Positivity in point-free topology: positivity predicates

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Formal Topology

For our purpose, it is enough to say that the main idea is to reverse the traditional conceptual order of definitions in topology and define points as particular filters of neighbourhoods, rather than opens as particular sets of points

G. Sambin, Intuitionistic Formal Spaces, A first communication, 1986

Formal Topology

Analogy with some approach in physics, e.g. to thermodynamics, which consists in considering only *observable* notions

Measure of a physical quantity (real number)
Only rational approximations are known in general
One can observe that this real number is contained in a rational interval

Formal Topology

(topos theory) has also a role to play in suggesting what constructive mathematics ought to be – what results one should aim for, and even how one should try to prove them . . . even the message that constructive general topology ought to be about locales and not spaces . . . has had little impact on any of the traditional schools of constructive mathematics

Johnstone, Open locales and exponentiation, 1984

Formal Topology

Solves many problems of constructive analysis: definition of continuous functions, proof of Heine-Borel, of Tychonoff's Theorem, simpler statement and proof of Hahn-Banach, Gelfand representation theorem

Problem of composition of continuous functions in Bishop's framework: inverse function $(0, \infty) \to \mathbb{R}$ and $f : [0, 1] \to (0, \infty)$, is the composition continuous? (E. Palmgren)

In algebra, definition of Zariski spectrum, with good properties, Krull dimension

Some precursors

Dedekind-Weber 1882 algebraic definition of Riemann surfaces

Kreisel 1959 neighbourhoods system; Scott information system 1982

P. Martin-Löf *Notes on Constructive Mathematics*, 1968
Cantor space, real line described as formal spaces, but in the context of *recursive* mathematics, e.g. a collection of neighbourhoods has to be given by a recursively enumerable sequence

Formal Topology

Developped in Type Theory, in a predicative setting (by opposition to the theory of *locales* developped in topos theory)

A set S of basic open and a relation $a \lhd U$ between elements of S and subsets of S, represented by predicates on S. Intuitively: the basic open is a subset of the union of the basic open in U

Important difference of nature between elements $a, b, \ldots$ of S and subsets $U, V, \ldots$ of S

Basic open are (most often) concrete, syntactical, discrete objects

Notation $U \lhd V$ means $a \lhd V$ for all $a \epsilon U$
where $a \epsilon U$ means $U(a)$

Formal Topology

We only use two rules

Transitivity rule: if $a \lhd U$ and $U \lhd V$ then $a \lhd V$

Reflexivity rule: $a \lhd U$ if $a \epsilon U$

Formal Topology

If D is a distributive lattice take $S = D$ and define $a \lhd U$ by

$$\exists a_1, \ldots, a_n \epsilon U. \quad a \leqslant a_1 \vee \cdots \vee a_n$$

If R is a commutative ring, take $S = R$ and define $a \lhd U$ if and only if a power of a is in the ideal generated by U

In both cases all basic open are *compact*: if $a \lhd U$ then there exists $a_1, \ldots, a_n \epsilon U$ such that $a \lhd a_1, \ldots, a_n$

Example: real line

Proof theoretic approach to topology

Example $\mathbb{R}$: the basic open are rational intervals (r, s)
Deduction rules

$$\frac{}{(r,s) \lhd U} \quad (r,s)\epsilon U$$

$$\frac{(r',s') \lhd U}{(r,s) \lhd U}$$

where $r' \leqslant r < s \leqslant s'$

$$\frac{(r,s') \lhd U \qquad (r',s) \lhd U}{(r,s) \lhd U}$$

where $r < r' < s' < s$

Example: real line

Defines a finitary covering relation $(r, s) \lhd_\omega U$

Any derivation is a finite tree

If $(r, s) \lhd_\omega U$ then

$$(r, s) \subseteq \bigcup_{(p,q)\epsilon U} (p, q)$$

in $\mathbb{R}$ but the converse may not be valid

Formal Topology

One should add the infinitary rule

$$\frac{\ldots (r', s') \lhd U \ldots}{(r, s) \lhd U} (r < r' < s' < s)$$

Classically, one has $(r, s) \lhd U$ if and only if $(r, s) \subseteq \bigcup_{(p,q) \epsilon U} (p, q)$ in $\mathbb{R}$

