Univalent Type Theory

Thierry Coquand

Tutorial for the Logic Colloquium 2016, Leeds

Equivalence

$$\mathsf{Fib}\ f\ b = \qquad (a : A) \times \mathsf{Id}\ B\ b\ (f\ a)$$

$$\mathsf{isEquiv}\ f = \qquad (b : B) \to \mathsf{isContr}\ (\mathsf{Fib}\ f\ b)$$

$$\mathsf{Equiv}\ A\ B = \qquad (f : A \to B) \times \mathsf{isEquiv}\ f$$

We recall

$$\mathsf{isContr}\ T = \qquad (t : T) \times ((x : T) \to \mathsf{Id}\ T\ t\ x)$$

Univalent type theory

- Id $A\ a_0\ a_1$

- 1_a : Id $A\ a\ a$

- $t(p) : B(a_0) \to B(a_1)$ if p : Id $A\ a_0\ a_1$

- a proof of Id $B(a)\ (t(1_a)\ u)\ u$ if $u : B(a)$

- a proof of Id $S\ (a, 1_a)\ (x, p)$ for $S = (x : A) \times$ Id $A\ a\ x$ and $(x, p) : S$

- the univalence axiom

The canonical map Id $U\ A\ B \to$ Equiv $A\ B$ *is an equivalence*

Univalent type theory

So univalent type theory is

(1) simple type theory extended with one universe (or a sequence of cumulative universes)

(2) extended with the $\mathsf{Id}\ A\ a_0\ a_1$ introduced by Martin-Löf

(3) extended with the univalence axiom

Consistency of (1) + (2): interpretation of types as sets

The consistency strength of (1) + (2) is known

What about (1) + (2) + (3)? We answer this question later

Equality in dependent sums

If $B(x)$ is a family of types over A then

any $p : \mathsf{Id}\ A\ a_0\ a_1$ defines a transport function $t(p) : B(a_0) \rightarrow B(a_1)$

For instance, if A is the collection of sets, and $B(X)$ is the collection $X \rightarrow X$ then any isomorphism $p : X_0 \simeq X_1$ defines a transport function

$B(X_0) \rightarrow B(X_1)$

$t(p)\ u_0 = p \circ u_0 \circ p^{-1}$

This was the notion of transport of structures considered by Bourbaki

Two structures are identified in $\sum_{X:A} B(X)$ if, and only if, they are isomorphic

Univalent type theory

Univalent type theory is a suitable system for developing mathematics in such a way that

it is impossible to formulate a statement which is not invariant with respect to equivalences

Stratification of types

A type A is a *proposition*

$$\text{isProp } A \;=\; (x_0\ x_1 : A) \to \text{Id } A\ x_0\ x_1$$

Notice that this itself is a type

A type is a *set*

$$\text{isSet } A \;=\; (x_0\ x_1 : A) \to \text{isProp } (\text{Id } A\ x_0\ x_1)$$

A type is a *groupoid*

$$\text{isGroupoid } A \;=\; (x_0\ x_1 : A) \to \text{isSet } (\text{Id } A\ x_0\ x_1)$$

The Univalence Axiom

The univalence axiom also implies that

-two isomorphic sets are equal

-two isomorphic algebraic structures are equal

-two equivalent (in the categorical sense) groupoid are equal

-two equivalent categories are equal

The equality of a and b entails that any property of a is also a property of b

Motivation for the term ⟪groupoid⟫

If A is any type we have operations of types

$1_a : \mathsf{Id}\ A\ a\ a$

$\mathsf{sym} : \mathsf{Id}\ A\ a_0\ a_1 \rightarrow \mathsf{Id}\ A\ a_1\ a_0$

$\mathsf{comp} : \mathsf{Id}\ A\ a_0\ a_1 \rightarrow \mathsf{Id}\ A\ a_1\ a_2 \rightarrow \mathsf{Id}\ A\ a_0\ a_2$

and we have e.g. for $p : \mathsf{Id}\ A\ a_0\ a_1$

$\mathsf{Id}\ (\mathsf{Id}\ A\ a_0\ a_1)\ (\mathsf{comp}\ 1_{a_0}\ p)\ p$

This uses in a crucial way the new law for equality discovered by Martin-Löf

Motivation for the term «groupoid»

If each Id A a_0 a_1 is a set, we can think of A as a groupoid in the «usual» sense

