$\frac{\Gamma \vdash t.1 = u.1 : A \qquad \Gamma \vdash t.2 = u.2 : B(x/t.1)}{\Gamma \vdash t = u : (x : A) \times B}$$

Dependent type theory

The following rules are admissible

Weakening rules: a judgment valid in a context stays valid in any extension of this context.

Substitution rules:

$$\frac{\Gamma \vdash \mathsf{J} \qquad \Delta \vdash \sigma : \Gamma}{\Delta \vdash \mathsf{J}\sigma}$$

where $\Delta \vdash \sigma : \Gamma$ is defined by induction on Γ

Paths

$$\Gamma, \Delta \quad ::= \quad \ldots \mid \Gamma, i : \mathbb{I}$$

$$\frac{\Gamma \vdash}{\Gamma, i : \mathbb{I} \vdash} \; (i \notin dom(\Gamma))$$

$$r, s \quad ::= \quad 0 \mid 1 \mid i \mid 1 - r \mid r \wedge s \mid r \vee s$$

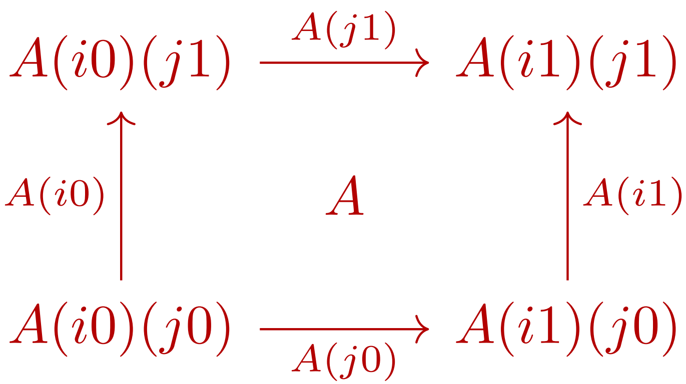
Paths

$$
\begin{array}{rcll}
t, u, A, B & ::= & \ldots & \\
& | & \mathsf{Path}\ A\ t\ u \mid \langle i \rangle\ t \mid t\ r & \text{Path types}
\end{array}
$$

Path abstraction, $\langle i \rangle\ t$, binds the name i in t

Path application, $t\ r$, applies a term t to an element $r : \mathbb{I}$

Paths

$() \vdash A$	$\bullet\ A$
$i : \mathbb{I} \vdash A$	$A(i0) \xrightarrow{A} A(i1)$
$i : \mathbb{I}, j : \mathbb{I} \vdash A$	
$\vdots$	$\vdots$

Paths

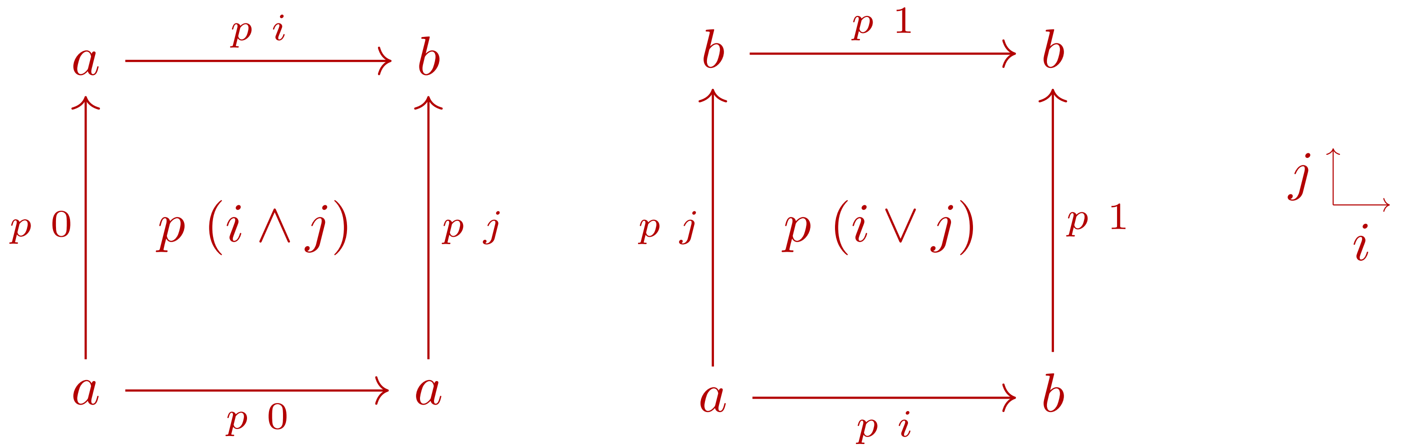
$$\frac{\Gamma \vdash A \qquad \Gamma \vdash t : A \qquad \Gamma \vdash u : A}{\Gamma \vdash \mathsf{Path}\ A\ t\ u} \qquad \frac{\Gamma \vdash A \qquad \Gamma, i : \mathbb{I} \vdash t : A}{\Gamma \vdash \langle i \rangle\ t : \mathsf{Path}\ A\ t(i0)\ t(i1)}$$

