What shall we do?

Analysis of chapter 8 of Bas Spitters' thesis

Motivated by the question: what are the algorithms behind these proofs??

Spectral theorems/representation theorems: what should be the definition of a compact space in constructive mathematics??

Use of enumerations, dependent choices entails a lot of *non canonical choices*. Can we avoid to have to make these choices??

Cf. the thought provoking review of Bridges "Constructive functional analysis." by Kreinovic MR 82k:03094

The Spectral Theorem

Two fundamental papers

M.H. Stone "A General Theory of Spectra I, II" 1940 Proc. N.A.S.

Algebraization of spectral theory

"Treatment of any system of real, simultaneously observable quantities as envisaged in the quantum theory"

What the spectral theorem says?

We have a commutative algebra R of operators (on a preHilbert space), we can consider R as a dense subalgebra of continuous functions $C(X)$ on a compact Hausdorff space $Sp(R)$

$Sp(R)$ can be seen as a set of maps $\phi : R \rightarrow \mathbb{R}$ such that

$$A \geq 0 \rightarrow \phi(A) \geq 0$$

Here we give a purely phenomenological description of $Sp(R)$

All the proofs here are constructive, most of them don't require dependent choices

Key Example

G compact group, $I : C(G) \to \mathbb{C}$ Haar measure

$$I(f) = \int f(x)dx$$

We have the convolution product on $C(G)$

$$(f \times g)(y) = \int f(x)g(x^{-1}y)dx$$

and scalar product

$$(f, g) = \int f(x)\overline{g(x)}dx$$

we write $g^*(x) = \overline{g(x^{-1})}$ and $|f|_2^2 = (f, f)$

Lemma 1

Lemma 1: The operator $T(f) : g \longmapsto f \times g$ is compact, and hence $T(f)$ is *normable*

The proof is elementary

Let B be the set of g such that $(g, g) \leq 1$

We prove that if $x_1, \ldots, x_n \in G$ then

$$\{(f \times g(x_1), \ldots, f \times g(x_n)) \mid g \in B\}$$

is totally bounded. Since $f \times g, g \in B$ is equicontinuous, the claim follows from Ascoli.

Key sublemma

Notice $f \times g(x) = (T(x)f, g)$ we are reduced to show, that in a preHilbert space

$$\{(h_1, g), \ldots, (h_n, g)) \mid g \in B\}$$

is totally bounded, which follows from the existence, for all $r > 0$ of a finite dimensional X such that $d(h_i, X) < r$

Lemma: In a preHilbert space for any $x_1, \ldots, x_n$ and $r > 0$ there exists a finite dimensional X such that $d(x_i, X) < r$

Proof: By induction on n

If we have X and x_{n+1} we do a case analysis on

$$d(x_{n+1}, X) < r \quad \vee \quad 0 < d(x_{n+1}, X)$$

Key Example (continued)

The elements of R are formal expressions $A = \lambda - f$ with $f \in Z(G)$ and $\lambda \in \mathbb{R}$

$$(\lambda - f)(\mu - g) = \lambda\mu - \lambda g - \mu f + f \times g$$

$A \geq 0$ iff $\lambda(g, g) \geq (f \times g, g)$ for all g

Lemma 2 (Riesz): if $A \geq 0$ and $B \geq 0$ and $AB = BA$ then $AB \geq 0$

Aside: center of $C(G)$

We let $Z(G)$ be the set of *central* functions $f(xy) = f(yx)$ *and* $f = f^*$

We have $f \times g = g \times f$ if $f \in Z(G)$

We have the explicit projection operator

$$P\ f\ x = \int f(y^{-1}xy)dy$$

such that $P\ f \in Z(G)$ if $f = f^*$ and

$$(f - P\ f, g) = 0$$

for all $g \in Z(G)$

Aside: center of $C(G)$

It is quite remarkable that the order on R can be defined without mention to the Haar measure. A direct definition is that $\lambda - f \geq 0$ iff

