Logic and Topology

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topos

(Tierney-Lawvere) *A topos is a presentable locally cartesian closed category with a subobject classifier*

⟪Simple⟫ type theory

1940 A. Church *A Formulation of the Simple Theory of Types*

Extremely simple and natural

A type $bool$ as a type of ⟪propositions⟫

A type I for ⟪individuals⟫

Function type $A \rightarrow B$

Natural semantics of *types* as *sets*

Functions in simple type theory

In set theory, a function is a *functional graph*

In type theory, a function is given by an *explicit definition*

If $t : B$, we can introduce f of type $A \to B$ by the definition

$$f(x) = t$$

$f(a)$ 《reduces》 to $(a/x)t$ if a is of type A

Functions in simple type theory

We have two notions of function

-*functional graph*

-*function explicitly defined* by a term

What is the connection between these two notions?

Church introduces a special operation $\iota x.P(x)$ and the ⟪axiom of description⟫

If $\exists ! x : A.P(x)$ then $P(\iota x.P(x))$

Functions in simple type theory

We can then define a function from a functional graph

$\forall x.\exists ! y.R(x,y) \rightarrow \exists f.\forall x.R(x,f(x))$

by taking $f(x) = \iota y.R(x,y)$

By contrast, Hilbert's operation $\epsilon x.P(x)$ (also used by Bourbaki) satisfies

if $\exists x : A.P(x)$ then $P(\epsilon x.P(x))$

To use $\exists ! x : A.\varphi$ presupposes a notion of equality on the type A

Rules of equality

Equality can be specified by the following purely logical rules

(1) $a =_A a$

(2) if $a_0 =_A a_1$ and $P(a_0)$ then $P(a_1)$

Equality in mathematics

The first axiom of set theory is the axiom of *extensionality* stating that two sets are equal if they have the same element

In Church's system we have two form of the axiom of extensionality

(1) two equivalent propositions are equal

$$(P \equiv Q) \ \rightarrow \ P =_{bool} Q$$

(2) two pointwise equal functions are equal

$$(\forall x : A.f(x) =_B g(x)) \quad \rightarrow \quad f =_{A \rightarrow B} g$$

The univalence axiom is a generalization of (1)

Limitation of simple type theory

We can form

$I \to bool, \ (I \to bool) \to bool, \ ((I \to bool) \to bool) \to bool, \ldots$

but not talk *internally* about the family of such types

We cannot introduce an *arbitrary* structure (ring, group, $\ldots$)

Dependent types

The basic notion is the one of *family of types* $B(x),\ x : A$

We describe directly some *primitive* operations

$(\Pi x : A)B(x)$	$f \quad \text{where } f(x) = b$
$(\Sigma x : A)B(x)$	(a, b)
$A + B$	$\mathsf{i}(a), \mathsf{j}(b)$

which are *derived* operations in set theory

Dependent types

Logical operations are reduced to constructions on types by the following dictionnary

$A \wedge B$	$A \times B = (\Sigma x : A)B$
$A \vee B$	$A + B$
$A \to B$	$A \to B = (\Pi x : A)B$
$(\forall x : A)B(x)$	$(\Pi x : A)B(x)$
$(\exists x : A)B(x)$	$(\Sigma x : A)B(x)$

Dependent types

de Bruijn (1967) notices that this approach is suitable for representation of mathematical proofs on a computer (AUTOMATH)

Proving a proposition is reduced to building an element of a given type

《This reminds me of the very interesting language AUTOMATH, invented by N. G. de Bruijn. AUTOMATH is not a programming language, it is a language for expressing proofs of mathematical theorems. The interesting thing is that AUTOMATH works entirely by type declarations, without any need for traditional logic! I urge you to spend a couple of days looking at AUTOMATH, since it is the epitome of the concept of type.》

D. Knuth (1973, letter to Hoare)

Dependent types

Two ways of introducing dependent types

(1) Universes

(2) Path types

Universes

A *universe* is a type the element of which are types, and which is closed by the operations

$$(\Pi x : A)B(x) \qquad (\Sigma x : A)B(x) \qquad A + B$$

Russell's paradox does not apply directly since one *cannot* express $X : X$ as a *type*

However, Girard (1971) shows how to represent Burali-Forti paradox if one introduces a type of all types

Univers

Martin-Löf (1973), following Grothendieck, introduces of hierarchy of universe

$$U_0 : U_1 : U_2 : \ldots$$

Each universe U_n is closed by the operations

$$(\Pi x : A)B(x) \qquad (\Sigma x : A)B(x) \qquad A + B$$

Universes and dependent sums

We can formally represent the notion of structure

$(\Sigma X : U_0)((X \times X \to X) \times X)$

collection of types with a binary operation and a constant

$(X \times X \to X) \times X$ family of types for $X : U_0$

This kind of representation is used by Girard for expressing Burali-Forti paradox

Equality type

It is now represented by a dependent family of type $\mathsf{Path}(A, a, b)$

We have the constant path $1_a : \mathsf{Path}(A, a, a)$ and if $p : \mathsf{Path}(A, a, b)$ the transport function