Theorem: $(r, s) \lhd U$ if and only if for all $r < r' < s' < s$ we have $(r', s') \lhd_\omega U$

This means that one can always put the infinitary rule at the end and use it at most once

Heine-Borel is a simple corollary

Formal points

A *point* α will be a predicate on basic open

We write $\alpha \in a$ for $\alpha(a)$ and $\alpha \in U$ for $\exists a \epsilon U . \alpha \in a$

We should have $\alpha \in U$ if $\alpha \in a$ and $a \lhd U$

What are the points in $\mathbb{R}$? Dedekind reals

$$(r, s) \lhd U \leftrightarrow \forall \alpha.\ \alpha \in (r, s) \rightarrow \alpha \in U$$

is equivalent to Brouwer's Fan Theorem
It does not hold in Type Theory

The definition of $(p, q) \lhd U$ captures the "right" covering notion

Positivity

Notion only interesting in a constructive framework

We want to express constructively that an open is inhabited

pos(a) if and only if $\exists \alpha.\ \alpha \in a$

Positivity Predicate

Rules for $pos(a)$

$$\frac{a \lhd U \qquad pos(a)}{pos(U)} \; monotonicity$$

$$\frac{pos(a) \rightarrow a \lhd U}{a \lhd U} \; positivity$$

Positivity Predicate

$$\frac{a \lhd U \qquad pos(a)}{pos(U)} \; monotonicity$$

$$\frac{\exists\alpha.\alpha \in a \qquad \forall\alpha.\;\; \alpha \in a \to \alpha \in U}{\exists\alpha.\alpha \in U}$$

$$\frac{pos(a) \to a \lhd U}{a \lhd U} \; positivity$$

$$\frac{(\exists\alpha.\alpha \in a) \to \forall\alpha.\;\; \alpha \in a \to \alpha \in U}{\forall\alpha.\;\; \alpha \in a \to \alpha \in U}$$

Are these rules complete?

Positivity Predicate

Write a^+ the subset $\{a \mid pos(a)\}$

$a \lhd a^+$ because $pos(a) \to a \lhd a^+$ because $a \lhd a$ by reflexivity

Conversely if $a \lhd a^+$ then we have positivity

Notice that $pos(a) \to a \lhd U$ can be written as $a^+ \lhd U$ and we have

$$\frac{a^+ \lhd U}{a \lhd U}$$

if $a \lhd a^+$ by transitivity

Positivity Predicate

Theorem: The following are equivalent

(1) the positivity rule

$$\frac{pos(a) \to a \lhd U}{a \lhd U}$$

(2) the rule

$$\frac{a^+ \lhd U}{a \lhd U}$$

(3) $a \lhd a^+$

(4) the rule

$$\frac{a \lhd U}{a \lhd U^+}$$

(5) $U \lhd U^+$ where $U^+ = \{b \epsilon U \mid pos(b)\}$

Positivity Predicate

Classically we can define $pos(a)$ by $\neg(a \lhd \emptyset)$

We have $a \lhd a^+$ in both cases $a \lhd \emptyset$ or $\neg(a \lhd \emptyset)$

So classically any formal space has a positivity predicate

Positivity Predicate

Impredicatively one can try to define $POS(a)$ as

$$\forall U. \;\; a \lhd U \rightarrow \exists b. b \epsilon U$$

This may not satisfy monotonicity and positivity

If we have $a \lhd \{a \mid POS(a)\}$ then POS is a positivity predicate in an *impredicative* framework (Fourman-Grayson)

A locale having this property is called *open* or *overt*

Positivity Predicate

Theorem: (P. Aczel) If there exists a positivity predicate *pos* then $pos(a) \leftrightarrow POS(a)$

