Dependent type theory

$$
\begin{array}{lcll}
\Gamma, \Delta & ::= & () \mid \Gamma, x : A & \text{Contexts} \\
 & & & \\
t, u, A, B & ::= & x & \\
 & \mid & \lambda x.\ t \mid t\ u \mid (x : A) \to B & \Pi\text{-types} \\
 & \mid & (t, u) \mid t.1 \mid t.2 \mid (x : A) \times B & \Sigma\text{-types}
\end{array}
$$

We write

$A \to B$ for the non-dependent product type and

$A \times B$ for the non-dependent sum type

Identity types

Inductive family with one constructor $\mathsf{refl}\ a : \mathsf{Id}\ A\ a\ a$

In general $(n : \mathsf{N}) \to \mathsf{Id}\ \mathsf{N}\ (f\ \ n)\ (g\ n)$ does not imply $\mathsf{Id}\ (\mathsf{N} \to \mathsf{N})\ f\ \ g$

We can then have f, g of type $\mathsf{N} \to \mathsf{N}$

-$(n : \mathsf{N}) \to \mathsf{Id}\ \mathsf{N}\ (f\ \ n)\ (g\ n)$ has a proof, i.e. f and g are pointwise equal

-$P(f)$ has a proof

-$P(g)$ does not have a proof

Example: $P(h) \quad = \quad \mathsf{Id}\ (\mathsf{N} \to \mathsf{N})\ f\ \ h$

Identity types

In the 1989 *Programming Methodology Group* meeting (Båstad), D. Turner suggested an extension of type theory with function extensionality, adding a new constant of type

$$((x : A) \to \mathsf{Id}\ B\ (f\ x)\ (g\ x)) \quad \to \quad \mathsf{Id}\ ((x : A) \to B)\ f\ g$$

that's one appeal of functional programming, that you can code a function in two different ways and know that they are interchangeable in all contexts

New axiom, how to make sense of it?

Identity types

you can make perfectly good sense of these axioms, but you will do that in a way which is analogous to what I think Gandy was the first to give: an interpretation of extensional simple type theory into the intensional version of simple type theory ... You can formulate an extensional version of type theory and make sense of it by giving a formal interpretation into the intensional version

P. Martin-Löf, from a recorded discussion after D. Turner's talk

R. Gandy's 1953 PhD thesis
On Axiomatic Systems in Mathematics and Theories in Physics

Identity types

Elimination rule $x : A,\ p : \mathsf{Id}\ A\ a\ x$

$C(a, \mathsf{refl}\ a)\ \rightarrow C(x, p)$

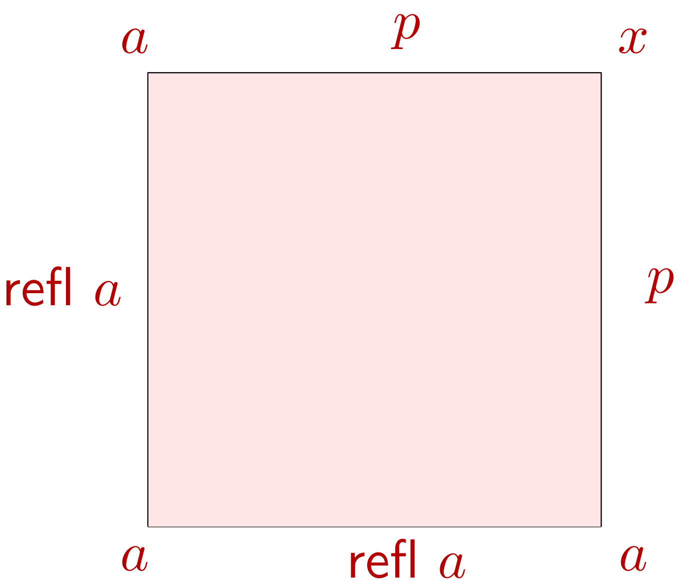
Special case

$C(a) \rightarrow C(x)$

For getting the general elimination rule from the special case, we need

$\mathsf{Id}\ ((x : A) \times \mathsf{Id}\ A\ a\ x)\ (a, \mathsf{refl}\ a)\ (x, p)$

Singleton types are contractible



Any element x, p in the type $(x : A) \times \mathsf{Id}\ A\ a\ x$ is equal to $a, \mathsf{refl}\ a$

