Cubical Type Theory

Interval and Face lattice

$$r, s \ ::= \ 0 \mid 1 \mid i \mid 1-i \mid r \wedge s \mid r \vee s \qquad \varphi, \psi \ ::= \ 0 \mid 1 \mid (r=0) \mid (r=1) \mid \varphi \wedge \psi \mid \varphi \vee \psi$$

The equality on the inverval $\mathbb{I}$ is the equality in the free bounded distributive lattice on generators $i, 1-i$. The equality in the face lattice $\mathbb{F}$ is the one for the free distributive lattice on formal generators $(i=0), (i=1)$ with the relation $(i=0) \wedge (i=1) = 0$. We have $[(r \vee s) = 1] = (r=1) \vee (s=1)$ and $[(r \wedge s) = 1] = (r=1) \wedge (s=1)$. An irreducible element of this lattice is a *face*, a conjunction of elements $(i=0)$ and $(j=1)$ and any element is a disjunction of irreducible elements (unique up to the absorption law).

Contexts

$$\Delta, \Gamma \ ::= \ () \mid \Gamma, x : A \mid \Gamma, i : \mathbb{I} \mid \Gamma, \varphi$$

Semantics

Let $\mathcal{C}$ the following category. The objects are finite sets $I, J, \ldots$. A morphism $\mathsf{Hom}(J, I)$ is a map $I \to \mathsf{dM}(J)$ where $\mathsf{dM}(J)$ is the free de Morgan algebra on J. The presheaf $\mathbb{I}$ is defined by $\mathbb{I}(J) = \mathsf{dM}(J)$. The presheaf $\mathbb{F}$ is defined by taking $\mathbb{F}(J)$ to be the free distributive lattice generated by formal elements $(j=0),\ (j=1)$ for j in J, with the relations $(j=0) \wedge (j=1) = 0$.

We interpret context as presheaves over the category $\mathcal{C}$. A dependent type $\Gamma \vdash A$ is interpreted as a family of sets $A\rho$ for each I and $\rho \in \Gamma(I)$ together with restriction maps $A\rho \to A\rho f,\ u \longmapsto uf$ for $f : J \to I$, satisfying $u1_I = u$ and $(uf)g = u(fg) \in \Gamma(K)$ if $g : K \to J$. An element $\Gamma \vdash a : A$ is interpreted by a family $a\rho \in A\rho$ for I and $\rho \in \Gamma(I)$, such that $(a\rho)f = a(\rho f) \in A\rho f$ if $f : J \to I$.

If $\Gamma \vdash A$, we interpret $\Gamma, x : A$ as the cubical set defined by taking $(\Gamma, x : A)(I)$ to be the set of element $\rho, x = u$ such that $\rho \in \Gamma(I)$ and $u \in A\rho$. If $f : J \to I$ the restriction map is defined by $(\rho, x = u)f = \rho f, x = uf$.
If $\Gamma, x : A \vdash B$ and $\Gamma \vdash a : A$ we define $\Gamma \vdash B(x = a)$ by taking $B(x=a)\rho$ to be the set $B(\rho, x = a\rho)$.

If $\Gamma \vdash \varphi : \mathbb{F}$ then $\varphi\rho \in \mathbb{F}(I)$ for each $\rho \in \Gamma(I)$. We define $(\Gamma, \varphi)(I)$ to be the set $\rho \in \Gamma(I)$ such that $\varphi\rho = 1$. (In particular $(\Gamma, 0)(I)$ is empty.)

Basic typing rules

$$\frac{\Gamma \vdash A}{\Gamma, x : A \vdash} \qquad \frac{\Gamma \vdash}{\Gamma, i : \mathbb{I} \vdash} \qquad \frac{\Gamma \vdash \varphi : \mathbb{F}}{\Gamma, \varphi \vdash} \qquad \frac{\Gamma \vdash r : \mathbb{I}}{\Gamma \vdash (r=1) : \mathbb{F}} \qquad \frac{\Gamma \vdash r : \mathbb{I}}{\Gamma \vdash (r=0) : \mathbb{F}}$$

$$\frac{\Gamma \vdash}{\Gamma \vdash x : A}\ (x : A \ in\ \Gamma) \qquad \frac{\Gamma \vdash}{\Gamma \vdash i : \mathbb{I}}\ (i : \mathbb{I}\ in\ \Gamma)$$

$$\frac{\Gamma, x : A \vdash B}{\Gamma \vdash (x : A) \to B} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x : A.\ t : (x : A) \to B} \qquad \frac{\Gamma \vdash t : (x : A) \to B \qquad \Gamma \vdash u : A}{\Gamma \vdash t\ u : B(u)}$$

The following observation will be useful for defining composition for glueing.

Lemma 0.1 *If* $\Gamma, i : \mathbb{I} \vdash \varphi : \mathbb{F}$ *we can build* $\Gamma \vdash \forall i.\varphi : \mathbb{F}$ *such that, if* $\Gamma \vdash \psi$ *we have* $\psi \leqslant \forall i.\varphi$ *iff* $\psi \leqslant \varphi$

We shall then use the fact that if a judgement is valid in a context $\Gamma, i : \mathbb{I}, \varphi$ then it holds in the context $\Gamma, \forall i.\varphi, i : \mathbb{I}$ as well, and that $\Gamma \vdash (\forall i.\varphi) = 1$ if we have $\Gamma, i : \mathbb{I} \vdash \varphi = 1$.