$$\Sigma f(x_i x_j^{-1}) r_i \overline{r_j} \leq \lambda(\Sigma r_i \overline{r_j})$$

for all $x_i \in G, r_i \in \mathbb{C}$

This ordering has been further analysed by Krein

Proof of Lemma 2

If $AB = BA$ and $A \geq 0, B \geq 0$ then $AB \geq 0$

We can assume $0 \leq A \leq 1$

We notice $BC^2 \geq 0$ since $(BC^2 g, g) = (BCg, Cg) \geq 0$

We define $A_0 = A, A_{n+1} = A_n - A_n^2$

One shows $0 \leq A_{n+1} \leq A_n \leq 1$ and $A_{n+1}^2 \leq A_n^2$

Since $A = A_1^2 + \ldots + A_n^2 + A_{n+1}$ we have $A_n^2 \to 0$

Key Example (continued)

Thus to a compact group G we associate an algebra R of elements of the form $A = \lambda - f,\ f \in Z(G)$

Because of lemma 1, all elements of R are *normable*

To R we shall associate a compact space $Sp(R)$, such that the elements A can also be seen as continuous functions on $Sp(R)$

$$\hat{A}(\phi) = \phi(A)$$

It will turned out that the space $Sp(R)$ has a positivity predicate (open locale)

Aside: centrum of $C(G)$

We are going also to define a formal space Σ of *characters* that are nonzero maps $\sigma : Z(G) \to \mathbb{C}$ such that

$$\sigma(f \times g) = \sigma(f)\sigma(g)$$

This space will be locally compact and *discrete*, and $Sp(R)$ is its Alexandrov compactification (we add one point)

It is very interesting to understand what discrete means here in a formal way

What is a point-free compact space?

A space is described as a *logical theory*

The Lindenbaum-Tarski algebra of this theory forms a distributive lattice (of basic open sets)

The models form a spectral space

The maximal models form a compact Hausdorff space if the lattice is *normal*

$u \ll v$ iff $(\exists x)[0 = ux \quad \& \quad 1 = v \vee x]$

normal: if $1 = a \vee b$ then $1 = a' \vee b$ for some $a' \ll a$

Example I

R commutative ring of elements $A, B, C, \ldots$

A subset of "positive" elements: R is an ordered group

A special element 1, so that R is divisible: for each $n > 0$ the equation $nX = 1$ has a solution and R is archimedian: for any $A \in R$ there exists k such that $A \leq k.1$

Finally, no "infinitesimal": if $n.A \leq 1$ for all n then $A \leq 0$

Spectral Space I

In the case of an ordered ring R we consider the theory T_1

1. $D(A), D(-A) \vdash$
2. $D(A+B) \vdash D(A), D(B)$
3. $D(A) \vdash$ if $A \leq 0$
4. $\vdash D(1)$
5. $D(A), D(B) \vdash D(AB)$
6. $D(AB) \vdash D(A), D(-B)$

The *models* of this theory define exactly a *total ordering* on R extending the given ordering

Spectral Space I

The Lindenbaum-Tarski algebra of T_1 is a distributive lattice L_1

The lattice L_1 is normal

Hence L_1 defines a compact Hausdorff space: the spectrum of R

One can completely characterise the order in L_1

For instance $D(A) \vdash D(B)$ iff we have $A^n(-B)^m \leq 0$ for some n, m

"Phenomenological" description of the spectrum $Sp(R)$ of R

Aside: space of characters

The *same* basic open will describe the space Σ of characters of $Z(G)$

Notice that the basic open of L_1 are of the form

$$D(\lambda - f)$$

An intuitive interpretation is that it represents the set of all characters σ such that

$$\sigma(f) < \lambda$$

This is a basic observation that we can make about a character σ

Spectral Space I

Proposition: (Krivine) If $1 \leq AB$ and $0 \leq A$ then there exists $r > 0$ such that $r \leq B$