Loop space

"Indeed, to apply Leray's theory I needed to construct fibre spaces which did not exist if one used the standard definition. Namely, for every space X, I needed a fibre space E with base X and with trivial homotopy (for instance contractible). But how to get such a space? One night in 1950, on the train bringing me back from our summer vacation, I saw it in a flash: just take for E the space of paths on X (with fixed origin a), the projection $E \to X$ being the evaluation map: path $\to$ extremity of the path. The fibre is then the loop space of (X, a). I had no doubt: this was it! $\ldots$ *It is strange that such a simple construction had so many consequences.*"

J.-P. Serre, describing the *loop space method* from his 1951 thesis

Problems for making sense of extensionality

-The equality type can be iterated Id (Id A a b) p q

-Internalisation: the constant for extensionality should satisfy extensionality

-How to express extensionality for universes?

We remark, however, on the possibility of introducing the additional axiom of extensionality, $p \equiv q \supset p = q$*, which has the effect of imposing so broad a criterion of identity between propositions that they are in consequence only two propositions, and which, in conjunction with* $10^{\alpha\beta}$*, makes possible the identification of classes with propositional functions* (A. Church, 1940)

Equivalence

Remarquable refinement of the notion of logical equivalence (Voevodsky 2009)

$\mathsf{isContr}\ B = (b : B) \times ((y : B) \to \mathsf{Id}\ B\ b\ y)$

$\mathsf{isEquiv}\ T\ A\ w = (a : A) \to \mathsf{isContr}((t : T) \times \mathsf{Id}\ A\ (w\ t)\ a)$

$\mathsf{Equiv}\ T\ A = (w : T \to A) \times \mathsf{isEquiv}\ T\ A\ w$

Generalizes in an uniform way notions of

-logical equivalence between propositions

-bijection between sets

-(categorical) equivalence between groupoids

Equivalence

The proof of

isEquiv $A\ A\ (\lambda x.x)$

is exactly the proof that "singleton" types are contractible

So we have a proof of

Equiv $A\ A$

Stratification

proposition $(x : A) \to (y : A) \to \mathsf{Id}\ A\ x\ y$

set $(x : A) \to (y : A) \to \mathsf{isProp}\ (\mathsf{Id}\ A\ x\ y)$

groupoid $(x : A) \to (y : A) \to \mathsf{isSet}\ (\mathsf{Id}\ A\ x\ y)$

Hedberg's Theorem: *A type with a decidable equality is a set*

One of the first result in the formal proof of the 4 color Theorem

univalence axiom

the canonical map $\mathsf{Id}\ U\ A\ B \rightarrow \mathsf{isEquiv}\ A\ B$ *is an equivalence*

This generalizes A. Church's formulation of "propositional" extensionality

two logically equivalent propositions are equal

This is *provably* equivalent to

$\mathsf{isContr}\ ((X : U) \times \mathsf{isEquiv}\ X\ A)$

Gandy's interpretation

"setoid" interpretation

A type with an equivalence relation

To each type we associate a relation, and show by induction on the type that the associated relation is an equivalence *relation*

Gandy's interpretation

This can be seen as an "internal version" of Bishop's notion of *sets*

A set *is defined when we describe how to construct its members and describe what it means for two members to be equal*

The equality relation *on a set is conventional: something to be determined when the set is defined, subject only to the requirement that it be an equivalence relation*

Mines, Richman and Ruitenburg *A Course is Constructive algebra*

Gandy's interpretation for type theory

Actually we have to use a more complex notion than Bishop's notion of sets

Propositions-as-types:

each $R(a, b)$ should itself be a *type*, with its own notion of equality

So, what we need to represent is the following "higher-dimensional" notion:

a collection, an equivalence relation on it, a relation between these relations, and so on

And we need a corresponding "higher-order" version of equivalence relations

Cubical sets

A cubical set is a “higher-order” version of a binary relation

Representation using the notion of *presheaf*

Idea originating from Eilenberg and Zilber 1950 (for simplicial sets)