An object is an element of type A

A morphism between a_0 and a_1 is an element of the *set* Id A a_0 a_1

Any morphism is an isomorphism

Some propositions

We can prove, i.e. build terms of type

$(A : U) \to \mathsf{isProp}\ (\mathsf{isContr}\ A)$

$(A : U) \to \mathsf{isProp}\ (\mathsf{isProp}\ A)$

$(A : U) \to \mathsf{isProp}\ (\mathsf{isSet}\ A)$

$(A\ B : U) \to (f : A \to B) \to \mathsf{isProp}\ (\mathsf{isEquiv}\ f)$

But in general $\mathsf{hasInv}\ f$ is *not* a proposition

The type representing the univalence axiom is a *proposition* since it states that a given map is an equivalence

Function Extensionality

We can state, for $C = (x : A) \to B$

$((x : A) \to \mathsf{Id}\ B(x)\ (f\ x)\ (g\ x)) \to \mathsf{Id}\ C\ f\ g$

but this is not a proposition

Function Extensionality

One can state function extensionality as a *proposition*

The canonical map

$$\mathsf{Id}\ C\ f\ g \to ((x : A) \to \mathsf{Id}\ B(x)\ (f\ x)\ (g\ x))$$

is an equivalence

Function Extensionality

Function extensionality can also be stated as

$((x : A) \to \mathsf{isProp}\ B(x)) \to \mathsf{isProp}\ ((x : A) \to B(x))$

It implies

$((x : A) \to \mathsf{isSet}\ B(x)) \to \mathsf{isSet}\ ((x : A) \to B(x))$

This holds for A *arbitrary* type

If A is a set, the intuition is that it is the usual set of sections

If A is a groupoid, for instance, a groupoid defined by a group G with one point 0, the intuition is that a family of sets over A is a set $B(0)$ with a G-action, and the dependent product is the set of fixed points $B(0)^G$

Algebraic structures

An algebraic structure is an element of a type of the form

$(X : U) \times (\mathsf{isSet}\ X) \times T(X)$

sets with operations and properties

The *type* S of all these structures is *not* a set

It has a more complex notion of equality: each type $\mathsf{Id}\ S\ s_0\ s_1$ is a *set* and not a *proposition*

Representation of structures

The type of *structures of semigroup* on a type A

$$\mathsf{SemiG}\ A = \mathsf{isSet}\ A \times (f : A \to A \to A) \times (x\ y\ z : A) \to \mathsf{Id}\ A\ (f\ (f\ x\ y)\ z)\ (f\ x\ (f\ y\ z))$$

This type is always a *set*

The type of all semigroups is $\mathsf{SG} = (A : U) \times \mathsf{SemiG}\ A$

A semigroup is a pair (A, p) with $A : U$ and $p : \mathsf{SemiG}\ A$

This type is a *groupoid*

Representation of structures

If $f : A \rightarrow B$ is an equivalence, we have by univalence Id U A B

s_f : SemiG $A \rightarrow$ SemiG B

Represents transport of (semigroup) structures along an equivalence f

Isomorphisms

If we have two semigroups (A, a) and (B, b) then an equivalence $f : A \to B$ is an *isomorphism* if, and only if, we have a proof of

$\mathsf{Id}\ (\mathsf{SemiG}\ B)\ (s_f\ a)\ b$

Isomorphisms and equality

Using univalence, one can show

Theorem: *If* $f : A \rightarrow B$ *is an isomorphism between* (A, a) *and* (B, b) *then* Id SG (A, a) (B, b)

This implies that we have Id T $P(A, a)$ $P(B, b)$ for any $P : \mathsf{SG} \rightarrow T$

$P(A, a) : T$ does not need to be a proposition

Any property/structure is transportable along an isomorphism

Isomorphisms and equality

If (A, a) is commutative then so is (B, b)

If instead of semigroups, we consider commutative monoids

$P(A, a)$ may be the associated group of fractions

If (A, a) and (B, b) are isomorphic then so are $P(A, a)$ and $P(B, b)$

Differences with set theory

Any property is transportable

No need of ⟪critères de transportabilité⟫ as in set theory

⟪Only practice can teach us in what measure the identification of two sets, with or without additional structures, presents more advantage than inconvenient. It is necessary in any case, when applying it, that we are not lead to describe non transportable relations.⟫ Bourbaki, Théorie des Ensembles, Chapitre 4, Structures (1957)

$\pi \in A$ is a non transportable property of a group A

⟪to be solvable⟫ is a transportable property

Isomorphisms and equality

Let us define Iso (A, a) (B, b) to be the type of pairs (f, p) where p is a proof that f is an isomorphism

We have a canonical map

Id SG (A, a) $(B, b) \rightarrow$ Iso (A, a) (B, b)