$$\frac{\Gamma \vdash t : \mathsf{Path}\ A\ u_0\ u_1 \qquad \Gamma \vdash r : \mathbb{I}}{\Gamma \vdash t\ r : A} \qquad \frac{\Gamma \vdash A \qquad \Gamma, i : \mathbb{I} \vdash t : A}{\Gamma \vdash (\langle i \rangle\ t)\ r = t(i/r) : A}$$

$$\frac{\Gamma, i : \mathbb{I} \vdash t\ i = u\ i : A}{\Gamma \vdash t = u : \mathsf{Path}\ A\ u_0\ u_1} \qquad \frac{\Gamma \vdash t : \mathsf{Path}\ A\ u_0\ u_1}{\Gamma \vdash t\ 0 = u_0 : A} \qquad \frac{\Gamma \vdash t : \mathsf{Path}\ A\ u_0\ u_1}{\Gamma \vdash t\ 1 = u_1 : A}$$

Paths

Given p : Path A a b we can build



Paths

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : A \qquad \Gamma \vdash f : A \to B \qquad \Gamma \vdash p : \mathsf{Path}\ A\ a\ b}{\Gamma \vdash \langle i \rangle\ f\ (p\ i) : \mathsf{Path}\ B\ (f\ a)\ (f\ b)}$$

$$\frac{\Gamma \vdash p : (x : A) \to \mathsf{Path}\ B\ (f\ x)\ (g\ x)}{\Gamma \vdash \langle i \rangle\ \lambda x : A.\ p\ x\ i : \mathsf{Path}\ ((x : A) \to B)\ f\ g}$$

Paths

$$\frac{\Gamma \vdash p : \mathsf{Path}\ A\ a\ b}{\Gamma \vdash \langle i \rangle\ (p\ i, \langle j \rangle\ p\ (i \wedge j)) : \mathsf{Path}\ ((x : A) \times (\mathsf{Path}\ A\ a\ x))\ (a, 1_a)\ (b, p)}$$

where $1_a : \mathsf{Path}\ A\ a\ a\ =\ \langle i \rangle\ a$

Face lattice

$$\varphi, \psi \quad ::= \quad 0_{\mathbb{F}} \mid 1_{\mathbb{F}} \mid (i = 0) \mid (i = 1) \mid \varphi \wedge \psi \mid \varphi \vee \psi$$

Any element of $\mathbb{F}$ is the union of the irreducible elements below it. An irreducible element of this lattice is a *face*, a conjunction of elements of the form $(i = 0)$ and $(j = 1)$.

This provides a disjunctive normal form for elements of $\mathbb{F}$, and it follows from this that the equality on $\mathbb{F}$ is decidable.