Sigma types

$$\frac{\Gamma, x : A \vdash B}{\Gamma \vdash (x : A, B)} \qquad \frac{\Gamma \vdash a : A \quad \Gamma \vdash b : B(a)}{\Gamma \vdash (a, b) : (x : A, B)} \qquad \frac{\Gamma \vdash z : (x : A, B)}{\Gamma \vdash z.1 : A} \qquad \frac{\Gamma \vdash z : (x : A, B)}{\Gamma \vdash z.2 : B(z.1)}$$

Path types

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash a_0 : A \qquad \Gamma \vdash a_1 : A}{\Gamma \vdash \mathsf{Path}\ a_0\ a_1} \qquad \frac{\Gamma \vdash A \qquad \Gamma, i : \mathbb{I} \vdash t : A}{\Gamma \vdash \langle i \rangle t : \mathsf{Path}\ A\ t(i0)\ t(i1)}$$

$$\frac{\Gamma \vdash t : \mathsf{Path}\ A\ a_0\ a_1}{\Gamma \vdash t\ 0 = a_0 : A} \qquad \frac{\Gamma \vdash t : \mathsf{Path}\ A\ a_0\ a_1}{\Gamma \vdash t\ 1 = a_1 : A}$$

We define $1_a : \mathsf{Path}\ A\ a\ a$ as $1_a = \langle i \rangle a$.

With these rules we also can justify function extensionality

$$\frac{\Gamma \vdash t : (x : A) \to B \qquad \Gamma \vdash u : (x : A) \to B \qquad \Gamma \vdash p : (x : A) \to \mathsf{Path}\ B\ (t\ x)\ (u\ x)}{\Gamma \vdash \langle i \rangle \lambda x : A.\ p\ x\ i : \mathsf{Path}\ ((x : A) \to B)\ t\ u}$$

We also can justify the fact that any element in $(x : A, \mathsf{Path}\ A\ a\ x)$ is equal to $(a, 1_a)$

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : A \qquad \Gamma \vdash p : \mathsf{Path}\ A\ a\ b}{\Gamma \vdash \langle i \rangle (p\ i, \langle j \rangle p\ (i \wedge j)) : \mathsf{Path}\ (x : A, \mathsf{Path}\ A\ a\ x)\ (a, 1_a)\ (b, p)}$$

For justifying the transitivity of equality, we need A to have *composition operations*.

Partial elements

$$\frac{\Gamma \vdash \varphi \leqslant \psi \qquad \Gamma, \psi \vdash A}{\Gamma, \varphi \vdash A} \qquad \frac{\Gamma \vdash \varphi \leqslant \psi \qquad \Gamma, \psi \vdash u : A}{\Gamma, \varphi \vdash u : A}$$

$$\frac{\Gamma, \psi_1 \vdash A_1 \quad \dots \quad \Gamma, \psi_k \vdash A_k \qquad \Gamma, \psi_i \wedge \psi_j \vdash A_i = A_j}{\Gamma, \psi_1 \vee \cdots \vee \psi_k \vdash \psi_1 A_1 \vee \cdots \vee \psi_k A_k}$$

$$\frac{\Gamma, \psi_1 \vdash u_1 : A_1 \quad \dots \quad \Gamma, \psi_k \vdash u_k : A_k \qquad \Gamma, \psi_i \wedge \psi_j \vdash A_i = A_j \qquad \Gamma, \psi_i \wedge \psi_j \vdash u_i = u_j : A_i}{\Gamma, \psi_1 \vee \cdots \vee \psi_k \vdash \psi_1 u_1 \vee \cdots \vee \psi_k u_k : \psi_1 A_1 \vee \cdots \vee \psi_k A_k}$$

We can have $k = 0$ in which case we get a dummy element of type A in the context $\Gamma, 0$.
We also have $\psi_1 u_1 \vee \cdots \vee \psi_k u_k = u_i : A$ if $\psi_i = 1$ and $\psi_1 \vee \cdots \vee \psi_k \vdash u = \psi_1 u \vee \cdots \vee \psi_k u$

If $\Gamma, \varphi \vdash u : A$ then $\Gamma \vdash a : A[\varphi \mapsto u]$ is an abreviation for $\Gamma \vdash a : A$ and $\Gamma, \varphi \vdash a = u : A$. In this case, we see this element a as a witness that the partial element u (which is defined at least on the extent φ) is *connected*.

For instance if $\Gamma, i : \mathbb{I} \vdash A$ and $\Gamma, i : \mathbb{I}, \varphi \vdash u : A$ where $\varphi = (i = 0) \vee (i = 1)$ then the element u is determined by two element $\Gamma \vdash a_0 : A(i0)$ and $\Gamma \vdash a_1 : A(i1)$ and an element $\Gamma, i : \mathbb{I} \vdash a : A[\varphi \mapsto u]$ gives a path connecting a_0 and a_1.

We may write $\Gamma \vdash a : A[\psi_1 \mapsto u_1, \dots, \psi_k \mapsto u_k]$ for $\Gamma \vdash a : A[\psi_1 \vee \cdots \vee \psi_k \mapsto \psi_1 u_1 \vee \cdots \vee \psi_k u_k]$. This means that $\Gamma \vdash a : A$ and $\Gamma, \psi_i \vdash a = u_i : A$ for $i = 1, \dots, k$.