From this follows

Main Theorem: We have $\vdash D(A)$ iff $A \geq r$ for some $r > 0$

The proof of the theorem is constructive, and similar to arguments used in proof theory (cut-elimination)

Stone-Weierstrass

Lemma 3: If $A \geq 0$ then there exists $B_n \geq 0$ such that $B_n^2 \to A$

The proof is elementary

Proof of Lemma 3

We can assume $0 \leq A \leq 1$

We define $B_0 = 0$ and $B_{n+1} = (1 - A + B_n^2)/2$

We define also $C_0 = 0, \;\; C_{n+1} = (1 + C_n^2)/2$

Then

$$0 \leq B_n \leq B_{n+1}, \;\; 0 \leq C_n \leq C_{n+1}, \quad B_{n+1} - B_n \leq C_{n+1} - C_n$$

$C_n \to 1$ and $(1 - B_n)^2 \to A$

Spectral Space II

If we Cauchy complete R we have an operation $A \vee B$

We can give another description of the spectrum

Inspired by F. Riesz "Sur la décomposition des opérations fonctionelles linéaires" 1928

The theory T_2 is

1. $D(A), D(-A) \vdash$
2. $D(A) \vdash$ if $A \leq 0$
3. $D(A+B) \vdash D(A), D(B)$
4. $D(A \vee B) \vdash D(A), D(B)$
5. $\vdash D(1)$

Spectral Space II

Actually, in the case we are analysing, it seems that we do not have to complete

$Z(G)$ should be itself closed under binary sup operations

This would mean that $Z(G)$ and R are natural example of *Riesz spaces*, i.e. ordered vector spaces that are lattices

The Spectrum as a Formal Space

For instance in T_2 one can show

$$D(A) \vee D(B) = D(A \vee B) \qquad D(A) \wedge D(B) = D(A \wedge B)$$

We have two descriptions T_1 and T_2 of two lattices that are *normal*.

They both define the *same* compact Hausdorff space $Sp(R)$, whose points are models of the corresponding theories with the extra "continuity" axiom

$$D(A) \vdash \bigvee_{r>0} D(A - r)$$

These points correspond to the maximal points in the spectral spaces

Spectral Theorem

The points of the spectrum can be also seen as continuous linear maps $\phi : R \to \mathbb{R}$ such that

$\phi(AB) = \phi(A)\phi(B)$ and $\phi(A \vee B) = \phi(A) \vee \phi(B)$

Main Theorem: We have $\phi(A) > 0$ for all ϕ iff $A \geq r$ for some $r > 0$

This can be proved constructively in a point-free way

Aside: elimination of choice sequences

What is the meaning of

For all $\phi \in Sp(R)$ we have $\phi(A) > 0$

in a point-free way???

Cf. introduction of Martin-Löf "Notes on Constructive Mathematics" and elimination of choice sequences

It means that

$$\vdash D(A)$$

is provable in the theory describing $Sp(R)$

Spectral Theorem

The spectral theorem in this point-free form holds without having to suppose that the elements in R are *normable* i.e. that

$$\{r > 0 \mid -r \leq A \leq r\}$$

has a g.l.b. $||A||$

In this sense, the statement is more general than in Bishop's (also R not given as an algebra of operators)

Also no separability hypotheses

BUT without extra-hypotheses we cannot "build" any points of $Sp(R)$. We know only that the theory describing $Sp(R)$ is consistent.

(It may be that for actual computations, this is all that is needed.)

