Presheaf and sheaf models of type theory

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Goal of the talk

I am going to present some models of *univalent type theory*

These models can be used to extract some *proof theoretic* informations on this formal system

-what is its proof theoretic strength?

-consistency of the notion of higher inductive types?

-independence results, e.g. countable choice cannot be proved

-consistency results, e.g. consistency with Brouwer's fan theorem

Goal of this talk

The first part (proof theoretic power) is part of joint work with

Simon Huber, Marc Bezem, Anders Mörtberg, Cyril Cohen

with contributions from Dan Licata, Ian Orton, Andy Pitts, Nicola Gambino, Christian Sattler

The second part (sheaf models) is work in progress, from several discussions with Christian Sattler and previous joint work with Bassel Mannaa and Fabian Ruch

Univalent type theory

These models will be “inner” models inside suitable *presheaf models* of type theory (inspired from Voevodsky’s simplicial set model)

For suitable base category, we can then consider further internal models, which can be seen as *sheaf* models for type theory

These models can be seen as generalization of sheaf models for simple type theory

Sheaf models for simple type theory used for independence or consistency results can in this way be extended to univalent type theory

All these definitions will take place in a constructive set theory known to have the same proof theoretic strength as dependent type theory

Basic type theory

$\Pi(x : A)B$ and $\Sigma(x : A)B$ given a family of types B over $x : A$

Special case: $A \to B$ and $A \times B$ (if B is a constant family)

Inductive data types: boolean, natural numbers N, lists (finitary) and ordinal notations, $W(x : A)B$ (infinitary)

Use propositions-as-types to express logical operations

Universes and Identity types

One needs to add two notions to these basic operations

(1) universes

(2) identity types or (maybe better?) *identification types*

One also adds two new principles

-univalence axiom

-propositional truncation

Universes

By analogy with the notion of Grothendieck universes, one adds types $U_0, U_1, U_2, \ldots$ with the rules

A is a type if $A : U_k$

$A : U_{k+1}$ if $A : U_k$

$U_k : U_{k+1}$

$\Pi(x : A)B : U_k$ if $A : U_k$ and $B : U_k \ (x : A)$

$\Sigma(x : A)B : U_k$ if $A : U_k$ and $B : U_k \ (x : A)$

Universes

This is used to represent collection of structures

E.g. $\Sigma(X : U_0)X \times (X \to X)$ collection of small types with one constant and one unary function

Universes

There are remarkable mutual interpretation of this type theory with extensions of CZF, using the representation of *sets-as-trees*

$\mathrm{CZF}^{+}\mathrm{u}_{<\omega} = \mathrm{CZF} + \mathrm{REA} +$ a cumulative sequence of inaccessible sets

On relating type theories and set theories

Peter Aczel, Proceedings of TYPES 1998, pp. 1-18

Identification types

If we want to express the collection of all small rings, or all small groups we need a notion of equality

Id $A\ a_0\ a_1$ type of possible *identifications* of a_0 and a_1

Using this notion, we can introduce Voevodsky's stratification

isProp A is $\Pi(a_0\ a_1 : A)$ Id $A\ a_0\ a_1$

isSet A is $\Pi(a_0\ a_1 : A)$ isProp (Id $A\ a_0\ a_1$)

isGroupoid A is $\Pi(a_0\ a_1 : A)$ isSet (Id $A\ a_0\ a_1$)

Identification types

The type of small *semigroups* will be

$G = \Sigma(X : U_0)(\Sigma(f : X \to X \to X)\ A(X, f)) \times \mathsf{isSet}\ X$

where

$A(X, f) = \Pi(x_0\ x_1\ x_2 : X)\mathsf{Id}\ X\ (f\ x_0\ (f\ x_1\ x_2))\ (f\ (f\ x_0\ x_1)\ x_2)$

One can prove $G : U_1$

An element of this type is an object of the form $(X, (f, q), p)$ where $X : U_0$ and p is a proof that X is a set and f a binary operation and q is a proof that this operation is associative

Identification types

One would expect $A(X, f)$ to be a *proposition* and G to be a *groupoid*

An element of Id $G\ (X_0,(f_0,q_0),p_0)\ (X_1,(f_1,q_1),p_1)$ should represent an *isomorphism* between the semigroups X_0 and X_1

This holds, but only as a consequence of the *univalence axiom*

Univalence axiom

For $f : A \to B$ and $b : B$ define $\mathsf{Fiber}(f, b) = \Sigma(a : A)\mathsf{Id}\ B\ b\ (f\ a)$ and

$\mathsf{isEquiv}(f) = \Pi(b : B)\mathsf{isContr}\ \mathsf{Fiber}(f, b)$

where

$\mathsf{isContr}\ T = T \times \mathsf{isProp}\ T$

and then $\mathsf{Equiv}\ A\ B = \Sigma(f : A \to B)\mathsf{isEquiv}(f)$

The Univalence Axiom can be stated as

the canonical map $\mathsf{Id}\ U_k\ A\ B \to \mathsf{Equiv}\ A\ B$ *is an equivalence*

Univalence axiom

How does it compare to *simple type theory* as formulated by Church?

