Forcing and non principal ultrafilter

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Abstract

The goal of this note is to present a simple proof of the fact that analysis extended with the existence of a non principal ultrafilter of natural number is conservative over analysis with dependent choice. The proof is purely syntactical and is a variation of an argument presented by Levin [1].

A.M. Levin "One conservative extension of formal mathematical analysis with a scheme of dependent choice" (1977)

Forcing over the system $\mathsf{HA}^\omega + \mathsf{EM} + \mathsf{DC}$ (for well-ordering of the reals)

Theorem: *If* $\mathsf{HA}^\omega + \mathsf{EM} + \mathsf{DC} + \mathsf{SUF} \vdash A$ *then* $\mathsf{HA}^\omega + \mathsf{EM} + \mathsf{DC} \vdash A$

The terms of the language are simply typed lambda terms. We have two basic types N (natural numbers) and N_2 (booleans). The atomic formulae are simply the terms of type N_2. There are two terms $0, 1$ of type N_2 and we identify 1 with the true formula $\top$ and 0 with the false formula $\bot$.

The formulae are

$$\varphi \ ::= \ \varphi \rightarrow \varphi \mid t \mid \forall x.\varphi$$

where t is a term of type N_2 (decidable atomic formula)

We use $n, m, \ldots$ for variables over the type N. Example: $\forall n.\exists^c m.n < m$.

$\neg\varphi$ to be $\varphi \rightarrow \bot$

$\exists^c x.\varphi$ is $\neg\forall x.\neg\varphi$

The system HA^ω is intuitionistic with the usual rules of natural deduction and induction over natural numbers and boolean. The rule EM is $(\neg\neg\varphi) \rightarrow \varphi$ which is equivalent to $\varphi \vee \neg\varphi$. The rule DC is

$$\forall n.\forall x.\exists y.\varphi(n, x, y) \rightarrow \forall u.\exists f.\varphi(0, u, f(0)) \wedge \forall n.\varphi(n, f(n), f(n+1))$$

The rule CC is

$$\forall n.\exists y.\varphi(n, y) \rightarrow \exists f.\forall n.\varphi(n, f(n))$$

We add a new symbol μ and new atomic formula $\mu(f)$ for f of type $N \rightarrow N_2$

We consider now the extension of the theory HA^ω with the axioms (we could add the selectivity axiom)

$$\mu(1) \qquad \mu(fg) \leftrightarrow (\mu(f) \wedge \mu(g))$$

$$\mu(f) \vee^c \mu(1 - f) \qquad \mu(f) \rightarrow \forall m.\exists^c n > m.f(m)$$

We use letters $p, q, r, \ldots$ to denote *forcing conditions*, here simply terms of type $N \rightarrow N_2$. One can think of forcing conditions as *decidable subsets* of $\mathbb{N}$.

We define a formula $p \Vdash \varphi$ by induction on φ where φ is an extended formula (which may contain the new symbol μ) and p is of type $N \rightarrow N_2$.

$I(p)$ is $\forall n.\exists m > n.\ p(m)$ $\qquad$ $F(p)$ is $\exists n.\forall m > n.\ \neg p(m)$

$\mu(f) \to I(f)$
$p \leqslant q$ is $F(p(1-q))$
$p \Vdash \mu(f)$ is $p \leqslant f$
$p \Vdash \varphi$ is $I(p) \to \varphi$ if φ is a boolean
$p \Vdash \varphi_0 \to \varphi_1$ is $\forall q \leqslant p.(q \Vdash \varphi_0) \to (q \Vdash \varphi_1)$
$p \Vdash \forall x.\varphi$ is $\forall x.(p \Vdash \varphi)$
We can add other connectives and existential quantification
Not needed if we are only interested in classical logic
Proposition: *If* $\varphi_1, \dots, \varphi_n \vdash \varphi$ *and* $p \Vdash \varphi_1, \dots, p \Vdash \varphi_n$ *then* $p \Vdash \varphi$
Using EM
Proposition: *We have* $p \Vdash \varphi_0 \vee^c \varphi_1$ *iff*

$$\forall q \leqslant p.\exists r \leqslant q.\ (r \Vdash \varphi_0) \vee^c (r \Vdash \varphi_1)$$

and $p \Vdash \exists^c x.\varphi$ *iff*

$$\forall q \leqslant p.\exists r \leqslant q.\exists^c x.\ r \Vdash \varphi$$

Proposition: *We have (classical version of the comprehension axiom)*

$$p \Vdash (\forall n.\varphi(n,0) \vee^c \varphi(n,1)) \to \exists^c f.\forall n \varphi(n, f(n))$$

This expresses that there are no more decidable functions in the extension than in the ground model
Proposition: *We have (countable choice)*

$$p \Vdash (\forall n.\exists^c x.\varphi(n,x) \to \exists^c f.\forall n \varphi(n, f(n))$$

All the axioms of non principal ultrafilters are forced
We have $\mathsf{HA}^\omega \vdash (I(p) \to \varphi) \leftrightarrow (p \Vdash \varphi)$ if φ does not mention μ
$\mathsf{HA}^\omega + \mathsf{EM} + \mathsf{DC} + \mathsf{SUF} \vdash \varphi$ implies $\mathsf{HA}^\omega + \mathsf{EM} + \mathsf{DC} \vdash\ (\Vdash \varphi)$ and hence $\mathsf{HA}^\omega + \mathsf{EM} + \mathsf{DC} \vdash \varphi$
So we have a computational interpretation of non principal ultrafilters
Levin (1977) does the same with a well-ordering of the reals, which justifies also the continuum hypothesis

References

[1] A.M. Levin. One conservative extension of formal mathematical analysis with a scheme of dependent choice 1977