Internal version of the uniform Kan filling condition

Introduction

We present a notion of fibration of cubical sets. This is formulated in term of the notion of *partial element*, which has a natural semantics in a presheaf model. We define a partial element to be *connected* if it can be extended to a total element. (The justification of this terminology is that this would generalize in the present framework the notion of two points being connected by a path.) To be fibrant can then be defined internally as the fact that if a partial path is connected at 0 then it also is connected at 1.

Cubical sets

Base category

Let $\mathcal{C}$ the following category. The objects are finite sets $I, J, \ldots$. A morphism $\mathsf{Hom}(J, I)$ is a map $I \to \mathsf{dM}(J)$ where $\mathsf{dM}(J)$ is the free de Morgan algebra on J. We write $f : J \to I$ for $f \in \mathsf{Hom}(J, I)$. We write $1_I : I \to I$ the identity map of I. If $f : J \to I$ and $g : K \to J$ we write $fg : K \to I$ their composition.

A presheaf X on $\mathcal{C}$ can be described as a family of sets $X(I)$ together with restriction maps $X(I) \to X(J),\ u \longmapsto uf$ for $f : J \to I$, satisfying $u1_I = u$ and $(uf)g = u(fg)$. (This notation for the restriction map is motivated by the canonical isomorphism between $X(I)$ and $I \to X$, where I is the presheaf represented by I.)

The presheaf $\mathbb{I}$ is defined by $\mathbb{I}(J) = \mathsf{dM}(J)$. We can think of an element of $\mathsf{dM}(I)$ as a lattice formula ψ on atoms $i, 1-i$ for i in I. If $f : J \to I$, and ψ is in $\mathsf{dM}(I)$, then the restriction operation ψf can be thought as a substitution: we replace the atom i by $f(i)$ in the formula ψ.

A *sieve* on I is a collection L of maps of codomain I such that $fg \in L$ whenever $f : J \to I$ is in L and $g : K \to J$. If L is a sieve on I then we let Lf be the sieve on J of all maps $g : K \to J$ such that fg is in L. We define in this way a presheaf Ω, taking $\Omega(I)$ be the set of sieves on I. Each element ψ in $\mathsf{dM}(I)$ determines the sieve of $f : J \to I$ such that $\psi f = 1$. This defines a natural transformation $\mathbb{I} \to \Omega$.

In each $\mathsf{dM}(I)$ there is a greatest element < 1, the disjunction of all i and $1-i$ for i in I. The sieve associated to this element is the boundary of I.

A *cubical set* is a presheaf on C.

Partial elements and connectedness

Ω is internally the set of truth-values. To each element p in Ω we can associate a subobject $[p]$ of the constant cubical set 1. A *partial* element of a cubical set X can be defined as a pair p, u where p is in Ω and u is a map $[p] \to X$. The element p is called the *extent* of the partial element p, u. (Alternatively, a partial element of X can be defined as a subsingleton of X.)

We think of $\mathbb{I}$ as a formal representation of the real interval $[0, 1]$. It has a structure of a de Morgan algebra. The map $\mathbb{I} \to \Omega$ can be described internally as the map $i \longmapsto i = 1$, which associates to an element i the truth-value $i = 1$. Using the disjunction property of free de Morgan algebra (which results from the conjunctive normal form representation of formulae), we see that we have

$$(i \wedge j = 1) = (i = 1) \wedge (j = 1) \qquad (i \vee j = 1) = (i = 1) \vee (j = 1)$$

and the map $\mathbb{I} \to \Omega,\ i \mapsto i = 1$ is an injective lattice map. We identify ψ with the truth-value $\psi = 1$.

Lemma 0.1 *If we have $\psi : \mathbb{I} \to \mathbb{I}$ we can define $\forall i.\psi(i)$ in $\mathbb{I}$ such that*

$$(1 = \forall i.\psi(i)) = \forall i : \mathbb{I}.(1 = \psi(i))$$

Proof. This corresponds to a map $\mathsf{dM}(I, i) \to \mathsf{dM}(I)$ natural in I. We let $(\forall i.\psi(i))(I)$ be the disjunction of the conjunctions not mentionning i in the disjunctive normal form of $\psi(i)$ in $\mathsf{dM}(I, i)$. □

If ψ is an element in $\mathbb{I}$ and u is a partial element of X of extent ψ, we write $X[\psi \mapsto u]$ the subset of X of element in X that extends u. An element of this set is a witness that u is *connected*.