In simple type theory, we cannot express the notion of *arbitrary* structures that we can express using universes

The univalence axiom can be seen as a generalization of Church's extensionality principle for *propositions*: two equivalent propositions are equal

It also can be seen as providing an "explanation" of what should be the notion of identification for universes: it should be given by *equivalences*

Identification and transport

These notions were analysed in Bourbaki

Théorie des Ensembles, Chapitre 4, Structures (1957)

The discovery of *isomorphisms* between seemingly different structures and the fact that we can transport results/notions from one structure to another corresponds often to key steps in mathematics

This has been refined with the notion of *equivalences*

Propositional truncation

Operation $\|A\|$ on types

$\text{isProp}\ \|A\|$

$A \to \|A\|$

$((A \to B) \times \text{isProp}\ B) \to (\|A\| \to B)$

$\|A\|$ expresses that A is *inhabited*

We can introduce new quantification $\exists(x : A)B$ defined as $\|\Sigma(x : A)B\|$

Countable choice

$\Pi(A : N \to U_0)\ (\Pi(n : N)\ \|A\ n\|) \to \|\Pi(n : N) A\ n\|$

If we take $A\ n$ of the form $\Sigma(y : B) R(n, y)$ we get

$(\Pi(n : N) \exists(y : B) R(n, y)) \to \exists(f : N \to B) \Pi(n : N) R(n, f(n))$

which is a way to express countable choice

We can build a model with a particular family A where

-the hypothesis $\Pi(n : \mathsf{N})\ \|A\ n\|$ holds

-the conclusion $\|\Pi(n : N) A\ n\|$ does not hold

Univalence axiom and propositional truncation

One gets a formal system with notations and concepts appropriate for representing some abstract notions used in mathematics

Expresses some general laws of the notion of identification coming from mathematical practice (Voevodsky)

E.g. representation of additive/abelian categories, category of complexes of an additive category, homotopy of complexes, triangulated categories (Tomi Pannila)

Is it consistent to add these new operations to type theory?

Simplicial set model

For interpreting *one* univalent universe: requires ZFC + *two* Grothendieck universes

Natural question: can one modify this model so that it can be expressed in a weaker formal system?

Models of identification type and univalence

The only known models (so far) rely in an essential way on ideas coming from homotopy theory, interpreting the type of identifications as a type of *paths*

Grothendieck's intuition that the laws underlying the notion of identifications in mathematics are similar to the laws underlying homotopy theory

Awodey-Warren's homotopic interpretation of the laws of identity type

Models of identification type and univalence

-definition of *presheaf models* of type theory

-we assume that there is a special presheaf $\mathbb{I}$ which will play the role of an interval

-using the interval we can isolate the types having a refined form of the path lifting property (which has been isolated in homotopy theory)

-the types having this extension property form a model of type theory

Models of identification type and univalence

How does it compare with Gandy and Takeuti's model of extensionality principles in simple type theory?

R. Gandy *On axiomatic systems in mathematics and theories in physics* PhD thesis, University of Cambridge, 1953

Internal model: defines a relation by induction on the types (logical relation)

Proves by induction on the type that this is an equivalence relation

The first step corresponds to the presheaf model and the second step to checking the homotopy extension property by induction on the type

Presheaf models of type theory

We work in a (constructive) set theory with universes $\mathcal{U}_0 \in \mathcal{U}_1 \in \dots$

We have a base category $\mathcal{C}$ in $\mathcal{U}_0$

We write $I, J, K, \dots$ the objects of $\mathcal{C}$

$Yo(I)$ denotes the presheaf represented by I

A context $\Gamma, \Delta, \dots$ is a $\mathcal{U}_k$-presheaf (for some k) on $\mathcal{C}$

$\mathsf{Type}_n(\Gamma)$ set of $\mathcal{U}_n$-presheaves on the category of elements of Γ