Theorem: *This map is an equivalence*

Representation of structures

Notice that we can consider the structures of fixed-point functional

$$S\ A = \mathsf{isSet}\ A \times (Y : (A \to A) \to A) \times (f : A \to A) \times \mathsf{Id}\ A\ (Y\ f)\ (f\ (Y\ f))$$

or even simpler

$$S\ A = \mathsf{isSet}\ A \times \quad (A \to A) \to A$$

We can define what is an *isomorphism* for this notion of structures

Not so clear what should be a *morphism* for this notion of structure

Univalent type theory

We define

PROP $= (X : U) \times$ isProp$X$

SET $= (X : U) \times$ isSet$X$

One can show

isSet PROP and isGroupoid SET

Univalent type theory

This follows from

$\text{isProp } B \rightarrow \text{isProp } (\text{Id } U \ A \ B)$

which follows from

$\text{isProp } B \rightarrow \text{isProp } (\text{Equiv } A \ B)$

which follows from

$\text{isProp } B \rightarrow \text{isProp } (A \rightarrow B)$

For this argument we have used that $\text{Id } U \ A \ B$ and $\text{Equiv } A \ B$ are equal, which follows from the univalence axiom

Univalent type theory

SET is not a set

bool is a set (non trivial but provable!)

We have $(\mathsf{bool}, p) : \mathsf{SET}$ where $p : \mathsf{isSet\ bool}$

The type Id SET (bool, p) (bool, p) is equal to Id U bool bool

By univalence the type Id U bool bool is equal to Equiv bool bool

This type has two distinct elements

Univalent type theory

By a similar reasoning we can show that

The type of all groups/rings is a *groupoid* which is not a set

The collection of all groups/rings/posets forms a *groupoid*

Univalent type theory

If we consider structures *without* automorphisms, they form a *set*

E.g. we can define the type of well-order structures WO X and then

$(X : U) \times$ WO X is a set

In general we have isSet A as soon as $(a : X) \to$ isProp (Id A a a) (Streicher, 1991)

If we form the type of all linear orders of fixed size n, this will define a *contractible* type, since it is inhabited and any two elements are equal

Univalent type theory

This is an important difference with set theory where the collection of all sets of a given universe forms a set

《Qualitative》 difference between a type like bool and the type SET of all sets

An element of SET can be thought of as 《a set up to bijection》

Posets and categories

In this approach

the notion of groupoid is more fondamental than the notion of category

A groupoid is defined as a type satisfying a property

In set theory, a groupoid is defined to be a category where any morphism is an isomorphism

Posets and categories

A *preorder* is a *set* A with a relation $R(x, y)$ satisfying

$$(x\ y : A) \to \mathsf{isProp}\ (R\ x\ y)$$

which is reflexive and transitive

A *poset* is a preorder such that the canonical implication

$$\mathsf{Id}\ A\ x\ y \to R\ x\ y \times R\ y\ x$$

is a logical equivalence

Posets and categories

A *category* is a *groupoid* A with a relation $\mathsf{Hom}\ x\ y$ satisfying

$$(x\ y : A) \to \mathsf{isSet}\ (\mathsf{Hom}\ x\ y)$$

This family of sets is «transitive» (associative composition operation) and «reflexive» (we have a neutral element)

This corresponds to the notion of *preorder*

This family of sets is «transitive» (associative composition operation) and «reflexive» (we have a neutral element)

This corresponds to the notion of *preorder*

Posets and categories

One can define Iso $x\ y$ which is a *set* and show Iso $x\ x$

$$\text{Iso}\ x\ y \ = \ (f : \text{Hom}\ x\ y) \times (g : \text{Hom}\ y\ x) \times \ldots$$

This defines a canonical map

$$\text{Id}\ A\ x\ y \to \text{Iso}\ x\ y$$

For being a *category* we require this map to be an equivalence (bijection) between the sets Id $A\ x\ y$ and Iso $x\ y$

Posets and categories

A category is defined as a *structure* on a groupoid

The univalence axiom implies that the groupoid of rings, for instance, has a categorical structure

It also implies that two equivalent categories are equal

Representation of categories

The notion of category has somewhat lost his special status

A category is a structure at the level of groupoids (among other structures)

Adjunction: Galois connection at the next level

Complexity of equality

Here, in the definition of category, Hom x_0 x_1 has to be a set

This is formally similar to the definition of a *locally small* category

But what is crucial here is the

complexity of equality

of the type Hom x_0 x_1 and not its

set theoretic ⟪size⟫

More general structures

To be a 2-groupoid can be defined as

$(x_0\ x_1 : A) \to \mathsf{isGroupoid}\ (\mathsf{Id}\ A\ x_0\ x_1)$

The collection of all groupoids (with equivalences) form a 2-groupoid

The definition of the notion of 2-groupoid in set theory is quite complex

What is a 3-groupoid, $\dots$?

