Grade and Linear Equations

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Introduction

We present a proof of a result of Sharpe [2], which gives a sufficient condition for a system of linear equations with coefficiens in a commutative ring to have a solution in term of the true grade of Hochster [1]

1 Statement of the result

Let R be a(n arbitrary) commutative ring. We consider $n+1$ column vectors $U_1, \ldots, U_n, V$ in R^m and the linear system

$$x_1 U_1 + \ldots + x_n U_n = V \tag{1}$$

We let A be the $m \times n$ matrix $U_1 \ldots U_n$ and B be the $m \times (n+1)$ matrix $AV = U_1 \ldots U_n V$. If M is a matrix we write $\Delta_l(M)$ the ideal generated by all $l \times l$ minors of M.

Theorem 1.1 *If we have $Gr(\Delta_n(A)) \geqslant 2$ and $\Delta_{n+1}(B) = 0$ then the system (1) has exactly one solution.*

For $R = \mathbb{Z}$ the system

$$2x = 3$$

$$4x = 6$$

satisfies the condition $\Delta_{n+1}(B) = 0$ but the grade of $\langle 2, 4 \rangle$ is only 1.

2 Proof of the statement

Let δ be a $n \times n$ minor of A. Since $\Delta_{n+1}(B) = 0$ we can find a solution $\lambda_1, \ldots, \lambda_n$ of the system (we precise this point in an appendix)

$$\lambda_1 U_1 + \ldots + \lambda_n U_n = \delta V \tag{2}$$

If δ' is another $n \times n$ minor, we have a solution

$$\lambda'_1 U_1 + \ldots + \lambda'_n U_n = \delta' V$$

We then have

$$\delta' \lambda_1 U_1 + \ldots + \delta' \lambda_n U_n = \delta \lambda'_1 U_1 + \ldots + \delta \lambda'_n U_n$$

and hence, for each i

$$\delta' \lambda_i U_1 \wedge \ldots \wedge U_n = \delta \lambda'_i U_1 \wedge \ldots \wedge U_n$$

and since $\Delta_n(A)$ is regular, this implies $\delta'\lambda_i = \delta\lambda'_i$.

Since $\Delta_n(A)$ is of grade $\geqslant 2$ this implies that there exists an unique x_i such that $\lambda_i = x_i\delta$ for all minors δ and all corresponding solution of the system (2). We then have, for all δ

$$\delta V = \lambda_1 U_1 + \ldots + \lambda_n U_n = \delta(x_1 U_1 + \ldots + x_n U_n)$$

and since $\Delta_n(A)$ is regular, this implies

$$V = x_1 U_1 + \ldots + x_n U_n$$

We then prove uniqueness of the solution. For this we need only that $\Delta_n(A)$ is regular. If we have also

$$V = y_1 U_1 + \ldots + y_n U_n$$

we then have

$$x_i U_1 \wedge \ldots \wedge U_n = y_i U_1 \wedge \ldots \wedge U_n$$

for each i, and hence $x_i = y_i$ since $\Delta_n(A)$ is regular. This shows that there exists at most one solution as soon as $\Delta_n(A)$ is regular.

3 Appendix: some exterior algebra

We let $e_1, \ldots, e_m$ be the canonical basis of R^m. If X is a column vector of R^m we write $X = X^1 e_1 + \ldots + X^n e_n$ and if I is a finite sequence $i_1, \ldots, i_k$ we write $X(I) = X^{i_1} e_{i_1} + \ldots + X^{i_k} e_{i_k}$ and $e_I = e_{i_1} \wedge \ldots \wedge e_{i_k}$. A $n \times n$ minor δ of the matrix A is determined by a strictly increasing sequence $I = i_1 < \ldots < i_n$. We have $\delta e_I = U_1(I) \wedge \ldots \wedge U_n(I)$ and

$$\lambda_1(I) e_I = V(I) \wedge U_2(I) \wedge \ldots \wedge U_n(I) \qquad \ldots \qquad \lambda_n(I) e_I = U_1(I) \wedge U_2(I) \wedge \ldots \wedge V(I)$$

We want to check that $V^i = \lambda_1(I) U_1^i + \ldots + \lambda_n(I) U_n^i$ for all i different from $i_1, \ldots, i_n$. For this we use that $\Delta_{n+1}(B) = 0$ and hence

$$(V(i) + V(I)) \wedge (U_1(i) + U_1(I)) \wedge \ldots \wedge (U_n(i) + U_n(I)) = 0$$

If we develop this equality we get

$$0 = V(i) \wedge U_1(I) \wedge \ldots \wedge U_n(I) + V(I) \wedge U_1(i) \wedge \ldots \wedge U_n(I) + \ldots + V(I) \wedge U_1(I) \wedge \ldots \wedge U_n(i)$$

which can be rewritten to

$$\delta V^i e_{i,I} - \lambda_1(I) U_1^i e_{i,I} - \ldots - \lambda_n(I) U_n^i e_{i,I} = 0$$

and hence

$$\delta V^i = \lambda_1(I) U_1^i + \ldots + \lambda_n(I) U_n^i$$

as desired.

References

[1] D.G. Northcott. *Finite Free Resolutions.* Cambridge Tracts in Mathematics, 1976.

[2] D.W. Sharpe Grade and the Theory of Linear Equations. Linear Algebra and its Applications, Vol. 18, Issue 1, 1977, p. 25-32.