$\mathsf{Elem}(\Gamma, A)$ set of global sections of $A \in \mathsf{Type}_n(\Gamma)$

Presheaf models of type theory

Composition gives a substitution operation $A\sigma$ in $\mathsf{Type}_n(\Delta)$ if $\sigma : \Delta \to \Gamma$

Similarly, we define $a\sigma$ in $\mathsf{Elem}(\Delta, A\sigma)$ if a is in $\mathsf{Elem}(\Gamma, A)$ and $\sigma : \Delta \to \Gamma$

We have a canonical context extension operation $\Gamma.A$ for A in $\mathsf{Type}_n(\Gamma)$

$p : \Gamma.A \to \Gamma$ and q in $\mathsf{Elem}(\Gamma.A, Ap)$

Any $\mathcal{U}_n$-presheaf F defines a constant family $\overline{F} \in \mathsf{Type}_n(\Gamma)$

Presheaf model of type theory: universes

Type_n with substitution defines a presheaf on the category of contexts

It is *continuous* and hence *representable* by $U_n(I) = \mathsf{Type}_n(Yo(I))$

We have natural bijections $\mathsf{Type}_n(\Gamma) \simeq \Gamma \to U_n \simeq \mathsf{Elem}(\Gamma, \overline{U_n})$

Definition due to Martin Hofmann and Thomas Streicher

Presheaf model of type theory: universes

Note that this does *not* work with sheaves

The problem is how to model the *universes*

《The collection of sheaves don't form a sheaf》

If we define $F(V)$ to be the collection of all $\mathcal{U}$-sheaves on V then F is a presheaf which is not a sheaf in general, since glueing will only be defined up to isomorphism

This basic fact was the motivation for the notion of *stacks*

We need a notion of identification

Base category

There are several possible choices for the base category

What matters is that we have a *segment* i.e. a presheaf $\mathbb{I}$ with two distinct elements 0 and 1 satisfying

(1) $\mathbb{I}$ has a connection structure, i.e. maps $(\wedge),\ (\vee) : \mathbb{I} \rightarrow \times\mathbb{I} \rightarrow \mathbb{I}$ satisfying $x \wedge 1 = x = 1 \wedge x,\ x \wedge 0 = 0 = 0 \wedge x$ and $x \vee 1 = 1 = 1 \vee x,\ x \vee 0 = x = 0 \vee x$ and

(2) We have a functor J^+ on $\mathcal{C}$ with a natural isomorphism $Yo(J^+) \simeq Yo(J) \times \mathbb{I}$

We get a notion of path by exponentiation to this interval $\mathbb{I}$

Base category

The axiomatic conditions required for getting a model of type theory have been analysed by Ian Orton and Andy Pitts

Axioms for Modelling Cubical Type Theory in a Topos, CSL 2016

A complementary analysis can be found in

The Frobenius condition, right properness, and uniform fibrations
Nicola Gambino and Christian Sattler, Journal of Pure and Applied Algebra, 221 (12), 2017, pp. 3027-3068.

«Inner» models

Using the segment $\mathbb{I}$ we can define a set of «filling structures» Fill(Γ, A), inspired from homotopy theory

An element of Fill(Γ, A) represents a generalized «path lifting» operation

It expresses that the type of all path liftings is a singleton up to homotopy (for a given path in the base and starting point)

«Inner» models

Define $\mathsf{Fib}_n(\Gamma)$ in $\mathcal{U}_{n+1}$

$\mathsf{Fib}_n(\Gamma)$ set of pairs (X, c) with $X \in \mathsf{Type}_n(\Gamma)$ and $c \in \mathsf{Fill}(\Gamma, X)$

$\mathsf{Elem}_{\mathsf{F}}(\Gamma, (X, c)) = \mathsf{Elem}(\Gamma, X)$

We get a new «proof relevant» inner model of the presheaf model

«Inner» models

We can lift the product operation at this level, using the connection structure on the interval

$\pi(c_A, c_B) \in \mathsf{Fill}(\Gamma, \Pi(A, B))$ if $c_A \in \mathsf{Fill}(\Gamma, A)$ and $c_B \in \mathsf{Fill}(\Gamma.A, B)$

Furthermore $\pi(c_A, c_B)\sigma = \pi(c_A\sigma, c_B(\sigma p, q))$

We can define a product operation for this new model

$\Pi((A, c_A), (B, c_B)) = (\Pi(A, B), \pi(c_A, c_B))$

We don't need to change the abstraction and application operations

$\mathsf{Elem}_{\mathsf{F}}(\Gamma, (X, c)) = \mathsf{Elem}(\Gamma, X)$ $\qquad\qquad$ $\Gamma.(X, c) = \Gamma.X$

