Infinite objects in constructive mathematics

Thierry Coquand

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Assume that the vector space $E$ is an ordered space which is a lattice (automatically distributive) and that it contains a special element $1$ which is a **strong unit**: for all $a \in E$, there exists $n$ such that $a \leq n$.1

**Example**: $C([0,1])$

Then we can define $N(r)$ by $x \in N(p/q)$ iff $qx \leq p.1$ and $-qx \leq p.1$

No reason why $|x| = \inf \{ r \mid x \in N(r) \}$ should computable (Dedekind real)

$x$ is **normable** iff $|x|$ is a Dedekind real
Riesz space

We can define the space of integrals $I(E)$: points of $Fn(E)$ such that $u(1) = 1$

We can replace $u(a) < r$ by $0 < u(r.1 - a)$.

Generators $I(a)$ and relations $I(a) = 0$ if $a \leq 0$ and

$I(a) \land I(-a) = 0, \ I(a + b) \leq I(a) \lor I(b), \ I(1) = 1$

$I(a) = \bigvee_{r>0} I(a - r.1)$
Spectrum of a Riesz space

We take the generators $D(a)$ and same relations

$D(a) = 0$ if $a \leq 0$

$D(a) \land D(-a) = 0$,  $D(a + b) \leq D(a) \lor D(b)$,  $D(1) = 1$

with the extra condition $D(a \lor b) = D(a) \lor D(b)$

We get a strongly normal lattice $Sp(E)$,

We add the relation $D(a) = \lor_{r>0} D(a - r)$

We get a compact space $X = Sp_r(E)$, subspace of $I(E)$. The space $I(E)$ can be thought of as the space of probability measure on $X$
We have a complete description of $Sp(E)$; notice that $a \in (p, q)$ is definable as $D(a - p.1) \land D(q.1 - b) = D((a - p.1) \land (q.1 - a))$

We take the set $P$ elements that are $\geq 0$ in $E$

We define the new relations $a \leq' b$ iff there exists $n$ such that $a \leq n.b$

$P$ for this relation is a distributive lattice, and this is a concrete description of $Sp(E)$

**Corollary:** We have $D(a) = 1$ in $Sp(E)$ iff there exists $n$ such that $1 \leq na$. 
Real spectrum of a Riesz space

If $X = Sp_r(E)$ then $X$ is compact regular.

There is a dense norm preserving injection $E \rightarrow C(X)$ (Stone-Weirstrass).

This is a representation theorem.

$X$ is overt iff all elements of $E$ are normable i.e. for all $x \in E$ we have that $|x|$ is a Dedekind real.

Using this, one can obtain a proof of Gelfand representation theorem (for commutative algebra of operators) in Bishop style mathematics, simpler than Bishop-Bridges' proof.
Let $L$ be a field (constructively $x = 0 \lor \exists y. xy = 1$)

We want a formal space whose points are the valuation rings of $L$

$$[x \in A] \land [y \in A] \leq [x + y \in A] \land [xy \in A]$$

$$1 = [x \in A] \lor [1/x \in A] \text{ if } x \neq 0$$

Interpret $[x \in A]$ symbolically: take the distributive lattice generated by these conditions

This defines a formal spectral space $V(L)$
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Space of valuations

More generally if $R$ is a subring of $L$ we define the space $V_R(L)$ of valuation rings containing $R$ by the theory

\begin{align*}
[x \in A] \land [y \in A] & \leq [x + y \in A] \land [xy \in A] \\
1 = [x \in A] \lor [1/x \in A] & \text{ if } x \neq 0 \\
1 = [x \in A] & \text{ if } x \in R
\end{align*}
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Space of valuations

**Theorem:** We have \([x_1 \in A] \land \cdots \land [x_n \in A] \leq [x \in A]\) in the space \(V_R(L)\) iff \(x\) is integral over \(R[x_1, \ldots, x_n]\)

In term of points: the intersection of all valuation rings containing \(x_1, \ldots, x_n\) is the set of elements integral over \(R[x_1, \ldots, x_n]\)
Let $R$ be an integral domain and $L = \text{Frac}(R)$

**Theorem:** The lattice map $D(x) \mapsto [1/x \in A]$ for $x \neq 0$ from the lattice $\text{Zar}(R)$ to the lattice $V_R(L)$ is conservative