Fibrations

A family of sets $A\rho$ for ρ in Γ is a *fibration* iff we have an operation which takes as argument a path γ in $\Gamma^{\mathbb{I}}$, an element ψ in $\mathbb{I}$, a partial section $u(i)$ of $A\gamma(i)$ of extent ψ, an element in $A\gamma(0)[\psi \mapsto u(0)]$, and produces an element in $A\gamma(1)[\psi \mapsto u(1)]$. This operation is thus an element of

$$(\gamma : \Gamma^{\mathbb{I}})\ (\psi : \mathbb{I})\ (u : ((i : \mathbb{I}) \to A\gamma(i))^{[\psi]})\ \to A\gamma(0)[\psi \mapsto u(0)] \to A\gamma(1)[\psi \mapsto u(1)]$$

Lemma 0.2 *If we have a composition operation*

$$\mathsf{comp} : (\gamma : \Gamma^{\mathbb{I}})\ (\psi : \mathbb{I})\ (u : ((i : \mathbb{I}) \to A\gamma(i))^{[\psi]})\ \to A\gamma(0)[\psi \mapsto u(0)] \to A\gamma(1)[\psi \mapsto u(1)]$$

then we have a filling operation: given γ in $\Gamma^{\mathbb{I}}$, ψ in $: \mathbb{I}$, u in $((i : \mathbb{I}) \to A\gamma(i))^{[\psi]}$ and a_0 in $A\gamma(0)[\psi \mapsto u(0)]$, we can find a section in

$$(i : \mathbb{I}) \to A\gamma(i)[\psi \mapsto u(i), (1 - i) \mapsto a_0]$$

Proof. We define

$$\mathsf{fill}\ \gamma\ \psi\ u\ a_0\ i = \mathsf{comp}\ \gamma\ (\psi \vee (1 - i))\ v\ a_0$$

where v is the partial element in $((j : \mathbb{I}) \to A\gamma(j))^{[\psi \vee (1-i)]}$ which is equal to $\lambda j.u(i \wedge j)$ on $[\psi]$ and $\lambda j.a_0$ on $[i = 0]$. This is well-defined since $u(i \wedge j) = u(0) = a_0$ on $[\psi] \cap [i = 0]$. □

Taking as a special case 0 for ψ, we see that if $\Gamma \vdash A$ is a fibration then we have the *path lifting property*: we have an operation taking as argument γ in $\Gamma^{\mathbb{I}}$ and a_0 in $A\gamma(0)$ and producing a section $a(i) : A\gamma(i)$ such that $a(0) = a_0$.

We say that a cubical set A is *fibrant* if it defines a fibration over the constant cubical set 1. Explicitely, it means that we have an operation taking as argument an element ψ in $\mathbb{I}$, a partial path u in $A^{\mathbb{I}}$ of extent ψ and producing a map $A[\psi \mapsto u(0)] \to A[\psi \mapsto u(1)]$. The previous Lemma shows that we then have another operation producing an element in

$$(a_0 : A[\psi \mapsto u(0)]) \to (i : \mathbb{I}) \to A[\psi \mapsto u(i), (1 - i) \mapsto a_0]$$

Model of type theory

Proposition 0.3 *If we have fibrations $\Gamma \vdash A$ and $\Gamma, x : A \vdash B$ then $\Gamma \vdash (x : A) \to B$ is a fibration.*

Proof. Let us write $C = (x : A) \to B$. Given γ in $\Gamma^{\mathbb{I}}$ and ψ in $\mathbb{I}$ and μ in $((i : \mathbb{I}) \to C\gamma(i))^{[\psi]}$ and λ_0 in $C\gamma(0)[\psi \mapsto \mu(0)]$, we define $\lambda_1 : C\gamma(1)[\psi \mapsto \mu(1)]$ by taking

$$\lambda_1\ a_1 = \mathsf{comp}\ (\lambda i.(\gamma(i), a(i)))\ \psi\ (\psi \mapsto \mu(i)\ a(i))\ (\lambda_0\ a(0))$$

where $a(i) : A\gamma(i)$ satisfying $a(1) = a_1$ is defined using the path lifting property of $\Gamma \vdash A$. □

We have a similar definition for the dependent sum $\Gamma \vdash (x : A, B)$.