Composition operations

$$\frac{\Gamma \vdash \varphi \qquad \Gamma, i : \mathbb{I} \vdash A \qquad \Gamma, i : \mathbb{I}, \varphi \vdash u : A \qquad \Gamma \vdash a_0 : A(i0)[\varphi \mapsto u(i0)]}{\Gamma \vdash \mathsf{comp}^i\ A\ [\varphi \mapsto u]\ a_0 : A(i1)[\varphi \mapsto u(i1)]}$$

Kan filling operation

We recover Kan filling operation

$$\Gamma, i : \mathbb{I} \vdash \mathsf{fill}^i\ A\ [\varphi \mapsto u]\ a_0 = \mathsf{comp}^j\ A(i \wedge j)\ [\varphi \mapsto u(i \wedge j), (i = 0) \mapsto a_0]\ a_0 : A$$

The element $i : \mathbb{I} \vdash v = \mathsf{fill}^i\ A\ [\varphi \mapsto u]\ a_0 : A$ satisfies

$$\Gamma \vdash v(i0) = a_0 : A(i0) \qquad \Gamma \vdash v(i1) = \mathsf{comp}^i\ A\ [\varphi \mapsto u]\ a_0 : A(i1) \qquad \Gamma, \varphi \vdash v(i1) = u : A$$

Recursive definition of composition

The operation $\mathsf{comp}^i\ A\ [\varphi \mapsto u]\ a_0$ is defined by induction on A.

Product type

In the case of a product type $i : \mathbb{I} \vdash (x : A) \to B = C$, we have $i : \mathbb{I}, \varphi \vdash \mu : C$ with and $\vdash \lambda_0 : C(i0)[\varphi \mapsto \mu(i0)]$ and we define, for $\vdash u_1 : A(i1)$

$$(\mathsf{comp}^i\ C\ [\varphi \mapsto \mu]\ \lambda_0)\ u_1 = \mathsf{comp}^i\ B(x = v)\ [\varphi \mapsto \mu\ v]\ (\lambda_0\ u_0)$$

where $i : \mathbb{I} \vdash v = \mathsf{fill}^i\ A(1 - i)\ []\ u_1 : A$ and $u_0 = v(i0) : A(i0)$.

Path type

In the case of path type $i : \mathbb{I} \vdash \mathsf{Path}\ A\ u\ v = C$ we have $i : \mathbb{I}, \varphi \vdash \mu : C$ and $\vdash p_0 : C(i0)[\varphi \mapsto \mu(i0)]$. We define

$$\mathsf{comp}^i\ C\ [\varphi \mapsto \mu]\ p_0 = \langle j \rangle \mathsf{comp}^i\ A\ [\varphi \mapsto \mu\ j, (j = 0) \mapsto u, (j = 1) \mapsto v]\ (p_0\ j)$$

Sum type

In the case of a sigma type $i : \mathbb{I} \vdash (x : A, B) = C$ given $i : \mathbb{I}, \varphi \vdash w : C$ and $\vdash w_0 : C(i0)[\varphi \mapsto w(i0)]$ we define

$$\mathsf{comp}^i\ C\ [\varphi \mapsto w]\ w_0 = (\mathsf{comp}^i\ A\ [\varphi \mapsto w.1]\ w_0.1, \mathsf{comp}^i\ B(x = a)\ [\varphi \mapsto w.2]\ w_0.2)$$

where $i : \mathbb{I} \vdash a = \mathsf{fill}^i\ A\ [\varphi \mapsto w.1]\ w_0.1 : A$.

Example

If $i : \mathbb{I} \vdash A$, composition for $\varphi = 0$ corresponds to a transport function $A(i0) \to A(i1)$.

If I is an object of $\mathcal{C}$ the lattice $\mathbb{F}(I)$ has a greatest element < 1 which is the disjunction of all $(i = 0) \vee (i = 1)$ for i in I. This element can be called the *boundary* of I. Composition w.r.t. this boundary gives the usual operation of Kan composition, which witnesses the existence of a lid for any open box.

Two derived operations

The first derived operation states that the image of a composition is path equal to the composition of the respective images.

Lemma 0.2 *If we have* $\Delta, i : \mathbb{I} \vdash \sigma : T \to A$, $\Delta \vdash \psi$ *and* $\Delta, \psi, i : \mathbb{I} \vdash t : T$ *with* $\Delta \vdash t_0 : T(i0)[\psi \mapsto t(i0)]$ *then we can build*

$$\Delta \vdash \mathsf{pres}(\sigma, [\psi \mapsto t], t_0) : \mathsf{Path}\ A(i1)\ (\mathsf{comp}^i\ A\ [\psi \mapsto a]\ a_0)\ \sigma(i1)\ (\mathsf{comp}^i\ T\ [\psi \mapsto t]\ t_0)$$

where $\Delta \vdash a_0 = \sigma(i0)\ t_0 : A(i0)$ *and* $\Delta, i : \mathbb{I}, \psi \vdash a = \sigma\ t : A$. *Furthermore, we have*

$$\Delta, \psi \vdash \mathsf{pres}(\sigma, [\psi \mapsto t], t_0) = \langle j \rangle a$$

We define the type of isomorphisms. Given $\Gamma \vdash f : A \to B$ and $\Gamma \vdash g : B \to A$ we have

$$\frac{\Gamma \vdash u : (y : B) \to \mathsf{Path}\ B\ (f\ (g\ y))\ y \qquad \Gamma \vdash v : (x : A) \to \mathsf{Path}\ A\ (g\ (f\ x))\ x}{\Gamma \vdash (f, g, u, v) : \mathsf{Iso}(A, B)}$$

The second operation corresponds to the fact that any isomorphism defines an equivalence.