Face lattice

$$\Gamma, \Delta \quad ::= \quad \ldots \mid \Gamma, \varphi$$

together with the rule:

$$\frac{\Gamma \vdash \varphi : \mathbb{F}}{\Gamma, \varphi \vdash}$$

Face lattice

$i : \mathbb{I}, (i = 0) \vee (i = 1) \vdash A$	$A(i0) \bullet \qquad \bullet A(i1)$
$i : \mathbb{I}, j : \mathbb{I}, (i = 0) \vee (j = 1) \vdash A$	$\begin{array}{ccc} A(i0)(j1) & \xrightarrow{A(j1)} & A(i1)(j1) \\ {\scriptstyle A(i0)}\uparrow & & \\ A(i0)(j0) & & \end{array}$

Face lattice

$$
i : \mathbb{I}, j : \mathbb{I}, (i = 0) \vee (i = 1) \vee (j = 0) \vdash A \quad \left| \quad \begin{array}{ccc} A(i0)(j1) & & A(i1)(j1) \\ {\scriptstyle A(i0)}\uparrow & & \uparrow{\scriptstyle A(i1)} \\ A(i0)(j0) & \xrightarrow[A(j0)]{} & A(i1)(j0) \end{array} \right.
$$

Systems

$$
\begin{array}{lcll}
t, u, A, B & ::= & \ldots & \\
 & \mid & S & \\
S & ::= & [\, \varphi_1\ t_1, \ldots, \varphi_n\ t_n \,] & \text{Systems}
\end{array}
$$

Systems

Assume $\Gamma \vdash \varphi_1 \vee \cdots \vee \varphi_n = 1_{\mathbb{F}} : \mathbb{F}$

$$\frac{\Gamma, \varphi_1 \vdash A_1 \qquad \cdots \qquad \Gamma, \varphi_n \vdash A_n \qquad \Gamma, \varphi_i \wedge \varphi_j \vdash A_i = A_j}{\Gamma \vdash [\ \varphi_1\ A_1, \ldots, \varphi_n\ A_n\ ]}$$

$$\frac{\Gamma \vdash A \qquad \Gamma, \varphi_1 \vdash t_1 : A \qquad \cdots \qquad \Gamma, \varphi_n \vdash t_n : A \qquad \Gamma, \varphi_i \wedge \varphi_j \vdash t_i = t_j : A}{\Gamma \vdash [\ \varphi_1\ t_1, \ldots, \varphi_n\ t_n\ ] : A}$$

Systems

$$\frac{\Gamma, \varphi_1 \vdash \mathsf{J} \qquad \cdots \qquad \Gamma, \varphi_n \vdash \mathsf{J}}{\Gamma \vdash \mathsf{J}}$$

$$\frac{\Gamma \vdash \varphi_i = 1_{\mathbb{F}} : \mathbb{F}}{\Gamma \vdash [\, \varphi_1 \ A_1, \ldots, \varphi_n \ A_n \,] = A_i}$$

$$\frac{\Gamma \vdash [\, \varphi_1 \ t_1, \ldots, \varphi_n \ t_n \,] : A \qquad \Gamma \vdash \varphi_i = 1_{\mathbb{F}} : \mathbb{F}}{\Gamma \vdash [\, \varphi_1 \ t_1, \ldots, \varphi_n \ t_n \,] = t_i : A}$$

Face lattice

Any judgement valid in Γ *is also valid in a restriction* Γ, ψ

E.g. if we have $\Gamma \vdash A$ we also have $\Gamma, \psi \vdash A$

Then $\Gamma, \psi \vdash u : A$ can be seen as a partial section of A

Any judgement valid in Γ *is also valid in an extension* $\Gamma, x : A$

Face lattice

We say that the partial element $\Gamma, \psi \vdash u : A$ is *connected*

iff we have $\Gamma \vdash a : A$ such that $\Gamma, \psi \vdash a = u : A$

We write $\Gamma \vdash a : A[\psi \mapsto u]$

a witnesses the fact that u is connected

This generalizes the notion of being *path-connected*

Take ψ to be $(i = 0) \vee (i = 1)$

Composition

$$t, u, A, B \quad ::= \quad \ldots$$
$$\quad | \quad \mathsf{comp}^i\ A\ [\varphi \mapsto u]\ a_0 \qquad \text{Compositions}$$

$$\frac{\Gamma \vdash \varphi \qquad \Gamma, i : \mathbb{I} \vdash A \qquad \Gamma, \varphi, i : \mathbb{I} \vdash u : A \qquad \Gamma \vdash a_0 : A(i0)[\varphi \mapsto u(i0)]}{\Gamma \vdash \mathsf{comp}^i\ A\ [\varphi \mapsto u]\ a_0 : A(i1)[\varphi \mapsto u(i1)]}$$