This shows that *pos* if it exists, is uniquely determined by $\lhd$

$pos(a) \rightarrow POS(a)$ by monotonicity
$POS(a) \rightarrow pos(a)$ since $a \lhd a^+$

Positivity

Assume that S is such that any basic open is compact

Reflexivity rule

$$\frac{}{a \lhd U} \quad a \epsilon U$$

Transistivity rule

$$\frac{a \lhd U \qquad U \lhd V}{a \lhd V}$$

Compactness: if $a \lhd U$ then $a \lhd a_1, \ldots, a_n$ for some $a_1, \ldots, a_n \epsilon U$

Positivity

With compactness we can prove

Lemma: We have $POS(a) \leftrightarrow \neg(a \lhd \emptyset)$

Theorem: S has a positivity predicate if and only if $a \lhd \emptyset$ is decidable and then $pos(a) \leftrightarrow \neg(a \lhd \emptyset)$

Proof: since $a \lhd a^+$ by compactness either $a \lhd \emptyset$ or $pos(a)$
We cannot have both $pos(a)$ and $a \lhd \emptyset$ by monotonicity
So for any a we have $a \lhd \emptyset$ or $\neg(a \lhd \emptyset)$

This provides examples of formal spaces *without* positivity predicate

Positivity predicate

Lemma: (Townsend-Thorensen Lemma, Johnstone 1984)
If $a \ll b$ then $a \lhd \emptyset$ or $pos(b)$

where $a \ll b$ means that if $b \lhd U$ then there exists $b_1, \ldots, b_n \epsilon U$ such that $a \lhd b_1, \ldots, b_n$

Proof: We have $b \lhd b^+$ hence $a \lhd \emptyset$ (if $n = 0$) or $pos(b)$

This property characterizes locally compact spaces with a positivity predicate

Reals

For the real line any basic open (r, s) with $r < s$ is positive

By proof tree induction, if $(r, s) \lhd U$ then U is inhabited

Compact regular space

Do we need the positivity predicate?

If $f : X \to \mathbb{R}$ and X is compact
Then there exists N such that $X = f^{-1}(-N, N)$
We can compute *sup* f only if X has a positivity predicate

The right notion of compact Hausdorff space seems to be

compact regular space *with* a positivity predicate

Non decidable pos

Let R be a divisible lattice ordered abelian group with a strong unit 1
For any a in R there exists n such that $|a| \leqslant n$ where $|a| = a \vee (-a)$
a is *normable* if and only if there exists $||a||$ in $\mathbb{R}$ such that

$$||a|| < s \quad \leftrightarrow \quad \exists r > 0.\ |a| \leqslant s - r$$

A representation $\sigma : R \to \mathbb{R}$ is a map preserving $\vee, +$ and the unit 1
cf. Stone A General Theory of Spectra, N.A.S. 1940

Non decidable pos

We define the *spectrum* of R by the rules

$$D(a+b) \leqslant D(a) \vee D(b) \qquad D(1) = 1$$

$$D(a) \wedge D(-a) = 0 \qquad D(a \vee b) = D(a) \vee D(b)$$

and

$$D(a) = \bigvee_{r>0} D(a-r)$$

Intuitively

$$D(a) = \{\sigma \in R \to \mathbb{R} \mid \sigma(a) > 0\}$$

where $\sigma : R \to \mathbb{R}$ is a representation of R

Non decidable pos

Theorem: The spectrum of R has a posivity predicate iff any element of R is normable

We define $pos(a)$ by $||a|| > 0$

In Bishop mathematics, if any element of R is normable and R has a dense countable subset, then for any a such that $||a|| > 0$ we can find a representation $\sigma : R \to \mathbb{R}$ such that $\sigma(a) > 0$

Intuitionistic notions

Three important notions that are only relevant in an intitionistic framework

Positivity

Normability

Locatedness