Lemma 0.3 *We have an operation*

$$\frac{\Delta \vdash \sigma : \mathsf{Iso}(T, A) \qquad \Delta, \delta \vdash t : T \qquad \Delta \vdash a : A[\delta \mapsto \sigma\ t]}{\Delta \vdash \mathsf{equiv}(\sigma, [\delta \mapsto t], a) : (x : T, \mathsf{Path}\ A\ a\ (\sigma\ x))[\delta \mapsto (t, \langle j \rangle a)]}$$

Glueing

$$\frac{\Gamma \vdash A \qquad \Gamma, \varphi \vdash T \qquad \Gamma, \varphi \vdash \sigma : \mathsf{Iso}(T, A)}{\Gamma \vdash \mathsf{glue}(A, [\varphi \mapsto (T, \sigma)]) \qquad \Gamma, \varphi \vdash \mathsf{glue}(A, [\varphi \mapsto (T, \sigma)]) = T}$$

$$\frac{\Gamma, \varphi \vdash \sigma : \mathsf{Iso}(T, A) \qquad \Gamma, \varphi \vdash t : T \qquad \Gamma \vdash a : A[\varphi \mapsto \sigma t]}{\Gamma \vdash \mathsf{glue}(a, [\varphi \mapsto t]) : \mathsf{glue}(A, [\varphi \mapsto (T, \sigma)])[\varphi \mapsto t]}$$

If $\sigma = (f, g, u, v)$ we write σa for $f\ a$.

Composition for glueing

Assume $\Gamma, i : \mathbb{I} \vdash A$ and $\Gamma, i : \mathbb{I} \vdash \varphi$ and $\Gamma, i : \mathbb{I}, \varphi \vdash \sigma : \mathsf{Iso}(T, A)$. We write $B = \mathsf{glue}(A, [\varphi \mapsto (T, \sigma)])$. Assume also $\Gamma \vdash \psi$ and $\Gamma, i : \mathbb{I}, \psi \vdash b = \mathsf{glue}(a, [\varphi \mapsto t]) : B$ and $\Gamma \vdash b_0 = \mathsf{glue}(a_0, [\varphi(i0) \mapsto t_0]) : B(i0)[\psi \mapsto b(i0)]$.

The goal is to build $\Gamma \vdash b_1 : B(i1)[\psi \mapsto b(i1)]$. Furthermore, we should have $b_1 = \mathsf{comp}^i\ T\ [\psi \mapsto t]\ t_0$ if $\Gamma, i : \mathbb{I} \vdash \varphi = 1$.

We have $\Gamma, \psi \vdash a(i0) = a_0 : A(i0)$ and $\Gamma, \psi \wedge \varphi(i0) \vdash t(i0) = t_0 : T(i0)$. Furthermore $\Gamma, \varphi(i0) \vdash a_0 = \sigma(i0) t_0 : A(i0)$ and $\Gamma, i : \mathbb{I}, \varphi \wedge \psi \vdash a = \sigma t : A$.

We define $a_1' = \mathsf{comp}^i\ A\ [\psi \mapsto a]\ a_0$ so that $\Gamma \vdash a_1' : A(i1)$ and $\Gamma, \psi \vdash a_1' = a(i1) : A(i1)$.

Take $\delta = \forall i.\varphi$. We have $\Gamma, \delta, \psi, i : \mathbb{I} \vdash a = \sigma\ t$ and $\Gamma, \delta \vdash a_0 = \sigma(i0)\ t_0$. Hence, using Lemma 0.2

$$\Gamma, \delta \vdash \omega = \mathsf{pres}\ \sigma\ [\psi \mapsto t]\ t_0 : \mathsf{Path}\ A(i1)\ a_1'\ (\sigma(i1)\ t_1')$$

where $t_1' = \mathsf{comp}^i\ T\ [\psi \mapsto t]\ t_0$. We can then define $a_1'' = \mathsf{comp}^j\ A(i1)\ [\delta \mapsto \omega\ j, \psi \mapsto a(i1)]\ a_1'$ so that $\Gamma \vdash a_1'' : A(i1)$ and $\Gamma, \psi \vdash a_1'' = a(i1) : A(i1)$ and $\Gamma, \delta \vdash a_1'' = \sigma(i1)\ t_1' : A(i1)$.

We have $\Gamma, \varphi(i1) \vdash \sigma(i1) : T(i1) \to A(i1)$ and $\Gamma \vdash a_1'' : A(i1)$ and $\Gamma, \delta \vdash a_1'' = \sigma(i1)\ t_1'$ and $\Gamma, \psi \wedge \varphi(i1) \vdash a_1'' = a(i1) = \sigma(i1)\ t(i1)$. Using Lemma 0.3 we get

$$t_1 = \mathsf{equiv}(\sigma(i1), [\delta \mapsto t_1', \psi \mapsto t(i1)], a_1'').1 \qquad \alpha = \mathsf{equiv}(\sigma(i1), [\delta \mapsto t_1', \psi \mapsto t(i1)], a_1'').2$$

so that $\Gamma, \varphi(i1) \vdash t_1 : T(i1)$ and $\Gamma, \varphi(i1) \vdash \alpha : \mathsf{Path}\ A(i1)\ a_1''\ (\sigma(i1)\ t_1)$. We then define

$$a_1 = \mathsf{comp}^j\ A(i1)\ [\varphi(i1) \mapsto \alpha\ j, \psi \mapsto a(i1)]\ a_1'' \qquad b_1 = (a_1, [\varphi(i1) \mapsto t_1])$$

We have $\Gamma \vdash b_1 : B(i1)[\psi \mapsto b(i1)]$ as required and, if $\Gamma, i : \mathbb{I} \vdash \varphi = 1$ we have $b_1 = \mathsf{comp}^i\ T\ [\psi \mapsto t]\ t_0$.