Composition

Composition expresses that to be connected is preserved along path

A partial path connected at 0 is connected at 1

Composition

If we have a substitution $\Delta \vdash \sigma : \Gamma$, then

$$(\mathsf{comp}^i\ A\ [\varphi \mapsto u]\ a_0)\sigma = \mathsf{comp}^j\ A(\sigma, i/j)\ [\varphi\sigma \mapsto u(\sigma, i/j)]\ a_0\sigma$$

where j is fresh for Δ

This corresponds semantically to the *uniformity* of the composition operation.

Cofibrations

$\Gamma, x : A$ defines a *fibration* over Γ

Γ, ψ defines a *cofibration* over Γ

Contractible types

$\mathsf{isContr}\ A = (x : A) \times ((y : A) \to \mathsf{Path}\ A\ x\ y)$

$\mathsf{isContr}\ A$ is inhabited iff there is an operation

$$\frac{\Gamma, \psi \vdash u : A}{\Gamma \vdash \mathsf{ext}\ u : A[\psi \mapsto u]}$$

Left lifting property of cofibrations w.r.t. trivial fibrations

Filling operation

We also have the left lifting property of *trivial cofibrations* w.r.t. any fibration

If $\Gamma \vdash \psi : \mathbb{F}$ we have an operation

$$\frac{\Gamma, i : \mathbb{I}, \psi \vee (i = 0) \vdash u : A}{\Gamma, i : \mathbb{I} \vdash \mathsf{fill}\ [\psi \vee (i = 0) \mapsto u] : A[\psi \vee (i = 0) \mapsto u]}$$

Composition operation

Defined by case on the type

$(x : A) \to B, \ (x : A) \times B, \ \mathsf{Path}\ A\ a\ b$

Classically, the fact that Kan simplicial sets are closed by dependent product is a non trivial fact

It reduces to the fact that trivial cofibrations are stable under pullbacks along Kan fibrations

Composition operation

Here we have an explicit definition, for $C = (x : A) \rightarrow B$

$\mathsf{comp}^i\ C\ [\varphi \mapsto \mu]\ \lambda_0 : C(i1)[\varphi \mapsto \mu(i1)]$

$(\mathsf{comp}^i\ C\ [\varphi \mapsto \mu]\ \lambda_0)\ u_1 \quad = \quad \mathsf{comp}^i\ B(x/v)\ [\varphi \mapsto \mu\ v]\ (\lambda_0\ v(i0))$

where

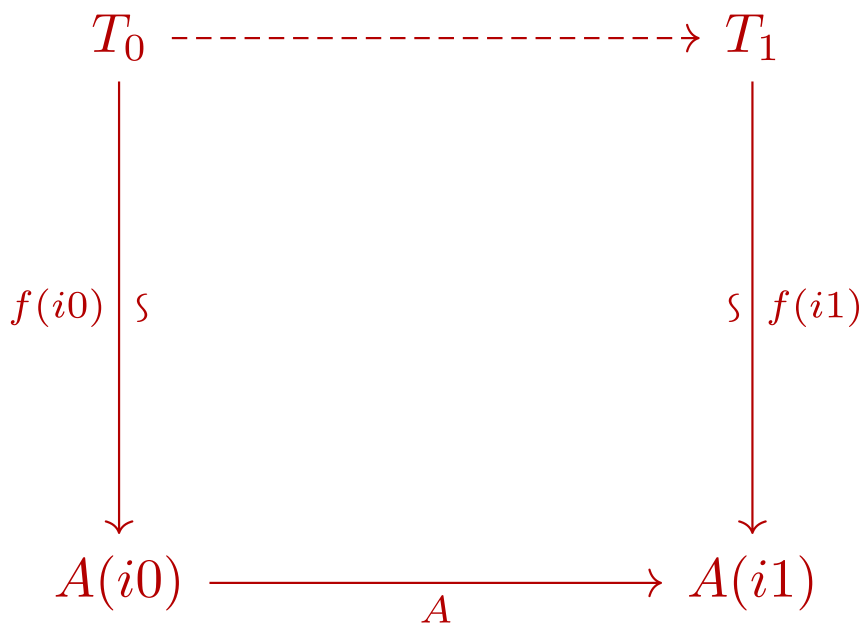
$i : \mathbb{I} \vdash w = \mathsf{fill}^i\ A(i/1-i)\ []\ u_1 : A(i/1-i)$

