Impredicative Definitions

One main goal of proof theory, since the debate between Poincaré and Russell, has been to analyse impredicative definitions.

Typically: a real number $x$ is defined as a set of rationals $q$ such that $q < x$. Given a formula $\phi(X)$, for instance

$$ (\exists q. X(q) \land q > 0) \land \forall q_1, q_2. q_1 < q_2 \rightarrow X(q_2) \rightarrow X(q_1) $$

the g.l.b. of the collection of all $X$ such that $\phi(X)$ is given by the predicate

$$ \forall X. \phi(X) \rightarrow X(q) $$

This predicate is defined by quantification over all possible predicates.
Impredicative Definitions

Such a definition looks circular (Poincaré)

In this case, the predicate can be rewritten as $q \leq 0$, so the circularity is only apparent
Impredicative Definitions

Takeuti formulated, in the 50s, a sequent calculus for second-order arithmetic, and conjectured cut-elimination.

He could prove cut-elimination for a restricted version to $\Pi^1_1$-comprehension.

Kreisel, by analysing the proof in a review, noticed that the argument can be represented in an intuitionistic system of inductive definitions.
Impredicative Definitions

Buchholz found a variation of the $\Omega$-rule that allows a more direct interpretation of $\Pi^1_1$-comprehension in terms of inductive definitions.

A particularly simple version of this reduction is obtained by showing the normalisation of a restricted fragment of system $F$ with only quantification over finite objects.
Impredicative Definitions

One main intuition can be found in Lorenzen (1958): it is possible to explain the classical truth of a statement

$$\forall X.\phi(X)$$

where $\phi$ does not have any quantification on predicates, by saying that

$$\phi(X)$$

is classically valid, where $X$ is a variable.

We know how to express this using inductive definitions.

For instance, it can be seen in this way that

$$\forall X.X(5) \rightarrow X(5)$$

is valid, without having to consider all subsets of $\mathbb{N}$.
Impredicative Definitions

To take another example, with

$$\phi(X) \equiv (\exists q. X(q) \land q > 0) \land \forall q_1, q_2. q_1 < q_2 \rightarrow X(q_2) \rightarrow X(q_1)$$

it is possible to show directly that

$$\vdash \phi(X) \rightarrow X(q)$$

is provable in $\omega$-logic, with $X$ variable predicate, iff $q \leq 0$.

Furthermore this reasoning will only involve inductive definitions, and not the explicit consideration of all subsets of $\mathbb{Q}$.
Impredicative Definitions

If

\[ \vdash X(q_0), \forall q > 0. \neg X(q), \exists p < q. X(q) \land \neg X(p) \]

is provable then, by inversion

\[ \vdash X(q_0), \neg X(q_1), \exists p < q. X(q) \land \neg X(p) \]

is provable for each \( q_1 > 0 \)

It is direct to see that if \( 0 < q_1 < q_0 \) then

\[ \vdash X(q_0), \neg X(q_1), \exists p < q. X(q) \land \neg X(p) \]

is not provable
A subsystem of system $F$

System $F$ was introduced by J.Y. Girard (1970) for giving a Dialectica interpretation of second-order arithmetic.

One can show the normalisation of system $F$, but only by using in the meta-language impredicative definitions

$$T ::= \alpha \mid T \to T \mid (\Pi \alpha)T$$

Each close type can be interpreted in a natural, but impredicative way, as a set of untyped $\lambda$-terms.

It is then direct to show that all terms in such a set are normalisable.
A subsystem of system $F$

This can be interpreted as a normalisation theorem for the following typing rules

\[
\begin{align*}
\Gamma \vdash x : T &\quad x : T \in \Gamma \\
\Gamma, x : T \vdash t : U &\quad \Gamma \vdash u : V \rightarrow T \quad \Gamma \vdash v : V \\
\Gamma \vdash \lambda x\ t : T \rightarrow U &\quad \Gamma \vdash u \ v : T \\
\Gamma \vdash t : (\Pi\alpha)T &\quad \Gamma \vdash t : T \\
\Gamma \vdash t : T[\!U\!] &\quad \Gamma \vdash t : (\Pi\alpha)T
\end{align*}
\]

where $\Gamma$ is a finite set of type declaration $x : T$, and in the last rule, $\alpha$ does not appear free in any type of $\Gamma$. 
A subsystem of system $F$

We consider the following types

$$T ::= \alpha \mid T \rightarrow T \mid (\Pi \alpha)T$$

where in the quantification, $T$ has to be built using only $\alpha$ and $\rightarrow$.

Then the normalisation theorem can be shown without impredicative definitions.
A subsystem of system $F$

General strategy: we define a Kripke model using only finite objects.

We interpret the usual proof of normalisation, interpreting each type as an $H$-valued predicate.

We build $H$ in such a way that, relative to this model, each impredicatively defined predicate required for this proof is *equivalent* to a predicate defined using only quantifications on finite objects.
References