Identity types

We explain how to define an identity type with the required computation rule, following an idea due to Andrew Swan.

We define a new type $\mathsf{Id}\ A\ a_0\ a_1$ with the introduction rule

$$\frac{\Gamma, \varphi \vdash a : A \qquad \Gamma \vdash \omega : \mathsf{Path}\ A\ a_0\ a_1[\varphi \mapsto \langle i \rangle a]}{\Gamma \vdash (\omega, [\varphi \mapsto a]) : \mathsf{Id}\ A\ a_0\ a_1}$$

We define $(\omega, [\varphi \mapsto a])\ i = \omega\ i : A$. Given $\Gamma \vdash \alpha : \mathsf{Id}\ A\ a\ x$ we define $\Gamma, i : \mathbb{I} \vdash \alpha^*(i) : \mathsf{Id}\ A\ a\ (\alpha\ i)$

$$\alpha^*(i) = (\langle j \rangle \omega(i \wedge j), [\varphi \vee (i = 0) \mapsto a])$$

This is well defined since $\Gamma, i : \mathbb{I}, (i = 0) \vdash \langle j \rangle \omega(i \wedge j) = \langle j \rangle a$ and $\Gamma, i : \mathbb{I}, \varphi \vdash \langle j \rangle \omega(i \wedge j) = \langle j \rangle a$. We can now define $\mathsf{r}(a) = (\langle j \rangle a, [1 \mapsto a]) : \mathsf{Id}\ A\ a\ a$.

If we have $\Gamma, x : A, \alpha : \mathsf{Id}\ A\ a\ x \vdash C$ and $\Gamma \vdash b : A$ and $\Gamma \vdash \beta : \mathsf{Id}\ A\ a\ b$ and $\Gamma \vdash d : C[a, \mathsf{r}(a)]$ we take, for $\beta = (\omega, [\varphi \mapsto a])$

$$J\ b\ \beta\ d = \mathsf{comp}^i\ C[\beta\ i, \beta^*(i)]\ d\ [\varphi \mapsto d] : C[b, \beta]$$

and we have $J\ a\ \mathsf{r}(a)\ d = d$ as desired.

If $i : \mathbb{I} \vdash \mathsf{Id}\ A\ a\ b$ and $p_0 = (\omega_0, [\psi_0 \mapsto a_0]) : \mathsf{Id}\ A(i0)\ a(i0)\ b(i0)$ and $\varphi, i : \mathbb{I} \vdash q = (\omega, [\psi \mapsto c]) : \mathsf{Id}\ A\ a\ b$ such that $\varphi \vdash q(i0) = p_0$ we define $\mathsf{comp}^i\ (\mathsf{Id}\ A\ a\ b)\ [\varphi \mapsto q]\ p_0$ to be $(\gamma, [\varphi \wedge \psi(i1) \mapsto c(i1)])$ where

$$\gamma = \langle j \rangle \mathsf{comp}^i\ A\ (p_0\ j)\ [\varphi \mapsto q\ j, (j = 0) \mapsto a, (j = 1) \mapsto b]$$

Factorization

The same idea of Andrew Swan can be used to factorize a map

$$\frac{\Gamma \vdash f : A \to B}{\Gamma \vdash \mathsf{G}(f)} \qquad \frac{\Gamma \vdash f : A \to B \quad \Gamma, \varphi \vdash a : A \qquad \Gamma \vdash b : B[\varphi \mapsto f\ a]}{\Gamma \vdash (b, [\varphi \mapsto a]) : \mathsf{G}(f)}$$

We define $p_G : \mathsf{G}(f) \to B$ by $p_G(b, [\varphi \mapsto a]) = b$ and $\mathsf{c}(a) = (f\ a, [1 \mapsto a])$ and we have a factorization of the map $f = p_G \circ \mathsf{c}$.

The composition for $G(f)$ is defined by

$$\mathsf{comp}^i\ G(f)\ [\varphi \mapsto (b, [\psi \mapsto a])]\ (b_0, [\psi_0 \mapsto a_0]) = (\mathsf{comp}^i\ B\ [\varphi \mapsto b]\ b_0, [\varphi \wedge \psi(i1) \mapsto a(i1)])$$

Here is one application of the type $G(f)$. Suppose that we have a dependent type $D(v)\ (v : B)$ with a section $g(v) : C(v)\ (v : B)$ and $h(a) : C(f\ a)\ (a : A)$ with $\omega(a) : \mathsf{Path}\ C(f\ a)\ g(f\ a)\ h(a)\ (a : A)$. We can define a new section $\tilde{g}(u) : C(p_G\ u)\ (u : \mathsf{G}(f))$ such that $\tilde{g}(\mathsf{c}\ a) = h(a)\ (a : A)$. The definition is

$$\tilde{g}(b, [\varphi \mapsto a]) = \mathsf{comp}^i\ C(b)\ g(b)\ [\varphi \mapsto \omega(a)\ i]$$

It can be checked that c has the lifting property w.r.t. any trivial fibrations. Also p_G is a trivial fibration, since $G(f)$ can be defined as the sigma type $(b : B, T_f(b))$ where $T_f(b)$ is the contractible type of element $\varphi \mapsto a$ with $\Gamma, \varphi \vdash a : A$ and $\Gamma, \varphi \vdash f\ a = b : B$.