This is called the center map

**Theorem:** If $R$ is arithmetical the center map is an isomorphism

$R$ arithmetical iff the lattice of ideals is distributive iff for any $x, y$ we can find $u, v, w$ such that $xu = yw, \ y(1-u) = xv$
Dedekind-Weber (1882); one early point-free description of a space

Let $L = \mathbb{Q}(x, y)$ with $y^2 = 1 - x^4$

We can consider the space $X$ of valuation rings containing $\mathbb{Q}$

This is a spectral space, and it has a formal covering $X = U_0 \cup U_1$

$U_0 = \{x \in A\} \quad U_1 = \{1/x \in A\}$
Riemann surface

$R_0$ integral closure of $\mathbb{Q}[x]$ in $L$

$R_1$ integral closure of $\mathbb{Q}[1/x]$ in $L$

**Theorem:** $R_0$ and $R_1$ are arithmetical ring

**Corollary:** $U_0 \equiv \text{Zar}(R_0)$ and $U_1 \equiv \text{Zar}(R_1)$
Towards formal sheaf theory

Over a space $\text{Zar}(R)$ we have a sheaf of rings, called the \textit{structure} sheaf $\mathcal{O}(D(a)) = R[1/a]$

If $R$ integral domain we have $\mathcal{O}(D(a_1, \ldots, a_n)) = R[1/a_1] \cap \cdots \cap R[1/a_n]$

The sheaf glueing property is what Henri calls the local-global principle

The structure $\text{Zar}(R), \mathcal{O}$ is called a (formal) \textit{affine} scheme
Towards formal sheaf theory

Over the space $X$ of valuations there is a natural sheaf $\mathcal{F}([u_1 \in A] \land \cdots \land [u_n \in A])$ is the integral closure of $\mathbb{Q}[u_1, \ldots, u_n]$

The fiber at the point $A$ is the ring $A$ itself!

Over the open $U_0 = [x \in A]$ the sheaf $\mathcal{F}$ reduces to the structure sheaf over the ring $R_0$.

Over the open $U_1 = [1/x \in A]$ the sheaf $\mathcal{F}$ reduces to the structure sheaf over the ring $R_1$.

We have a most natural example of a scheme: glueing of two affine scheme
Notice that the global sections of this sheaf are exactly the elements of $\mathbb{Q}$ since $\mathcal{O}(U_0)$ is elements integral over $\mathbb{Q}[x]$ and $\mathcal{O}(U_1)$ are elements integral over $\mathbb{Q}[1/x]$

This shows that this sheaf is not isomorphic to a structure sheaf of a ring

Indeed the global sections over a structure sheaf $\text{Zar}(R), \mathcal{O}$ form a ring isomorphic to $R$ itself
Towards formal (Čech) cohomology

If we have a space $X$ with a covering $X = U_0 \cup U_1$ and a sheaf $\mathcal{F}$ we can consider the map

$$\mathcal{F}(U_0) \oplus \mathcal{F}(U_1) \rightarrow \mathcal{F}(U_0 \cap U_1)$$

$$(a_0, a_1) \mapsto a_1|_{U_0 \cap U_1} - a_0|_{U_0 \cap U_1}$$

We define $H^1(U_0, U_1)$ the coker of this map

We say that $X, \mathcal{F}$ is acyclic iff $H^1(U_0, U_1) = 0$ for any covering $U_0, U_1$: any $b \in \mathcal{O}(U_0 \cap U_1)$ can be written of the form $a_1|_{U_0 \cap U_1} - a_0|_{U_0 \cap U_1}$
Towards formal cohomology

**Theorem:** *Any structure sheaf is acyclic*

**Theorem:** If $X = U_0 \cup U_1 = V_0 \cup V_1$ and $U_i, V_j$ are acyclic then $H^1(U_0, U_1)$ and $H^1(V_0, V_1)$ are isomorphic (as abelian group)
Towards formal cohomology

In this way one can define the genus of $L = \mathbb{Q}(x, y)$ as the dimension of the $\mathbb{Q}$ vector space $H^1([x \in A], [1/x \in A]) = H^1(X, \mathcal{O})$

This is an invariant of $L$ and is equal to 1 ($y/x$ is a generator)

It does not depend on the choice of the parameter $x$

**Theorem:** Over the field $K = \mathbb{Q}(t)$ we have $H^1([t \in A], [1/t \in A]) = 0$

**Corollary:** It is impossible to write $L$ of the form $\mathbb{Q}(t)$ with $t \in L$