Type Theory and Univalent Foundation

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This talk

Univalence axiom as an *extensionality axiom* for dependent types

Explain the effectivity problem with the Kan simplicial set semantics

Solution of this problem with a variation of the cubical set model

Connections with nominal sets

A new justification of the axiom of description

New view on the problem of “size” of collections
This talk

This fits in the theme of finding effective content of mathematical arguments.

The current justification of the axiom of univalence is not effective.

We present an effective version of the Kan simplicial set model.
Type Theory

1908 Russell *Mathematical Logic as Based on the Theory of Types*

1940 Church *A Formulation of the Simple Theory of Types*

1973 Martin-Löf *An Intuitionistic Theory of Types: Predicative Part*

2009 Voevodsky, axiom of univalence

1908 Zermelo *Investigations in the Foundations of Set Theory*
Church simple type theory

Simple types $o$ (type of propositions) and $\iota$ (type of individuals)

$\alpha \rightarrow \beta$ (function types) written $(\beta)\alpha$ by Church

10$^o$ Propositional extensionality $(p \equiv q) \rightarrow p = q$ (already in Russell 1925)

10$^{\alpha\beta}$ Function extensionality $(\forall x^\alpha. f\ x = g\ x) \rightarrow f = g$

9$^\alpha$ Axiom of Description $\forall f^{\alpha \rightarrow o}. \forall x^\alpha. f\ x \land (\forall y^\alpha. f\ y \rightarrow x = y) \rightarrow f(\iota f)$

11$^\alpha$ Axiom of Choice $\forall f^{\alpha \rightarrow o}. \forall x^\alpha. f\ x \rightarrow f(\iota f)$
Remarks

We can rewrite the extensionality axioms as

10° Propositional extensionality $p = q \equiv (p \equiv q)$

10αβ Function extensionality $f = g \equiv (\forall x^\alpha. f(x) = g(x))$

The axioms 1 – 6 are about basic laws of logic

The axioms 7 – 8 are about individuals (axiom of infinity)

Church introduced type of functions not necessarily proposition valued

E.g. $\iota \to \iota$ if $\iota$ primitive type of individuals
Propositions as Types

The next step occurs in the 70s through the work of Curry, de Bruijn, Howard, Tait, Scott, Martin-Löf, Girard, …

In *natural deduction* the laws for proving a proposition are the same as the laws for building an element of a given type

E.g. $\lambda x.t$ is of type $A \rightarrow B$ if $t$ is of type $B$ given $x$ of type $A$

$c \ u$ is of type $B$ if $c$ is of type $A \rightarrow B$ and $u$ of type $A$

It is natural to identify *propositions* and *types*

de Bruijn: this formalism is well-suited to represent proofs on a computer
Propositions as Types

So the type of propositions can be thought of as a type of (small) types

Universal quantification corresponds to an operation

\[(Πx : A)B\] if \(B(x)\) is a dependent type over \(x : A\)

E.g. \(\lambda x.t\) is of type \((Πx : A)B\) if \(t\) is of type \(B\) given \(x\) of type \(A\)

\(c\ u\) is of type \(B(u)\) if \(c\) is of type \((Πx : A)B\) and \(u\) of type \(A\)
Dependent Types and Extensionality

Highly desirable to add extensionality to systems with dependent types

E.g. for interactive proof systems

Main issue: what are the rules for equality with dependent types?

Given $A$ type and $a_0$, $a_1$ of type $A$ we have to introduce a new type $\text{Id}_A a_0 a_1$
Propositions as Types

New rules for equality

Reflexivity  \(1_a : \text{Id}_A a a\)

Leibnitz’ law of indiscernability of identicals

\(C(a)\) implies \(C(x)\) if \(p : \text{Id}_A a x\)

The new law (1973) is that if \(a : A\) any element \((x, \omega)\) of the type

\[ S = (\Sigma x : A)\text{Id}_A a x \]

is equal to the element \((a, 1_a)\), i.e. we have an element in

\[ \text{Id}_S (a, 1_a) (x, \omega) \]
**Voevodsky’s stratification**

Define $A$ to be a *proposition* if we have

$$(\Pi x_0 : A)(\Pi x_1 : A) \text{Id}_A x_0 x_1$$

Define $A$ to be a *set* if we have

$$(\Pi x_0 : A)(\Pi x_1 : A) \text{prop} (\text{Id}_A x_0 x_1)$$

Define $A$ to be a *groupoid* if we have

$$(\Pi x_0 : A)(\Pi x_1 : A) \text{set} (\text{Id}_A x_0 x_1)$$

*Type theory* can be seen as a generalization of *set theory*

Hedberg’s Theorem: a type with a *decidable equality* is a *set*
This stratification was motivated by the following semantics