Appendix 1: definition of the derived operations

A definition of equiv

We assume given

$$\Delta \vdash \sigma : \mathsf{Iso}(T, A) \qquad \Delta \vdash \delta \qquad \Delta, \delta \vdash t : T \qquad \Delta \vdash a : A[\delta \mapsto f\ t]$$

where $\sigma = (f, g, u, v)$ and

$$\Delta \vdash u : (y : A) \to \mathsf{Path}\ A\ (f\ (g\ y))\ y \qquad \Delta \vdash v : (x : T) \to \mathsf{Path}\ T\ (g\ (f\ x))\ x$$

We define

$$\Delta, i : \mathbb{I} \vdash \theta = \mathsf{fill}^i\ T\ [\delta \mapsto v\ t\ i]\ (g\ a) : T$$

so that $\theta(i0) = g\ a$ and $\delta, i : \mathbb{I} \vdash \theta = v\ t\ i$. We write $\Delta \vdash t' = \theta(i1) : T$ such that $\Delta, \delta \vdash t' = t$.

We define next

$$\Delta, i : \mathbb{I} \vdash \theta' = \mathsf{comp}^j\ T\ [(i = 0) \mapsto g\ a, (i = 1) \mapsto v\ t'\ (1 - j), \delta \mapsto v\ t\ (i \wedge (1 - j))]\ \theta$$

so that $\theta'(i0) = g\ a$, $\theta'(i1) = g\ (f\ t')$ and $\delta \vdash \theta' = g\ (f\ t)$ and

$$\Delta, i : \mathbb{I} \vdash \theta'' = \mathsf{comp}^j\ A\ [(i = 0) \mapsto u\ a\ j, (i = 1) \mapsto u\ (f\ t')\ j, \delta \mapsto u\ (f\ t)\ j]\ (f\ \theta')$$

and we can take $\mathsf{equiv}(\sigma, [\delta \mapsto t], a) = (t', \langle i \rangle \theta'')$.

A definition of pres

We assume given $\Delta, i : \mathbb{I} \vdash \sigma : T \to A$, $\Delta \vdash \psi$ and $\Delta, \psi, i : \mathbb{I} \vdash t : T$ with $\Delta \vdash t_0 : T(i0)[\psi \mapsto t(i0)]$. We define $\Delta \vdash a_0 = \sigma(i0)\ t_0 : A(i0)$ and $\Delta, i : \mathbb{I}, \psi \vdash a = \sigma\ t : A$, and

$$\Delta, i : \mathbb{I} \vdash u = \mathsf{fill}^i\ A\ [\psi \mapsto a]\ a_0 : A \qquad \Delta, i : \mathbb{I} \vdash v = \mathsf{fill}^i\ T\ [\psi \mapsto t]\ t_0 : T$$

We define then $\mathsf{pres}(\sigma, [\psi \mapsto t], t_0) = \langle j \rangle \mathsf{comp}^i\ A\ [\psi \mapsto \sigma\ t, (j = 0) \mapsto \sigma\ v, (j = 1) \mapsto u]\ a_0$

Appendix 2: self-contained operational semantics

We use $\alpha, \beta, \gamma, \dots$ for the "faces", irreducible elements of the distributive $\mathbb{F}$. If we restrict context as follows

$$\Gamma \quad ::= \quad () \mid \Gamma, x : A \mid \Gamma, i : \mathbb{I} \mid \Gamma, \alpha$$

then any partial element in such a context is equal to a total element. This follows from the fact that faces are irreducible element. To test a judgement in a context Γ, φ is then reduced to test the judgement in the context Γ, α for all irreducible component α of φ.

$$\frac{\Gamma \vdash A}{\Gamma, x : A \vdash} \qquad \frac{\Gamma \vdash}{\Gamma, i : \mathbb{I} \vdash} \qquad \frac{\Gamma \vdash \varphi : \mathbb{F}}{\Gamma, \varphi \vdash} \qquad \frac{\Gamma \vdash}{\Gamma \vdash x : A}\ (x{:}A\ in\ \Gamma) \qquad \frac{\Gamma \vdash}{\Gamma \vdash i : \mathbb{I}}\ (i{:}\mathbb{I}\ in\ \Gamma)$$

$$\frac{\Gamma, x : A \vdash B}{\Gamma \vdash (x : A) \to B} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x : A.\ t : (x : A) \to B} \qquad \frac{\Gamma \vdash t : (x : A) \to B \qquad \Gamma \vdash u : A}{\Gamma \vdash t\ u : B(u)}$$

$$\frac{\Gamma, x : A \vdash B}{\Gamma \vdash (x : A, B)} \qquad \frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B(a)}{\Gamma \vdash (a, b) : (x : A, B)} \qquad \frac{\Gamma \vdash z : (x : A, B)}{\Gamma \vdash z.1 : A} \qquad \frac{\Gamma \vdash z : (x : A, B)}{\Gamma \vdash z.2 : B(z.1)}$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash a_0 : A \qquad \Gamma \vdash a_1 : A}{\Gamma \vdash \mathsf{Path}\ a_0\ a_1} \qquad \frac{\Gamma \vdash A \qquad \Gamma, i : \mathbb{I} \vdash t : A}{\Gamma \vdash \langle i \rangle t : \mathsf{Path}\ A\ t(i0)\ t(i1)}$$

$$\frac{\Gamma \vdash t : \mathsf{Path}\ A\ a_0\ a_1}{\Gamma \vdash t\ 0 = a_0 : A} \qquad \frac{\Gamma \vdash t : \mathsf{Path}\ A\ a_0\ a_1}{\Gamma \vdash t\ 1 = a_1 : A}$$