If $\Gamma \vdash A$ and we have two sections $\Gamma \vdash u : A$ and $\Gamma \vdash v : A$ then we define $\Gamma \vdash \mathsf{Id}\ A\ u\ v$ by taking $(\mathsf{Id}\ A\ u\ v)\rho$ to be the subset of path p in $(A\rho)^{\mathbb{I}}$ such that $p(0) = u\rho$ and $p(1) = v\rho$.

Proposition 0.4 *If $\Gamma \vdash A$ is fibrant then so is $\Gamma \vdash \mathsf{Id}\ A\ u\ v$.*

Proof. We suppose given γ in $\Gamma^{\mathbb{I}}$ and p_0 in $(\mathsf{Id}\ A\ u\ v)\gamma(0)\ \psi$ in $\mathbb{I}$ and a partial section $q(i)$ in $(\mathsf{Id}\ A\ u\ v)\gamma(i)$ of extent ψ such that $q(0) = p_0$ on ψ. This means that we have $q(i,j)$ in $A\gamma(i)$ and $q(i,0) = u\gamma(i)$ and $q(i,1) = v\gamma(i)$. We define then p_1 in $(\mathsf{Id}\ A\ u\ v)\gamma(1)[\psi \mapsto q(1)]$ by $p_1(j) = \mathsf{comp}_A\ \gamma\ (\psi \vee j \vee (1-j))\ r\ p_0(j)$ where $r(i)$ is a partial section in $A\gamma(i)$ of extent $\psi \vee j \vee (1-j)$ defined as $r(i) = q(i,j)$ on $[\psi]$ and $r(i) = u\gamma(i)$ on $[1-j]$ and $r(i) = v\gamma(i)$ on $[j]$. □

Isomorphisms

If T and A are two cubical sets, an isomorphism $T \to A$ consists in two maps $f : T \to A$ and $g : A \to T$ and two sections $s : (x : T) \to \mathsf{Id}\ T\ (g\ (f\ x))\ x$ and $t : (x : A) \to \mathsf{Id}\ A\ (f\ (g\ x))\ x$. So we have a map $s : T \times \mathbb{I} \to T$ such that $s(x,0) = g\ (f\ x)$ and $s(x,1) = x$ and a map $t : A \times \mathbb{I} \to A$ such that $t(x,0) = f\ (g\ x)$ and $t(x,1) = x$.

Lemma 0.5 *If T and A are fibrant, and we have an isomorphism $(f,g,s,t) : T \to A$ then we have an operation taking as argument ψ in $\mathbb{I}$ and a partial element t in T of extent ψ and a in $A[\psi \mapsto f\ t]$ and producing an element in $(x : T, \mathsf{Id}\ A\ a\ (f\ x))[\psi \mapsto (t, 1_a)]$.*

Glueing operation

Lemma 0.6 *We assume given a section $\Gamma \vdash \sigma : T \to A$ where $\Gamma \vdash A$, $\Gamma \vdash T$ are two fibrations. Given γ in $\Gamma^{\mathbb{I}}$ and a partial section $t(i) \in T\gamma(i)$ of extent ψ and t_0 in $T\gamma(0)[\psi \mapsto t(0)]$, we can consider $a_0 = \sigma\gamma(0)\ t_0$ in $A\gamma(0)$ and the partial section $a(i) = \sigma\gamma(i)\ t(i)$ of extent ψ. There is a path connecting $a_1 = \mathsf{comp}_A\ \gamma\ \psi\ a\ a_0$ to $\sigma\gamma(1)\ t_1$ where $t_1 = \mathsf{comp}_T\ \gamma\ \psi\ t\ t_0$ in $T\gamma(1)$. This path is furthermore constant on the extent ψ.*