A type is interpreted as a homotopy type (space)

A dependent type is interpreted as a fiber space over a space

homotopy type: Kan simplicial set

fiber space: Kan fibration

Equality type: space of paths

Kan A Combinatorial Definition of Homotopy Groups, 1958
Voevodsky was able to define uniformly the notion of \textit{equivalence} of types

If $A$ and $B$ are sets we get back the notion of \textit{bijection} between sets

If $A$ and $B$ are groupoids notion of \textit{categorical equivalence} between groupoids

If $A$ and $B$ are propositions notion of \textit{logical equivalence} between propositions

The Axiom of Univalence can be stated roughly as

$$A =_{\text{Type}} B \simeq (A \simeq B)$$

where $A \simeq B$ means that there exists an equivalence between $A$ and $B$

\textit{Type} is a \textit{universe}, a type of small types
Axiom of Univalence

It implies that
- logically equivalent propositions are equal
- isomorphic sets are equal
- isomorphic algebraic structures are equal
- equivalent groupoids are equal
- equivalent categories are equal
Kan simplicial set model

The new laws of equality with dependent types express that

*the total space of the path fibration is contractible*

Crucial fact that started the loop space method in algebraic topology (Serre)
We have *classically* that

if $P(x)$ is a Kan fibration over $x : A$

there is a path between $a$ and $b$ in $A$

then $P(a)$ and $P(b)$ are homotopy equivalent

*This does not hold effectively*

Counter-model, Kripke models over $0 \leq 1 \leq 2$ (j.w.w. Marc Bezem)
Effectivity problem

Similarly we have a classical proof that if $B$ is Kan then $B^A$ is Kan.

This also seems to use classical logic in an essential way.

The core of the problem is that

\[ \text{to be a degenerate simplex} \]

may not be decidable.

Also, the definition of $\pi_n(X, a)$ for $X$ Kan simplicial set is quite complicated.
Cubical set models

For a model of type theory, *cubical* sets are more natural than *simplicial* sets.


The definition of $\pi_n(X, a)$ for $X$ Kan cubical set is simple and natural.

One can define directly the path space $\text{Id}_X\ a_0\ a_1$. 
Cubical sets, reformulated

We fix a countable set of names $x, y, z, \ldots$ distinct from $0, 1$

A name should be thought of as an abstract notion of direction

An object of $C$ is a finite set of names

A morphism $I \to J$ is a set map $I \to J \cup \{0, 1\}$ which is injective on its domain, i.e. if $i_0 \neq i_1$ and $f(i_0), f(i_1)$ in $J$ then $f(i_0) \neq f(i_1)$

Partition $I = I_0, I_1, I'$ with an injection $I' \to J$

This represents a substitution: we can assign the value 0 or 1 or do renaming or add new variables
Cubical sets, reformulated

**Definition:** a cubical set is a functor $C \rightarrow \text{Set}$.

**Definition:** If $X$ is a cubical set, an $I$-cube of $X$ is an element of $X(I)$.

Formal representation of singular cubical complexes

A cubical set $X$ is a presheaf on the category $C^{opp}$

Via Yoneda, the object $I$ can be thought of as a cubical set

We may think of this cubical set as a formal version of $[0,1]^I$

An $I$-cube is then a formal version of a map $[0,1]^I \rightarrow X$
Cubical sets, reformulated

A cube: an object which may depend on some (finite set of) names

This dependency relation may not be decidable

The fact that this is a “good” notion of dependency is expressed by the following result (similar to a result of Staton, Levy)

**Theorem:** any cubical set restricted to the category of finite sets and injection preserve pull-back, i.e. defines a nominal set

Cf. A. Pitts *Nominal Sets. Names and Symmetry in Computer Science*
Constructive Type Theory