$$\frac{\Gamma \vdash \varphi \qquad \Gamma, i : \mathbb{I} \vdash A \qquad \Gamma, i : \mathbb{I}, \varphi \vdash u : A \qquad \Gamma \vdash a_0 : A(i0)[\varphi \mapsto u(i0)]}{\Gamma \vdash \mathsf{comp}^i\ A\ [\varphi \mapsto u]\ a_0 : A(i1)[\varphi \mapsto u(i1)]}$$

$$\Gamma, i : \mathbb{I} \vdash \mathsf{fill}^i\ A\ [\varphi \mapsto u]\ a_0 = \mathsf{comp}^j\ A(i \wedge j)\ [\varphi \mapsto u(i \wedge j), (i = 0) \mapsto a_0]\ a_0 : A$$

For $i : \mathbb{I} \vdash C = (x : A) \to B$

$$(\mathsf{comp}^i\ C\ [\varphi \mapsto \mu]\ \lambda_0)\ u_1 = \mathsf{comp}^i\ B(x = v)\ [\varphi \mapsto \mu\ v]\ (\lambda_0\ u_0)$$

where $i : \mathbb{I} \vdash v = \mathsf{fill}^i\ A(1 - i)\ []\ u_1 : A$ and $u_0 = v(i0) : A(i0)$.

For $i : \mathbb{I} \vdash C = \mathsf{Path}\ A\ u\ v$

$$\mathsf{comp}^i\ C\ [\varphi \mapsto \mu]\ p_0 = \langle j \rangle \mathsf{comp}^i\ A\ [\varphi \mapsto \mu\ j, (j = 0) \mapsto u, (j = 1) \mapsto v]\ (p_0\ j)$$

For $i : \mathbb{I} \vdash C = (x : A, B)$

$$\mathsf{comp}^i\ C\ [\varphi \mapsto w]\ w_0 = (\mathsf{comp}^i\ A\ [\varphi \mapsto w.1]\ w_0.1, \mathsf{comp}^i\ B(x = a)\ [\varphi \mapsto w.2]\ w_0.2)$$

where $i : \mathbb{I} \vdash a = \mathsf{fill}^i\ A\ [\varphi \mapsto w.1]\ w_0.1 : A$.

$$\frac{\Gamma \vdash u : (y : B) \to \mathsf{Path}\ B\ (f\ (g\ y))\ y \qquad \Gamma \vdash v : (x : A) \to \mathsf{Path}\ A\ (g\ (f\ x))\ x}{\Gamma \vdash (f, g, u, v) : \mathsf{Iso}(A, B)}$$

$$\frac{\Gamma \vdash A \qquad \Gamma, \varphi \vdash T \qquad \Gamma, \varphi \vdash \sigma : \mathsf{Iso}(T, A)}{\Gamma \vdash \mathsf{glue}(A, [\varphi \mapsto (T, \sigma)]) \qquad \Gamma, \varphi \vdash \mathsf{glue}(A, [\varphi \mapsto (T, \sigma)]) = T}$$

$$\frac{\Gamma, \varphi \vdash \sigma : \mathsf{Iso}(T, A) \qquad \Gamma, \varphi \vdash t : T \qquad \Gamma \vdash a : A[\varphi \mapsto \sigma t]}{\Gamma \vdash \mathsf{glue}(a, [\varphi \mapsto t]) : \mathsf{glue}(A, [\varphi \mapsto (T, \sigma)])[\varphi \mapsto t]}$$

For $\Gamma, i : \mathbb{I} \vdash B = \mathsf{glue}(A, [\varphi \mapsto (T, \sigma)])$ we define

$$\mathsf{comp}^i\ B\ [\psi \mapsto \mathsf{glue}(a, [\varphi \mapsto t])]\ \mathsf{glue}(a_0, [\varphi(i0) \mapsto t_0]) = \mathsf{glue}(a_1, [\varphi(i1) \mapsto t_1])$$

where

$$\begin{array}{lcll}
a_1 & = & \mathsf{comp}^j\ A(i1)\ [\varphi(i1) \mapsto \alpha\ j, \psi \mapsto a(i1)]\ a_1'' & \Gamma \\
t_1 & = & \mathsf{equiv}(\sigma(i1), [\delta \mapsto t_1', \psi \mapsto t(i1)], a_1'').1 & \Gamma, \varphi(i1) \\
\alpha & = & \mathsf{equiv}(\sigma(i1), [\delta \mapsto t_1', \psi \mapsto t(i1)], a_1'').2 & \Gamma, \varphi(i1) \\
a_1'' & = & \mathsf{comp}^j\ A(i1)\ [\delta \mapsto \omega\ j, \psi \mapsto a(i1)]\ a_1' & \Gamma \\
\omega & = & \mathsf{pres}\ \sigma\ [\psi \mapsto t]\ t_0 & \Gamma, \delta \\
t_1' & = & \mathsf{comp}^i\ T\ [\psi \mapsto t]\ t_0 & \Gamma, \delta \\
a_1' & = & \mathsf{comp}^i\ A\ [\psi \mapsto a]\ a_0 & \Gamma \\
\delta & = & \forall i. \varphi & \Gamma
\end{array}$$

