Type Theory and Univalent Foundation

Thierry Coquand (j.w.w. M. Bezem and S. Huber)

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This talk

Univalence axiom as an *extensionality axiom* for dependent types

Explain the effectivity problem with the Kan simplicial set semantics

Solution of this problem with a variation of the cubical set model

Connections with nominal sets

A new justification of the axiom of description

New view on the problem of “size” of collections

This talk

This fits in the theme of finding effective content of mathematical arguments

The current justification of the axiom of univalence is *not* effective

We present an *effective* version of the Kan simplicial set model

Type Theory

1908 Russell *Mathematical Logic as Based on the Theory of Types*

1940 Church *A Formulation of the Simple Theory of Types*

1973 Martin-Löf *An Intuitionistic Theory of Types: Predicative Part*

2009 Voevodsky, axiom of univalence

1908 Zermelo *Investigations in the Foundations of Set Theory*

Church simple type theory

Simple types o (type of propositions) and ι (type of individuals)

$\alpha \to \beta$ (function types) written $(\beta)\alpha$ by Church

10^{o} Propositional extensionality $(p \equiv q) \;\to\; p = q$ (already in Russell 1925)

$10^{\alpha\beta}$ Function extensionality $(\forall x^{\alpha}.f\ x = g\ x) \;\to\; f = g$

9^{α} Axiom of Description $\forall f^{\alpha\to o}.\forall x^{\alpha}.\ f\ x \wedge (\forall y^{\alpha}.f\ y \to x = y) \;\to\; f\ (\iota\ f)$

11^{α} Axiom of Choice $\forall f^{\alpha\to o}.\forall x^{\alpha}.\ f\ x \;\to\; f\ (\iota\ f)$

Remarks

We can rewrite the extensionality axioms as

10^o Propositional extensionality $p = q \;\equiv\; (p \equiv q)$

$10^{\alpha\beta}$ Function extensionality $f = g \;\equiv\; (\forall x^\alpha . f\ x = g\ x)$

The axioms $1 - 6$ are about basic laws of logic

The axioms $7 - 8$ are about individuals (axiom of infinity)

Church introduced type of functions not necessarily proposition valued

E.g. $\iota \rightarrow \iota$ if ι primitive type of individuals

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Propositions as Types

The next step occurs in the 70s through the work of Curry, de Bruijn, Howard, Tait, Scott, Martin-Löf, Girard, $\ldots$

In *natural deduction* the laws for proving a proposition are the same as the laws for building an element of a given type

E.g. $\lambda x.t$ is of type $A \to B$ if t is of type B given x of type A

$c\ u$ is of type B if c is of type $A \to B$ and u of type A

It is natural to identify *propositions* and *types*

de Bruijn: this formalism is well-suited to represent proofs on a computer

Propositions as Types

So the type of propositions can be thought of as a type of (small) types

Universal quantification corresponds to an operation

$(\Pi x : A)B$ if $B(x)$ is a *dependent type* over $x : A$

E.g. $\lambda x.t$ is of type $(\Pi x : A)B$ if t is of type B given x of type A

$c\ u$ is of type $B(u)$ if c is of type $(\Pi x : A)B$ and u of type A

Dependent Types and Extensionality

Highly desirable to add extensionality to systems with dependent types

E.g. for interactive proof systems

Main issue: what are the rules for equality with dependent types?

Given A type and $a_0,\ a_1$ of type A we have to introduce a new type $\mathsf{Id}_A\ a_0\ a_1$