The Kan filling condition is simple for cubical sets

*any open box can be filled*

We refine the notion of Kan filling by requiring fillings to be invariant under *renaming* and *addition of new directions*

Effectively each type $X$ comes with filling operations $X \uparrow$ and $X \downarrow$

Classically we can always require to have this condition

Intuitionistically this refinement solves all effectivity problems
We define \( \text{inh} \ X \) for any Kan cubical set \( X \)

This is a *proposition* stating that \( X \) is inhabited

We add a constructor \( \alpha_x(a_0, a_1) \) connecting formally along the dimension \( x \) any two \( I \)-cubes \( a_0 \) and \( a_1 \) (with \( x \) not in \( I \))

\[
\alpha_x(a_0, a_1)(x = 0) = a_0 \quad \alpha_x(a_0, a_1)(x = 1) = a_1
\]

We define degeneracy of these new elements by commutation with substitution

We close it by adding also a filling operation

Induction principle of \( \text{inh} \ X \): if \( Y \) is a proposition and \( X \rightarrow Y \) then we have \( \text{inh} \ X \rightarrow Y \)
We define a new quantification operation

\((\exists x : A)B(x)\)

as

\(\text{inh } (\Sigma x : A)B(x)\)

This satisfies the usual elimination rule for existential quantification.

Contrary to \((\Sigma x : A)B(x)\) however, we cannot extract a witness.

In particular, choice does not hold when formulated with \(\exists\)
If $B(x)$ a family of propositions and
\[
B(x_0) \land B(x_1) \rightarrow \text{Id}_A \ x_0 \ x_1
\]
then we have
\[
\text{prop} \ (\Sigma x : A)B(x)
\]
and hence we have the Axiom of Description
\[
(\exists! x : A)B(x) \rightarrow (\Sigma x : A)B(x)
\]
while we don’t have in general
\[
(\exists x : A)B(x) \rightarrow (\Sigma x : A)B(x)
\]
Constructive Type Theory

We have implemented a prototype implementation in Haskell

-dependent types and product

-ordinary recursive types (natural numbers, booleans, lists, \ldots)

-equality

-inhabited modality

-function extensionality

(Not yet universe and univalence)

j.w.w. S. Huber, A. Mörtberg and C. Cohen
In particular we have tested the following example

Given \( A \) and a proposition valued relation \( R \) on \( A \) define \( A/R \)

An element of \( A/R \) is a proposition valued predicate on \( A \)

This should be compatible with \( R \) and \textit{inhabited}

We have the canonical surjection \( s : A \rightarrow A/R \)

In general we don’t have a section

However if \( f : A \rightarrow B \) and \( f \) compatible with \( R \)

\textit{and} \( B \) is a set then we can find \( g : A/R \rightarrow B \) such that \( g \circ s = f \)
Existential quantification

If $P(x)$ is a dependent types over $A$ we express that $P$ is inhabited as

$$\exists x : A. P(x)$$

From this we cannot in general extract any element of $A$

If $f : A \to B$ and $B$ is a set then the predicate on $B$

$$Q y = \exists x : A. P(x) \land \text{Id}_B y (f x)$$

satisfies

$$\exists! y : B. Q(y)$$

By the Axiom of Description we can extract this element and this is $g P$
Existential quantification

Hedberg’s theorem shows that types with a decidable equality are sets.

We have tested the previous example with $A = \text{Nat}$ and $B = \text{Bool}$.

Proofs of proposition have *computational content*.

On the other hand two proofs of the same proposition are equal.
Voevodsky suggested the following “resizing” axioms

(1) If $A$ is a type and prop $A$ then $A : \text{Type}_0$

(2) If $A, B$ are types $A : \text{Type}_n$ and $B : \text{Type}_m$ and $n < m$ and

$$\text{Id}_{\text{Type}_m} A B$$

then $B : \text{Type}_n$

E.g. by (2) we can consider that the category of all finite sets to be small
Resizing Axioms

We now have computational interpretation of these axioms

We know that they are consistent (Voevodsky)

Conjecture: all computations are terminating

The usual reducibility argument does not seem to apply
Some references


M. Hofmann and Th. Streicher *A groupoid model of type theory*, 1993

V. Voevodsky *Univalent foundation*, home page

HoTT book, 2013

M. Bezem, Th. C., S Huber
*A cubical set model of type theory*, preprint, 2013