Appendix 3: propositional truncation

$$\frac{\Gamma \vdash A}{\Gamma \vdash \mathsf{inh}\ A} \qquad \frac{\Gamma \vdash a : A}{\Gamma \vdash \mathsf{inc}\ a : \mathsf{inh}\ A} \qquad \frac{\Gamma \vdash u_0 : \mathsf{inh}\ A \quad \Gamma \vdash u_1 : \mathsf{inh}\ A \quad \Gamma \vdash r : \mathbb{I}}{\Gamma \vdash \mathsf{squash}(u_0, u_1, r) : \mathsf{inh}\ A}$$

$$\frac{\Gamma \vdash u_0 : \mathsf{inh}\ A \quad \Gamma \vdash u_1 : \mathsf{inh}\ A}{\Gamma \vdash \mathsf{squash}(u_0, u_1, 0) = u_0 : \mathsf{inh}\ A} \qquad \frac{\Gamma \vdash u_0 : \mathsf{inh}\ A \quad \Gamma \vdash u_1 : \mathsf{inh}\ A}{\Gamma \vdash \mathsf{squash}(u_0, u_1, 1) = u_1 : \mathsf{inh}\ A}$$

$$\frac{\Gamma, \varphi, i : \mathbb{I} \vdash u : \mathsf{inh}\ A \quad \Gamma \vdash u_0 : \mathsf{inh}\ A[\varphi \mapsto u(i0)]}{\Gamma \vdash \mathsf{hcomp}^i\ [\varphi \mapsto u]\ u_0 : \mathsf{inh}\ A}$$

We can then define two operations

$$\frac{\Gamma, i : \mathbb{I} \vdash A \quad \Gamma \vdash u_0 : \mathsf{inh}\ A(i0)}{\Gamma \vdash \mathsf{transp}\ u_0 : \mathsf{inh}\ A(i1)} \qquad \frac{\Gamma, i : \mathbb{I} \vdash A \quad \Gamma, i : \mathbb{I} \vdash u : \mathsf{inh}\ A}{\Gamma, i : \mathbb{I} \vdash \mathsf{squeeze}\ u : \mathsf{inh}\ A(i1)}$$

satisfying

$$\frac{\Gamma, i : \mathbb{I} \vdash A \quad \Gamma, i : \mathbb{I} \vdash u : \mathsf{inh}\ A}{\Gamma \vdash (\mathsf{squeeze}\ u)(i0) = \mathsf{transp}\ u(i0) : \mathsf{inh}\ A(i1)} \qquad \frac{\Gamma, i : \mathbb{I} \vdash A \quad \Gamma, i : \mathbb{I} \vdash u : \mathsf{inh}\ A}{\Gamma \vdash (\mathsf{squeeze}\ u)(i1) = u(i1) : \mathsf{inh}\ A(i1)}$$

by the equations

$$\begin{array}{lcl}
\mathsf{transp}\ (\mathsf{inc}\ a) & = & \mathsf{inc}\ (\mathsf{comp}^i\ A\ []\ a) \\
\mathsf{transp}\ (\mathsf{squash}(u_0, u_1, r)) & = & \mathsf{squash}(\mathsf{transp}\ u_0, \mathsf{transp}\ u_1, r) \\
\mathsf{transp}\ (\mathsf{hcomp}^j\ [\varphi \mapsto u]\ u_0) & = & \mathsf{hcomp}^j\ [\varphi \mapsto \mathsf{transp}\ u]\ (\mathsf{transp}\ u_0) \\
 & & \\
\mathsf{squeeze}\ (\mathsf{inc}\ a) & = & \mathsf{inc}\ (\mathsf{comp}^j\ A(i \vee j)\ [(i = 1) \mapsto a(i1)]\ a) \\
\mathsf{squeeze}\ (\mathsf{squash}(u_0, u_1, r)) & = & \mathsf{squash}(\mathsf{squeeze}\ u_0, \mathsf{squeeze}\ u_1, r) \\
\mathsf{squeeze}\ (\mathsf{hcomp}^j\ [\varphi \mapsto u]\ u_0) & = & \mathsf{hcomp}^j\ [\varphi \mapsto \mathsf{squeeze}\ u]\ (\mathsf{squeeze}\ u_0)
\end{array}$$

Using these operations, we can define

$$\frac{\Gamma, i : \mathbb{I} \vdash A \quad \Gamma, \varphi, i : \mathbb{I} \vdash u : \mathsf{inh}\ A \quad \Gamma \vdash u_0 : \mathsf{inh}\ A(i0)[\varphi \mapsto u(i0)]}{\Gamma \vdash \mathsf{comp}^i\ [\varphi \mapsto u]\ u_0 : \mathsf{inh}\ A(i1)[\varphi \mapsto u(i1)]}$$

by the equation

$$\Gamma \vdash \mathsf{comp}^i\ [\varphi \mapsto u]\ u_0 = \mathsf{hcomp}^i\ [\varphi \mapsto \mathsf{squeeze}\ u]\ (\mathsf{transp}\ u_0) : \mathsf{inh}\ A(i1)$$

Given $\Gamma \vdash B$ and $\Gamma \vdash q : (x\ y : B) \to \mathsf{Path}\ B\ x\ y$ and $f : A \to B$ we define $g : \mathsf{inh}\ A \to B$ by the equations

$$\begin{array}{lcl}
g\ (\mathsf{inc}\ a) & = & f\ a \\
g\ (\mathsf{squash}(u_0, u_1, r)) & = & q\ (g\ u_0)\ (g\ u_1)\ r \\
g\ (\mathsf{hcomp}^j\ [\varphi \mapsto u]\ u_0) & = & \mathsf{comp}^j\ B\ [\varphi \mapsto g\ u]\ (g\ u_0)
\end{array}$$