$i : \mathbb{I} \vdash v = w(i/1-i) : A$

Glueing

t, u, A, B	$::=$	$\dots$		
	$\mid$	Glue $[\varphi \mapsto (T, f)]\ A$		Glue type
	$\mid$	glue $[\varphi \mapsto t]\ u$		Glue term
	$\mid$	unglue $[\varphi \mapsto (T, f)]\ u$		Unglue term

Glueing



Glueing

This operation expresses that to be connected is preserved under equivalence

The main algorithm builds a composition for Glue $[\varphi \mapsto (T, f)]\ A$

Glueing

$$\frac{\Gamma \vdash A \qquad \Gamma, \varphi \vdash T \qquad \Gamma, \varphi \vdash f : \mathsf{Equiv}\ T\ A}{\Gamma \vdash \mathsf{Glue}\ [\varphi \mapsto (T, f)]\ A} \qquad \frac{\Gamma \vdash b : \mathsf{Glue}\ [\varphi \mapsto (T, f)]\ A}{\Gamma \vdash \mathsf{unglue}\ b : A[\varphi \mapsto f\ b]}$$

$$\frac{\Gamma, \varphi \vdash f : \mathsf{Equiv}\ T\ A \qquad \Gamma, \varphi \vdash t : T \qquad \Gamma \vdash a : A[\varphi \mapsto f\ t]}{\Gamma \vdash \mathsf{glue}\ [\varphi \mapsto t]\ a : \mathsf{Glue}\ [\varphi \mapsto (T, f)]\ A}$$

Glueing

$$\frac{\Gamma \vdash T \qquad \Gamma \vdash f : \mathsf{Equiv}\ T\ A}{\Gamma \vdash \mathsf{Glue}\ [1_{\mathbb{F}} \mapsto (T, f)]\ A = T} \qquad \frac{\Gamma \vdash t : T \qquad \Gamma \vdash a : A}{\Gamma \vdash \mathsf{glue}\ [1_{\mathbb{F}} \mapsto t]\ a = t : T}$$

$$\frac{\Gamma \vdash b : \mathsf{Glue}\ [\varphi \mapsto (T, f)]\ A}{\Gamma \vdash b = \mathsf{glue}\ [\varphi \mapsto b]\ (\mathsf{unglue}\ b) : \mathsf{Glue}\ [\varphi \mapsto (T, f)]\ A}$$

$$\frac{\Gamma, \varphi \vdash t : T \qquad \Gamma \vdash a : A[\varphi \mapsto f\ t]}{\Gamma \vdash \mathsf{unglue}\ (\mathsf{glue}\ [\varphi \mapsto t]\ a) = a : A}$$

Universe

$$\frac{\Gamma \vdash}{\Gamma \vdash U} \qquad \frac{\Gamma \vdash A : U}{\Gamma \vdash A}$$

We reflect all typing rules, e.g.

$$\frac{\Gamma \vdash A : U \qquad \Gamma, x : A \vdash B : U}{\Gamma \vdash (x : A) \to B : U}$$

Composition for the universe

If $\Gamma, i : \mathbb{I} \vdash E$ then we can build an equivalence $E(i0) \to E(i1)$