⟪Inner⟫ models

What about universes? This is where the second condition on the interval is used

Fib_n *is* continuous *and hence* representable *by* $F_n(I) = \mathsf{Fib}_n(Yo(I))$

We have a natural isomorphism $\Gamma \to F_n \simeq \mathsf{Fib}_n(\Gamma)$

We can then build c_n in $\mathsf{Fill}(\Gamma, \overline{F_n})$

In this way we define $\mathsf{U}_n = (\overline{F_n}, c_n)$ in $\mathsf{Fib}_{n+1}(\Gamma)$

Theorem: *We get a model of type theory with the univalence axiom and higher inductive types*

Presheaf extension of the cubical set model

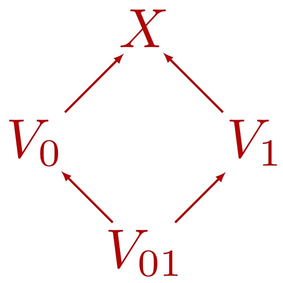
Given another category $\mathcal{D}$ in $\mathcal{U}_0$ with objects $X, V, L, \ldots$ we now consider a new model, where the base category is now $\mathcal{D} \times \mathcal{C}$

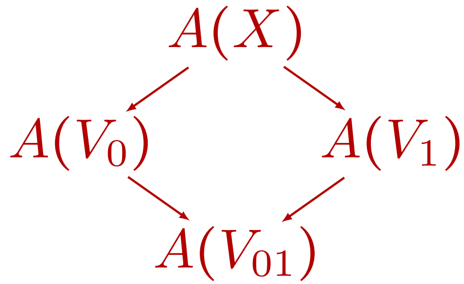
This is similar to iterated forcing

A context Γ is given by a family of sets $\Gamma(X|I)$ with restriction maps

$\mathbb{I}_{\mathcal{D}}(X|J) = \mathbb{I}(J)$ defines an interval for this presheaf extension

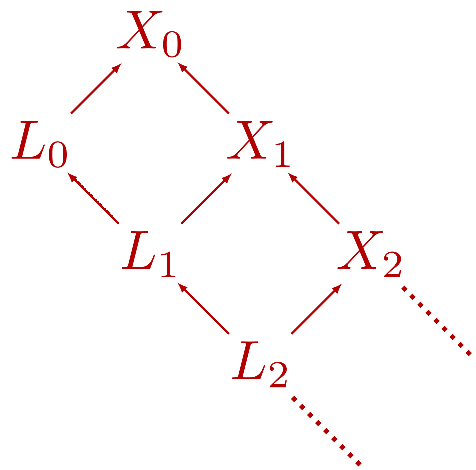
Some examples





Some examples: Countable choice

We now consider the following space, where X_n is covered by L_n and X_{n+1}



Sheaf models as internal models

In all these models, we can express the notion of sheaf internally in the presheaf model

We get a type operation $S(X)$ which expresses that the presheaf X is a sheaf (and which is a proposition)

“Any compatible collection of local data can be glued in a unique way”

This can be expressed (internally) by the fact that some maps are equivalences

Sheaf models as internal models

Furthermore this operation satisfies the following closure conditions

$c_\Pi : (\Pi(x : A)S(B)) \rightarrow S(\Pi(x : A)B)$

$c_\Sigma : S(A) \times (\Pi(x : A)S(B)) \rightarrow S(\Sigma(x : A)B)$

$c_{\mathsf{Path}} : S(A) \rightarrow \Pi(a_0\ a_1 : A)S(\mathsf{Path}\ A\ a_0\ a_1)$

$c_{U_k} : S(\Sigma(X : U_k)S(X))$

Whenever we have such an operation, we can define a new model of type theory by internalisation, where a type is now interpreted by a type with a proof that this type is a sheaf

Sheaf models as internal models

Inspired from *An effectful way to eliminate addiction to dependences*, P.-M. Pédrot and N. Tabareau, LICS 2017