Spectral Theorem

To connect this to Bishop-Bridges theory: if all elements of R are *normable* then $Sp(R)$ is *open* that is admits a *positivity predicate* defined by

$Pos(D(A))$ iff $||A^+|| > 0$ (written $A^+ > 0$)

This follows from

Lemma 4: $D(A) \ll D(B) \rightarrow [D(A) = 0 \vee Pos(D(B))]$

Using Pos, we can build (with dependent choices) as many points as we want if we can enumerate R

Intuitively, whenever $||A^+|| > 0$ we can build ϕ, effectively, but with maybe non canonical choices, such that $\phi(A) > 0$

Spectral Theorem

If we can enumerate a dense subset f_n of $Z(G)$ then we take $r_n \to 0$ and using dependent choices we build a sequence of rationals q_n such that

$$|f_0 - q_0| < r_0 \wedge |f_1 - q_1| < r_1 \wedge \ldots \wedge |f_n - q_n| < r_n$$

is positive

Given such a sequence we build then ϕ such that $|\phi(f_n) - q_n| < r_n$ for all n

Spectral Theorem

If all elements of R are normable, we have a much nicer formulation of the main theorem

Main Theorem: If $A \in R$ then $||A||$ is equal to the uniform norm of the continuous map

$$\hat{A} : C(Sp(R)) \to \mathbb{R} \qquad \phi \longmapsto \phi(A)$$

defined on the spectrum

This is Gelfand's theorem (for *real* C^*-algebras)

A discrete space??

In $Sp(R)$ there is a special point ϕ_0 such that

$$\phi_0(\lambda - f) = \lambda$$

The *space of characters* of G is the space Σ that we get by removing ϕ_0

We get Σ by adding the axiom

$$\vdash \vee_{f \in Z(G)} D(f)$$

A discrete space??

We prove first *with points* that Σ is discrete

That is for any given model σ of the theory Σ we build a function f_σ such that the open $D(f_\sigma)$ is the singleton $\{\sigma\}$

Here we give only the explicit formula: if $f \in Z(G)$ such that $\sigma(f) \neq 0$ then

$$\sigma(f)f_\sigma(x) = \sigma(P\ f_x)$$

where $f_x(y) = f(xy)$

A discrete space??

It is possible to show that $f_\sigma \times f_\sigma = f_\sigma$ and $D(f_\sigma) = \{\sigma\}$

But notice that f_σ is defined *in term of* σ

There is thus a kind a *circularity*: a basic open is defined in term of a point

Similar situation in intuitionism, when the definition of a spread may depend on a choice sequence

A discrete space??

We conjecture that without dependent choices, the space Σ may fail to have enough points

It is likely also that Σ has a natural measure that we can define in a point-free way, and that the corresponding Plancherel formula holds (even if we cannot have access to the points)

$$ {}_{G} |f|^2 dx = {}_{\Sigma} |\hat{f}|^2 d\sigma $$

With points this becomes

$$ {}_{G} |f|^2 dx = \Sigma |f_\sigma|^2 $$

Plancherel Formula??

The commutative algebra $Z(G)$ with the map

$$I : Z(G) \to \mathbb{R} \quad I(f) = f(e)$$

is a (constructive) example of an *integration algebra* (Segal)

The map I is positive: $I(f) \geq 0$ if $\hat{f} \geq 0$

I can be seen as a *measure* on the point-free space Σ

For this measure, the corresponding Plancherel formula holds

Enough characters??

In a point-free way, we expect that we can express most of the known theorems about irreducible representations

For instance the set of functions $f \in C(G)$ such that

$$f_\sigma \times f = f$$

should be a finite dimensional space

Such a statement makes sense over the space Σ

It can be expected that, for applications, we need only to talk about a generic character, and not to build all characters effectively

λ-notation

We just illustrate the use of λ-notation in the proof and statement similar to lemma 3.4 of Bishop-Bridges

Lemma: If $F : C(G) \to \mathbb{C}$ is continuous then

$$F(f \times g) = \quad f(x^{-1})F(g^x)dx$$

Proof: We consider $h(x, y) = f(x^{-1})g(xy)$. The lemma can be expressed as

$$F(\lambda y.I(\lambda x.h(x, y))) = I(\lambda x.F(\lambda y.h(x, y)))$$

We only have to check it in the case where $h(x, y) = u(x)v(y)$, since the functions of the form $\Sigma_i u_i(x)v_j(y)$ are dense in $C(G \times G)$, by Stone-Weierstrass and it is direct in this case.