Using the glueing operation we build a composition for U

Univalence axiom

Let $B = \mathsf{glue}\ [\varphi \mapsto (T, f)]\ A$

One proves that $\mathsf{unglue} : B \to A$ is an *equivalence*

This maps extends $\varphi \vdash f.1 : T \to A$

It follows that $(X : U) \times \mathsf{Equiv}\ X\ A$ *is contractible*

This is one way to state the univalence axiom

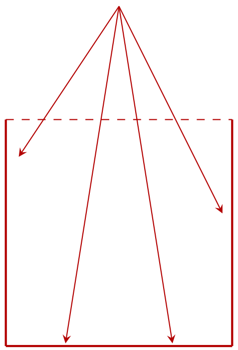
Identity type

Path $A\ a\ b$ satisfies the computation rule for Id only as a path equality

Geometrically, we cannot expect to have a judgemental equality

Fibrant cubical sets

Theorem: *Any singular cubical set has a uniform composition structure*



Identity type

We have Path A a_0 $b_0 \rightarrow$ Path A a_1 b_1 given

p : Path A a_0 a_1

q : Path A b_0 b_1

but we cannot expect this map to be the identity map if p and q are constant

Identity type

A. Swan (Leeds)

$$\frac{\Gamma \vdash \omega : \mathsf{Path}\ A\ a_0\ a_1[\varphi \mapsto \langle i \rangle\ a_0]}{\Gamma \vdash (\omega, \varphi) : \mathsf{Id}\ A\ a_0\ a_1}$$

Cofibration-trivial fibration factorization

Given $\Gamma \vdash f : A \to B$ we define the type T_f

$$\frac{\Gamma, y : B, \psi \vdash a : A \qquad \Gamma, y : B, \psi \vdash f\ a = y : B}{\Gamma, y : B \vdash [\psi \mapsto a] : T_f}$$

$\Gamma, y : B \vdash T_f$ is contractible

$A \to (y : B) \times T_f, \quad a \longmapsto (f\ a, [1_F \mapsto a])$ is a cofibration

Spheres

$$\frac{\Gamma \vdash}{\Gamma \vdash \mathsf{S}^1} \qquad \frac{\Gamma \vdash}{\Gamma \vdash \mathsf{base} : \mathsf{S}^1} \qquad \frac{\Gamma \vdash r : \mathbb{I}}{\Gamma \vdash \mathsf{loop}(r) : \mathsf{S}^1}$$

with the equalities $\mathsf{loop}(0) = \mathsf{loop}(1) = \mathsf{base}$.

Spheres

$$\frac{\Gamma, \varphi, i : \mathbb{I} \vdash u : \mathsf{S}^1 \qquad \Gamma \vdash u_0 : \mathsf{S}^1[\varphi \mapsto u(i0)]}{\Gamma \vdash \mathsf{hcomp}^i\ [\varphi \mapsto u]\ u_0 : \mathsf{S}^1}$$

with the equality $\mathsf{hcomp}^i\ [1_{\mathbb{F}} \mapsto u]\ u_0 = u(i1)$.

Spheres

Given $x : \mathsf{S}^1 \vdash A$ and $a : A(x/\mathsf{base})$ and $l : \mathsf{Path}^i\ A(x/\mathsf{loop}(i))\ a\ a$ we define

$g : (x : \mathsf{S}^1) \to A$

$$
\begin{aligned}
g\ \mathsf{base} &= a \\
g\ \mathsf{loop}(r) &= l\ r \\
g\ (\mathsf{hcomp}^i\ [\varphi \mapsto u]\ u_0) &= \mathsf{comp}^i\ A(x/v)\ [\varphi \mapsto g\ u]\ (g\ u_0)
\end{aligned}
$$

where

$$
\begin{aligned}
v &= \mathsf{fill}^i\ \mathsf{S}^1\ [\varphi \mapsto u]\ u_0 \\
&= \mathsf{hcomp}^j\ [\varphi \mapsto u(i/i \wedge j), (i = 0) \mapsto u_0]\ u_0.
\end{aligned}
$$