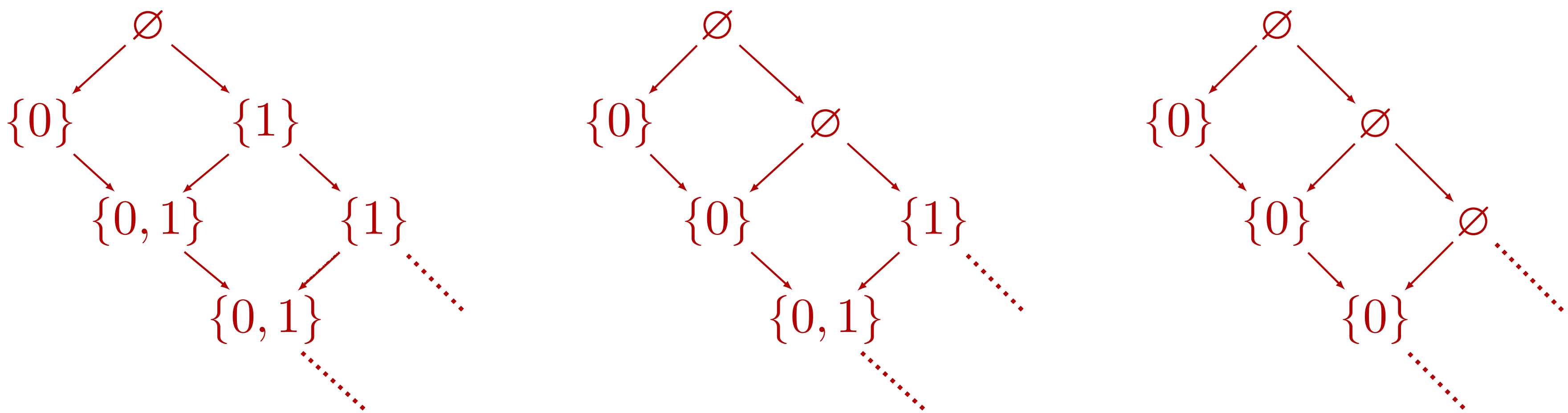
$$
\begin{array}{lcl}
[x] & = & x \\
[M\ N] & = & [M]\ [N] \\
[\lambda(x:A)M] & = & \lambda(x:[A].1)[M] \\
[\Pi(x:A)B] & = & (\Pi(x:[A].1)[B].1,\ c_\Pi\ (\lambda(x:[A].1)[B].2)) \\
[\Sigma(x:A)B] & = & (\Sigma(x:[A].1)[B].1,\ c_\Sigma\ [A].2\ (\lambda(x:[A].1)[B].2)) \\
[U_k] & = & (\Sigma(X:U_k)S(X),\ c_{U_k})
\end{array}
$$

Sheaf models as internal models

We get in this way new models of univalence

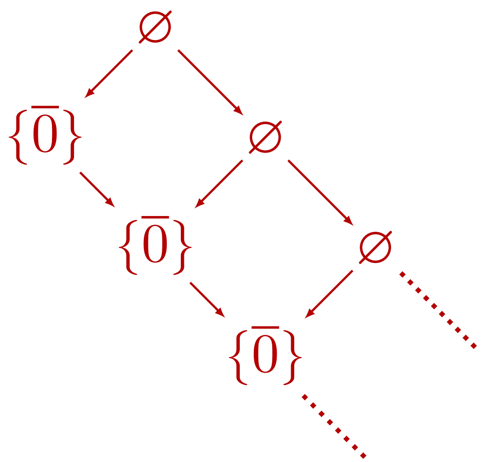
Example 2: Countable choice

We then can define a family of sets (stacks) $A\ n$, e.g. for $A\ 0,\ A\ 1$ and $A\ 2$



Example 2: Countable choice

$\Pi(n:N)A\ n$ is (a proposition) is *not* globally inhabited and $\|A\ n\|$ *is* globally inhabited *because* of the stack condition



Example 3: Markov principle

Let C be the Boolean algebra corresponding to Cantor space

The base category is the poset of nonzero elements of C

A covering is a partition of unity.

Theorem: *Markov's principle does not hold in the corresponding stack model of type theory. Actually, its* negation *holds (Bassel Mannaa).*

Corollary: *Markov's principle cannot be proved in type theory with univalence*

Example 4: Fan theorem

Let $\mathcal{D}^{op}$ be a full subcategory of the category of Boolean algebra having for objects localizations of finite power of C

A covering of an object is given by a partition of unity and corresponding localizations (Zariski topology)

Lemma: $2(B|J) = B$ *and* 2^N *is represented by* $(C, \varnothing)$

Theorem: *Brouwer's fan theorem holds in the corresponding stack model of type theory*

Conclusion

For representing in a natural way collections of mathematical objects, it seems necessary to extend simple type structure with universes

Understanding what notion of identification we should have on these universes seems to be involve ideas similar to the ones used in the theory of homotopy