We are going to consider a *presheaf extension* of type theory

Cubical sets as presheafs

The idea is to allow elements and types to depend on "names"

$$u(i_1, \ldots, i_n)$$

Purely *formal* objects which represent elements of the unit interval $[0, 1]$

Cubical sets as presheafs

At any point we can do a "re-parametrisation"

$$i_1 = f_1(j_1, \ldots, j_m)$$

$$\ldots$$

$$i_n = f_n(j_1, \ldots, j_m)$$

We then have

$$a(i_1, \ldots, i_n) = a(f_1(j_1, \ldots, j_m), \ldots, f_n(j_1, \ldots, j_m))$$

Cubical sets, reformulated

$i, j, k, \ldots$ formal symbols/names representing abstract *directions*

New context extension $\Gamma, i : \mathbb{I}$

If $\vdash A$ then $i : \mathbb{I} \vdash t : A$ represents a line

$$t(i/0) \xrightarrow{\quad t \quad}_i t(i/1)$$

in the direction i

$i : \mathbb{I}, j : \mathbb{I} \vdash t : A$ represents a square, and so on

Cubical sets

Extension of ordinary *type theory* e.g. the rules for introduction and elimination of function is the same as in ordinary type theory

$$\frac{\Gamma \vdash w : (x : A) \to B \qquad \Gamma \vdash u : A}{\Gamma \vdash w\ u : B(x/u)} \qquad \qquad \frac{\Gamma, x : A \vdash v : B}{\Gamma \vdash \lambda x.v : (x : A) \to B}$$

Simpler than in *set theory*

Compare with definition of exponential of two presheafs: an element t in $G^F(I)$ is a a family of functions $t_f : F(J) \to G(J)$ such that $(t_f\ u)g = t_{fg}\ (ug)$ for $f : J \to I$ and $g : K \to J$ in the base category

Cubical types

We can introduce a new type, the type of *paths* Path $A\ a_0\ a_1$

New operations: *name* abstraction and application

Cubical types

$$\frac{\Gamma \vdash A \qquad \Gamma, i : \mathbb{I} \vdash t : A}{\Gamma \vdash \langle i \rangle t : \mathsf{Path}\ A\ t(i/0)\ t(i/1)}$$

$$\frac{\Gamma \vdash p : \mathsf{Path}\ A\ a_0\ a_1}{\Gamma, i : \mathbb{I} \vdash p\ i : A}$$

$$\frac{\Gamma \vdash p : \mathsf{Path}\ A\ a_0\ a_1}{\Gamma \vdash p\ 0 = a_0 : A} \qquad \frac{\Gamma \vdash p : \mathsf{Path}\ A\ a_0\ a_1}{\Gamma \vdash p\ 1 = a_1 : A}$$

Cubical types

$$t(0) \xrightarrow{\quad t(i) \quad}_i t(1)$$

A line in the direction i

$$t(0) \xrightarrow{\quad \langle i \rangle t(i) \quad} t(1)$$

A line where the direction is *abstracted away*

Cubical types

Reflexivity is provable

$(x : A) \to \mathsf{Path}\ A\ x\ x$

$\lambda x.\langle i \rangle x$

If $f : A \to B$ we have

$\mathsf{Path}\ A\ a_0\ a_1 \to \mathsf{Path}\ B\ (f\ a_0)\ (f\ a_1)$

$\lambda p.\langle i \rangle f\ (p\ i)$

Cubical types

Function extensionality is provable

$$((x : A) \rightarrow \mathsf{Path}\ B\ (f_0\ x)\ (f_1\ x)) \quad \rightarrow \quad \mathsf{Path}\ ((x : A) \rightarrow B)\ f_0\ f_1$$

$$\lambda p. \langle i \rangle \lambda x.\ p\ x\ i$$

Cubical types

We can in this way formulate a presheaf extension of type theory

In this extension, each type has a *cubical* structure: points, lines, squares, ...

Recall Gandy's extensionality model

To each type we associate a relation, and show by induction on the type that the associated relation is an equivalence relation

Can we do the same here, e.g. can we show transitivity of the relation corresponding to the type Path $A\ a_0\ a_1$ by induction on A?