Proof. By filling in T, we find an extension of the partial section t to a total section $\tilde{t}$ such that $\tilde{t}(1) = t_1$. By filling in A, we find an extension of the partial section a to a total section $\tilde{a}$ such that $\tilde{a}(1) = a_1$. Given i we define the partial section u of extent $\varphi = \psi \vee (1-i) \vee i$ by taking $u(j) = \sigma\gamma(j)\ t(j)$ on ψ and $u(j) = \tilde{a}(j)$ on $i = 0$ and $u(j) = \sigma\gamma(j)\ \tilde{t}(j)$ on $i = 1$. The path joining a_1 to $\sigma\gamma(1)\ t_1$ is then $\lambda i.\mathsf{comp}_A\ \varphi\ u\ a_0$. □

If $\Gamma \vdash \psi : \mathbb{I}$, i.e. we have $\psi : \Gamma \to \mathbb{I}$, we define Γ, ψ to be the subset of elements ρ in Γ such that $\psi(\rho) = 1$.

The rules for the glueing operation are

$$\frac{\Gamma \vdash A \qquad \Gamma, \varphi \vdash T \qquad \Gamma, \varphi \vdash \sigma : \mathsf{Iso}(T, A)}{\Gamma \vdash \mathsf{glue}(A, [\varphi \mapsto (T, \sigma)])} \qquad \frac{}{\Gamma \vdash \mathsf{glue}(A, [1 \mapsto (T, \sigma)]) = T}$$

We write $B = \mathsf{glue}(A, [\varphi \mapsto (T, \sigma)])$ and we explain the composition operation for B.

If ρ in Γ any element of $B\rho$ can be written uniquely on the form $\mathsf{glue}(a, [\varphi\rho \mapsto t])$ with a in $A\rho$, t a partial element of $T\rho$ of extent $\varphi\rho$ such that $\sigma\rho\ t = a$.

We assume given γ in $\Gamma^{\mathbb{I}}$, and element $b_0 = (a_0, [\varphi\gamma(0) \mapsto t_0])$ and a partial section $v(i) = (u(i), [\varphi\gamma(i) \mapsto w(i)])$ of extent ψ. We want to define $b_1 = (a_1, [\varphi\gamma(1) \mapsto t_1])$ in $B\gamma(1)[\psi \mapsto v(1)]$.

We first consider a_0 in $A\gamma(0)$ and $u(i)$ in $A\gamma(i)$ of extent ψ, and such that $a_0 = u(0)$. Since $\Gamma \vdash A$ is a fibration, we get a_1' in $A\gamma(1)$, such that $a_1' = u(1)$ on ψ.

Using Lemma 0.1 we define $\delta = \forall i.\varphi\gamma(i)$ in $\mathbb{I}$. We have $\delta \leqslant \varphi\gamma(1)$. On the extent $\psi \wedge \delta$ we can consider t_0 in $T\gamma(0)$ and the partial section $w(i)$ in $T\gamma(i)$. Since $\Gamma, \varphi \vdash T$ is a fibration we define t_1' in $T\gamma(1)$ of extent δ and such that $t_1' = w(1)$ on $\delta \wedge \psi$. Using Lemma 0.6, we have a path between a_1' and $\sigma\gamma(1)\ t_1'$ of extent δ. Since $A\gamma(1)$ is fibrant we can then find a_1'' in $A\gamma(1)$ such that $a_1'' = a_1'$ on ψ and $a_1'' = \sigma\gamma(1)\ t_1'$ on δ.

Using the fact that $\sigma\gamma(1)$ is an isomorphism and Lemma 0.5, we can extend t'_1 in $T\gamma(1)$ to an element t_1 of extent $\varphi\gamma(1)$ such that $t_1 = w(1)$ on ψ. Using that $A\gamma(1)$ is fibrant, we find a_1 in $A\gamma(1)$ such that $\sigma\gamma(1)\ t_1 = a_1$ on $\varphi\gamma(1)$ and and $a_1 = a''_1 = a'_1$ on ψ. The element $b_1 = \mathsf{glue}(a_1, [\varphi\gamma(1) \mapsto t_1])$ is in $B\gamma(1)[\psi \mapsto v(1)]$ and satisfies $b_1 = t'_1$ on the extent δ.