More general structures

《the intuition appeared that ∞-groupoids should constitute particularly adequate models for homotopy types, the n-groupoids corresponding to truncated homotopy types (with $\pi_i = 0$ for $i > n$)》

Grothendieck, *Sketch of a program*, 1984

An ∞-groupoid should be considered to be a space up to homotopy

This is used in the reverse direction: use a combinatorial way to represent homotopy types, due to Kan 1958, to define a model of type theory

Connection with homotopy theory

A connection between Identity type as introduced by Martin-Löf and homotopy theory was indicated in the work of Steve Awodey and Michael Warren

Homotopy theoretic models of identity types

pointing out the analogy between the elimination rule for identity type and some notion used in an abstract framework for homotopy theory

Nicola Gambino and Richard Garner
The identity type weak factorisation system

Benno van den Berg and Richard Garner
Types are weak ω-groupoid

Loop space

《Indeed, to apply Leray's theory I needed to construct fibre spaces which did not exist if one used the standard definition. Namely, for every space X, I needed a fibre space E with base X and with trivial homotopy (for instance contractible). But how to get such a space? One night in 1950, on the train bringing me back from our summer vacation, I saw it in a flash: just take for E the space of paths on X (with fixed origin a), the projection $E \to X$ being the evaluation map: path $\to$ extremity of the path. The fibre is then the loop space of (X, a). I had no doubt: this was it! $\ldots$ *It is strange that such a simple construction had so many consequences.*》

J.-P. Serre, describing the 《loop space method》 introduced in his thesis (1951)

Connection with homotopy theory

How do we know that the laws for univalent type theory are consistent?

For ordinary type theory, the proof theoretic strength is known (Γ_0 if we don't have generalized inductive definitions)

We cannot use an interpretation where a type is interpreted as a set

Model where types are interpreted by Kan simplicial sets (Voevodsky 2010)

This model reduces the consistency of univalent type theory to ZFC with $\omega+1$ inaccesibles

Connection with homotopy theory

The work

Cubical type theory: a constructive model of type theory,
Cyril Cohen, T.C., Simon Huber, Anders Mörtberg,
to appear in postproceeding of TYPES 2015

provides a constructive model of univalent type theory, inspired by the simplicial set model

This work has been *formalized* in the proof assistent NuPrl (Mark Bickford) establishing in this way that the axiom of univalence *does not* add any proof theoretic power to type theory

Connection with constructive mathematics

This model is closely connected to questions that appear in Bishop's approach to constructive mathematics

-the notion of dependent sums Cf. Exercice 3.2 in Bishop's book and *A course in constructive algebra*, Mines, Richman, Ruitenburg, p. 18

« An element of the disjoint union of a family $(A_i)_{i \in I}$ is a pair (i, x) such that $i \in I$ and $x \in A_i$. Two elements (i, x) and (j, y) of the disjoint sum are equal if $i = j$ and $A^i_j(x) = y$ »

-what should be a category?

Representing mathematics

Bourbaki tried to represent in a rigourous way the theory of categories

They stopped

Will the framework of univalent type theory help for a rigourous presentation of category theory?

Connection with homotopy theory

This connection also suggests a ⟪purely logical⟫ way to develop homotopy theory

E.g. one can prove, in univalent type theory, that composition of paths is commutative in any $\mathsf{Id}\ (\mathsf{Id}\ A\ a\ a)\ 1_a\ 1_a$ and this can be seen as a purely logical explanation of the fact that higher homotopy groups (i.e. $\pi_n(X, x)$ for $n > 1$) are commutative

All notions used in such a development are invariant by homotopy equivalence (e.g. cohomology groups, themselves invariant, but are usually defined using non invariant notions)

Resizing axiom

Connections between complexity of equality and 《set-theoretic》 size?

$U_0 : U_1 : U_2 : \ldots$

Theorem: (Nicolai Kraus and Christian Sattler) U_0 *is not a set,* U_1 *is not a groupoid,* $\ldots$

If a type is a proposition, can we consider it to be of type U?

E.g. $(X : U) \times \mathsf{Id}\ U\ A\ X$ is contractible, can we postulate that it is of type U without contradiction?

Can we postulate $\mathsf{PROP} : U$ without contradiction?