Propositions as Types

New rules for equality

Reflexivity $1_a : \mathsf{Id}_A\ a\ a$

Leibnitz' law of indiscernability of identicals

$C(a)$ implies $C(x)$ if $p : \mathsf{Id}_A\ a\ x$

The *new* law (1973) is that if $a : A$ any element (x, ω) of the type

$$S = (\Sigma x : A)\mathsf{Id}_A\ a\ x$$

is equal to the element $(a, 1_a)$, i.e. we have an element in

$$\mathsf{Id}_S\ (a, 1_a)\ (x, \omega)$$

Voevodsky's stratification

Define A to be a *proposition* if we have

$(\Pi x_0 : A)(\Pi x_1 : A)\mathsf{Id}_A\ x_0\ x_1$

Define A to be a *set* if we have

$(\Pi x_0 : A)(\Pi x_1 : A)\mathsf{prop}\ (\mathsf{Id}_A\ x_0\ x_1)$

Define A to be a *groupoid* if we have

$(\Pi x_0 : A)(\Pi x_1 : A)\mathsf{set}\ (\mathsf{Id}_A\ x_0\ x_1)$

Type theory can be seen as a generalization of *set theory*

Hedberg's Theorem: a type with a *decidable equality* is a *set*

Kan simplicial set model

This stratification was motivated by the following semantics

A *type* is interpreted as a *homotopy type* (space)

A *dependent type* is interpreted as a *fiber space* over a space

homotopy type: Kan simplicial set

fiber space: Kan fibration

Equality type: space of paths

Kan *A Combinatorial Definition of Homotopy Groups*, 1958

Axiom of Univalence

Voevodsky was able to define uniformly the notion of *equivalence* of types

If A and B are sets we get back the notion of *bijection* between sets

If A and B are groupoids notion of *categorical equivalence* between groupoids

If A and B are propositions notion of *logical equivalence* between propositions

The Axiom of Univalence can be stated roughly as

$$A =_{\mathsf{Type}} B \simeq (A \simeq B)$$

where $A \simeq B$ means that there exists an equivalence between A and B

Type is a *universe*, a type of small types

Axiom of Univalence

It implies that

-logically equivalent propositions are equal

-isomorphic sets are equal

-isomorphic algebraic structures are equal

-equivalent groupoids are equal

-equivalent categories are equal

Kan simplicial set model

The new laws of equality with dependent types express that

the total space of the path fibration is contractible

Crucial fact that started the loop space method in algebraic topology (Serre)

Effectivity problem

We have *classically* that

if $P(x)$ is a Kan fibration over $x : A$

there is a path between a and b in A

then $P(a)$ and $P(b)$ are homotopy equivalent

This does not hold effectively

Counter-model, Kripke models over $0 \leqslant 1 \leqslant 2$ (j.w.w. Marc Bezem)

Effectivity problem

Similarly we have a *classical* proof that if B is Kan then B^A is Kan

This also seems to use classical logic in an essential way

The core of the problem is that

to be a degenerate simplex

may not be decidable

Also, the definition of $\pi_n(X, a)$ for X Kan simplicial set is quite complicated

Cubical set models

For a model of type theory, *cubical* sets are more natural than *simplicial* sets

Cubical sets are better suited for studying fibrations, cf. PhD thesis of Serre

Kan *Abstract Homotopy*, 1955

The definition of $\pi_n(X, a)$ for X Kan cubical set is simple and natural

One can define directly the path space $\mathsf{Id}_X\ a_0\ a_1$

Cubical sets, reformulated

We fix a countable set of *names* $x, y, z, \dots$ distinct from $0, 1$

A name should be thought of as an abstract notion of direction

An object of $\mathcal{C}$ is a finite set of names

A morphism $I \to J$ is a set map $I \to J \cup \{0, 1\}$ which is injective on its domain, i.e. if $i_0 \neq i_1$ and $f(i_0), f(i_1)$ in J then $f(i_0) \neq f(i_1)$

Partition $I = I_0, I_1, I'$ with an injection $I' \to J$

This represents a *substitution*: we can assign the value 0 or 1 or do renaming or add new variables

Cubical sets, reformulated

Definition: *a* cubical set *is a functor* $\mathcal{C} \to \text{Set}$*.*

Definition: *If* X *is a cubical set, an* I-cube *of* X *is an element of* $X(I)$*.*

Formal representation of singular cubical complexes

A cubical set X is a presheaf on the category $\mathcal{C}^{opp}$

Via Yoneda, the object I can be thought of as a cubical set

We may think of this cubical set as a formal version of $[0,1]^I$

An I-cube is then a formal version of a map $[0,1]^I \to X$

Cubical sets, reformulated

A cube: an object which may *depend* on some (finite set of) names

This dependency relation may not be decidable

The fact that this is a "good" notion of dependency is expressed by the following result (similar to a result of Staton, Levy)