Cubical types

We can actually prove it, but we need to prove by induction a stronger property

Box principle: *any open box has a lid*

This generalizes the notion of equivalence relation

First formulated by D. Kan *Abstract homotopy I* (1955)

Suggested by algebraic topology (Alexandroff and Hopf 1935, Eilenberg 1939)

if X *is a subpolyhedron of a bottom lid* $C = [0,1]^n$ *then* $(X\times[0,1])\cup(C\times\{0\})$ *is a retract of* $C \times [0,1]$

Cubical types

For formulating the box principle, we add a new *restriction* operation

Γ, ψ where ψ is a "face" formula

If $\Gamma \vdash A$ and $\Gamma, \psi \vdash u : A$ then u is a *partial element* of A of *extent* ψ

If $\Gamma, \psi \vdash T$ then T is a *partial type* of extent ψ

Face lattice

$$
i : \mathbb{I}, j : \mathbb{I}, (i = 0) \vee (i = 1) \vee (j = 0) \vdash A \quad \left| \quad \begin{array}{ccc} A(i/0)(j/1) & & A(i/1)(j/1) \\ {\scriptstyle A(i/0)}\uparrow & & \uparrow{\scriptstyle A(i/1)} \\ A(i/0)(j/0) & \xrightarrow[A(j/0)]{} & A(i/1)(j/0) \end{array} \right.
$$

Distributive lattice generated by the formal elements $(i = 0)$, $(i = 1)$ with the relation $0_{\mathbb{F}} = (i = 0) \wedge (i = 1)$

Face lattice

Any judgement valid in Γ *is also valid in a restriction* Γ, ψ

E.g. if we have $\Gamma \vdash A$ we also have $\Gamma, \psi \vdash A$

This is similar to the following property

Any judgement valid in Γ *is also valid in an extension* $\Gamma, x : A$

The restriction operation is a type-theoretic formulation of the notion of *cofibration*

The extension operation $\Gamma, x : A$ of a context Γ is a type-theoretic formulation of the notion of *fibration*

Face lattice

We say that the partial element $\Gamma, \psi \vdash u : A$ is *connected*

iff we have $\Gamma \vdash a : A$ such that $\Gamma, \psi \vdash a = u : A$

We write $\Gamma \vdash a : A[\psi \mapsto u]$

a *witnesses* the fact that u is connected

This generalizes the notion of being *path-connected*

Take ψ to be $(i = 0) \vee (i = 1)$

A partial element u of extent ψ is determined by 2 points

An element $a : A[\psi \mapsto u]$ is a line connecting these 2 points

Contractible types

The type $\mathsf{isContr}\ A$ is inhabited iff we have an operation

$$\frac{\Gamma \vdash \psi \qquad \Gamma, \psi \vdash u : A}{\Gamma \vdash \mathsf{ext}\ [\psi \mapsto u] : A[\psi \mapsto u]}$$

i.e. any partial element is *connected*

Box principle

By induction on A we build a "lid" operation

$$\frac{\Gamma \vdash \varphi \qquad \Gamma, i : \mathbb{I} \vdash A \qquad \Gamma, \varphi, i : \mathbb{I} \vdash u : A \qquad \Gamma \vdash a_0 : A(i/0)[\varphi \mapsto u(i/0)]}{\Gamma \vdash \mathsf{comp}^i\ A\ [\varphi \mapsto u]\ a_0 : A(i/1)[\varphi \mapsto u(i/1)]}$$

We consider a partial path u (of extent φ) in the direction i

If $u(i/0)$, partial element of extend φ, is connected then so is $u(i/1)$

This is a type theoretic formulation of the box principle

Main operation and univalence axiom

Given $\Gamma \vdash A$, a partial type $\Gamma, \psi \vdash T$ and map $\Gamma, \psi \vdash w : T \to A$ we can find

a total type $\Gamma \vdash \tilde{T}$ and map $\Gamma \vdash \tilde{w} : \tilde{T} \to A$

such that $\tilde{T}, \tilde{w}$ is an *extension* of T, w i.e.

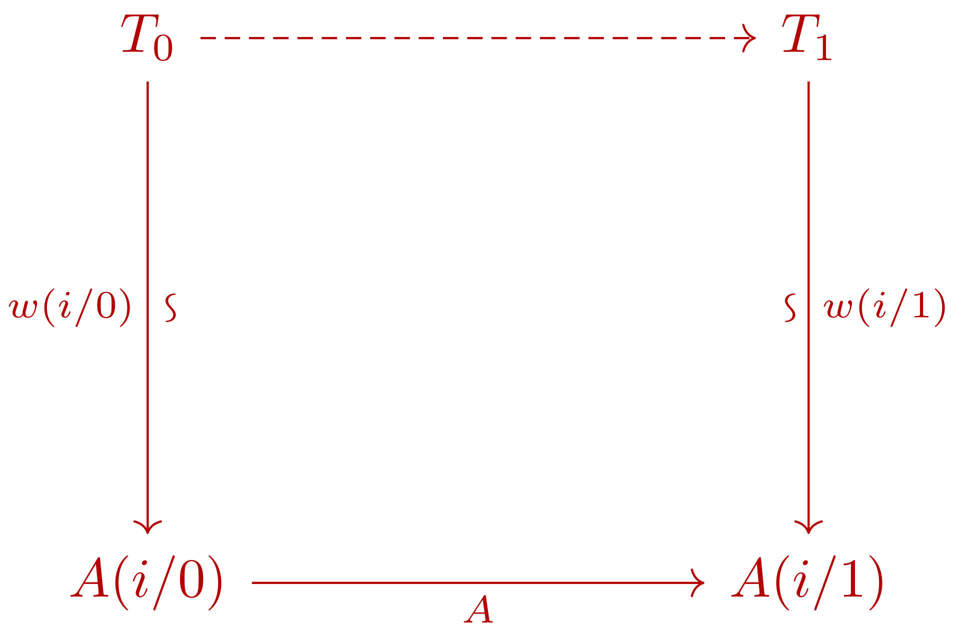
$\Gamma, \psi \vdash T = \tilde{T} \qquad \Gamma, \psi \vdash w = \tilde{w} : T \to A$

From this operation follows that

$(X : U) \times \mathsf{Equiv}\ X\ A$ is *contractible*

which is a way to state the univalence axiom

Main operation and univalence axiom



$$\psi \;=\; (i = 0) \vee (i = 1)$$

Main operation and univalence axiom

We define $\tilde{T} = \mathsf{Glue}\ [\psi \mapsto (T, w)]\ A$ with

If $\psi = 1_F$ then $\mathsf{Glue}\ [\psi \mapsto (T, w)]\ A = T$

If $\psi \neq 1_F$ then

$$\frac{\Gamma, \psi \vdash t : T \qquad \Gamma \vdash a : A[\psi \mapsto w\ t]}{\Gamma \vdash \mathsf{glue}\ [\psi \mapsto t]\ a : \mathsf{Glue}\ [\psi \mapsto (T, w)]\ A}$$

Cubical set model

This model actually suggests some *simplifications* of Voevodsky's model

Cf. work of Nicola Gambino and Christian Sattler (Leeds)

E.g. in both framework, to be contractible can be defined as

any partial element can be extended to a total element

Cubical type theory

Suggested by the cubical *set* model but now *independent of any set theory*

We can define an *evaluation* relation of terms, e.g.

$(\langle i \rangle t)\ 0 \ \rightarrow\ t(i/0)$

Theorem: (S. Huber) *Any term of type* N *reduces to a numeral in a context of the form* $i_1 : \mathbb{I}, \ldots, i_m : \mathbb{I}$

Only constant lines, squares, $\ldots$ in the cubical type of natural numbers

A prototype implementation (j.w.w. C. Cohen, S. Huber and A. Mörtberg)

Dependent type theory

We obtain a formulation of dependent type theory with *extensional equality*

In this formal system we can *prove* univalence

The extensionality problem is solved by ideas coming from algebraic topology

Dependent type theory

As stated above, we can prove (as a special case of univalence axiom) that two *equivalent propositions* are *path equal*

We can represent *basic set theory* in an interesting way

Unification of HOL with type theory (original motivation of Voevodsky)

Dependent type theory, new operations

Propositional truncation and existential quantification (which is a proposition)

Unique choice is provable

Representation of the notion of *category*

a category is the next level analog of a partially ordered set (Voevodsky, 2006)

Simpler than previous developments (e.g. G. Huet and A. Saïbi) since any type comes with its one notion of equality

Unique choice "up to isomorphism", e.g. the fact that a functor which is fully faithful and essentially surjective is an equivalence becomes provable in a constructive framework (in set theory we need the axiom of choice)