Theorem: *any cubical set restricted to the category of finite sets and injection preserve pull-back, i.e. defines a* nominal set

Cf. A. Pitts *Nominal Sets. Names and Symmetry in Computer Science*

Constructive Type Theory

The Kan filling condition is simple for cubical sets

any open box can be filled

We refine the notion of Kan filling by requiring fillings to be invariant under *renaming* and *addition of new directions*

Effectively each type X comes with filling operations $X \uparrow$ and $X \downarrow$

Classically we can always require to have this condition

Intuitionistically this refinement solves all effectivity problems

Modality

We define inh X for any Kan cubical set X

This is a *proposition* stating that X is inhabited

We add a constructor $\alpha_x(a_0, a_1)$ connecting formally along the dimension x any two I-cubes a_0 and a_1 (with x not in I)

$$\alpha_x(a_0, a_1)(x = 0) \;=\; a_0 \qquad\qquad \alpha_x(a_0, a_1)(x = 1) \;=\; a_1$$

We define degeneracy of these new elements by commutation with substitution

We close it by adding also a filling operation

Induction principle of inh X: if Y is a proposition and $X \to Y$ then we have inh $X \to Y$

Existential quantification

We define a *new* quantification operation

$(\exists x : A)B(x)$

as

$\mathsf{inh}\ (\Sigma x : A)B(x)$

This satisfies the usual elimination rule for existential quantification

Contrary to $(\Sigma x : A)B(x)$ however, we cannot extract a witness

In particular, choice does not hold when formulated with $\exists$

Axiom of Description

If $B(x)$ a family of *propositions* and

$$B(x_0) \wedge B(x_1) \rightarrow \mathsf{Id}_A\ x_0\ x_1$$

then we have

$$\mathsf{prop}\ (\Sigma x : A)B(x)$$

and hence we have the Axiom of Description

$$(\exists! x : A)B(x) \rightarrow (\Sigma x : A)B(x)$$

while we don't have in general

$$(\exists x : A)B(x) \rightarrow (\Sigma x : A)B(x)$$

Constructive Type Theory

We have implemented a prototype implementation in Haskell

-dependent types and product

-ordinary recursive types (natural numbers, booleans, lists, . . .)

-equality

-inhabited modality

-function extensionality

(Not yet universe and univalence)

j.w.w. S. Huber, A. Mörtberg and C. Cohen

Constructive Type Theory

In particular we have tested the following example

Given A and a proposition valued relation R on A define A/R

An element of A/R is a proposition valued predicate on A

This should be compatible with R and *inhabited*

We have the canonical surjection $s : A \to A/R$

In general we don't have a section

However if $f : A \to B$ and f compatible with R

and B is a set then we can find $g : A/R \to B$ such that $g \circ s = f$

Existential quantification

If $P(x)$ is a dependent types over A we express that P is inhabited as

$$\exists x : A.P(x)$$

From this we cannot in general extract any element of A

If $f : A \to B$ and B is a *set* then the predicate on B

$$Q\ y \quad = \quad \exists x : A.P(x) \wedge \mathsf{Id}_B\ y\ (f\ x)$$

satisfies

$$\exists ! y : B.Q(y)$$

By the Axiom of Description we can extract this element and this is $g\ P$

Existential quantification

Hedberg's theorem shows that types with a decidable equality are *sets*

We have tested the previous example with $A = Nat$ and $B = Bool$

Proofs of proposition have *computational content*

On the other hand two proofs of the same proposition are equal

Resizing Axioms

Voevodsky suggested the following "resizing" axioms

(1) If A is a type and prop A then $A : \mathsf{Type}_0$

(2) If A, B are types $A : \mathsf{Type}_n$ and $B : \mathsf{Type}_m$ and $n < m$ and

$$\mathsf{Id}_{\mathsf{Type}_m}\ A\ B$$

then $B : \mathsf{Type}_n$

E.g. by (2) we can consider that the category of all finite sets to be small

Resizing Axioms

We now have computational interpretation of these axioms

We know that they are consistent (Voevodsky)

Conjecture: all computations are terminating

The usual reducibility argument does not seem to apply

Some references

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M. Hofmann and Th. Streicher *A groupoid model of type theory*, 1993

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