$C(a) \rightarrow C(b)$

which is reminiscent of the path lifting condition

Equality type

Voevodsky introduced the definitions

$$\mathsf{isContr}(A) \;=\; (\Sigma a : A)(\Pi x : A)\mathsf{Path}(A, a, x)$$

$\mathsf{Fiber}(f, a) \;=\; (\Sigma x : T)\mathsf{Path}(A, f(x), a)$ for $f : T \to A$

$$\mathsf{isEquiv}(f) \;=\; (\Pi a : A)\mathsf{isContr}(\mathsf{Fiber}(f, a))$$

$$\mathsf{Equiv}(T, A) \;=\; (\Sigma f : T \to A)\mathsf{isEquiv}(f)$$

$$\mathsf{isProp}(X) \;=\; (\Pi a : X)(\Pi b : X)\mathsf{Path}(X, a, b)$$

$$\mathsf{isSet}(X) \;=\; (\Pi a : X)(\Pi b : X)\mathsf{isProp}(\mathsf{Path}(X, a, b))$$

Equivalence

Voevodsky proves for instance that

given $\psi : (\Pi a : A)\ B(a) \rightarrow C(a)$

we can define $\psi' : (\Sigma a : A)B(a) \rightarrow (\Sigma a : A)C(a)$

then

$\mathsf{isEquiv}(\psi') \leftrightarrow (\Pi a : A)\mathsf{isEquiv}(\psi(a))$

Equality type

Martin-Löf introduced, for purely formal logical reasons, the law

$(\Pi a : A)\mathsf{isContr}((\Sigma x : A)\mathsf{Path}(A, a, x))$

This expresses that the total space of the fibration defined by the space of paths having a given origin is *contractible*

This is exactly the starting point of the loop-space method in algebraic topology (J.P. Serre)

Univalence axiom

The canonical map

$\text{Path}(U, A, B) \rightarrow \text{Equiv}(A, B)$

is itself an equivalence (original statement)

This generalizes the fact that two equivalent propositions are equal!

Another (equivalent) statement is

$(\Pi A : U)\text{isContr}((\Sigma X : U)\text{Equiv}(A, X))$

Equality type

$\mathsf{isContr}((\Sigma x : A)B(x))$ is a uniform generalization of

$(\exists! x : A)B(x)$

and we have a description operator since

$\mathsf{isContr}((\Sigma x : A)B(x)) \to (\Sigma x : A)B(x)$

unique existence implies effective existence

Algebraic topology

In the 50s, development of a "combinatorial" notion of spaces

D. Kan: first with cubical sets (1955) then with simplicial sets

"A combinatorial definition of homotopy groups" (1958)

These spaces form a cartesian closed category
Moore (1955)

They form a model of type theory with the univalence axiom
Voevodsky (2009)

Algebraic topology

Proof of Moore's Theorem

"assez technique et délicate" (H. Cartan, Séminaire E.N.S. 56-57)

Algebraic topology

Existence of dependent product reduces to the fact that trivial cofibrations are stable under pullbacks along Kan fibrations

The proof of which is quite complex (uses minimal fibrations?)

Algebraic topology

All these *results* are intrinsically *non effective*

If one expresses the definitions as they are in IZF then the following facts are *not* provable

(1) If $E \to B$ fibration and b_0, b_1 path-connected then $E(b_0)$ and $E(b_1)$ are homotopy equivalent (j.w.w. M. Bezem, 2015)

(2) Moore's Theorem (E. Parmann, 2015)

Algebraic topology

The "reason" is that all these arguments use reasoning by case whether a complex is degenerate or not

(already in J.P. Serre's thesis 1951)

This is not decidable in general in an effective framework

The arguments are not "uniform" and non elementary

Univalent Foundations

I will now present a possible effective combinatorial notion of spaces with higher-order notion of connectedness

This is done in a constructive setting

We can extract from this a purely syntactical type system with no axioms

Category of cubes

For each finite set I we introduce a formal representation of the cube $[0,1]^I$

$[0,1]$ has a structure of *de Morgan algebra*

Bounded distributive lattice with a de Morgan involution

We can consider the free de Morgan algebra $\mathsf{dM}(S)$ on any set S

It is finite if S is finite

This defines a monad dM on the category of finite sets

$\mathcal{C}$ is the opposite of the Kleisli category of dM

Cubical sets

A map $J \to I$ in $\mathcal{C}$ is a set theoretic map $I \to \mathsf{dM}(J)$

A *cubical set* if a presheaf on $\mathcal{C}$

Family of sets $X(I)$ with transition functions

$X(I) \to X(J)$ for $f : J \to I$

$u \longmapsto uf$

If I finite set, we write also I the representable functor it defines

So I represents a cubical set

Category of cubes

We have direct face maps $(i0), (i1) : I \to I, i$ that are monos

A *face map* is a composition of direct face maps

Any map $f : J \to I$ can be uniquely decomposed

$f = gh$

where g is a face map and $h : J \to K$ is *strict* i.e. the corresponding map $K \to \mathsf{dM}(J)$ never takes the value 0 or 1

Singular cubical sets

We have a functor $\mathcal{C} \rightarrow \mathsf{Top}$

$I \longmapsto [0,1]^I$

Any topological space X defines a singular cubical set $S(X)$

$S(X)(I)$ is the set of all continuous maps $[0,1]^I \rightarrow X$

Interval

$\mathbb{I}(J) = \mathsf{dM}(J)$

This defines a cubical set, which represents the interval

$\mathbb{I}$ has a de Morgan algebra structure

Path

If X cubical set, the path "space" of X is $X^{\mathbb{I}}$

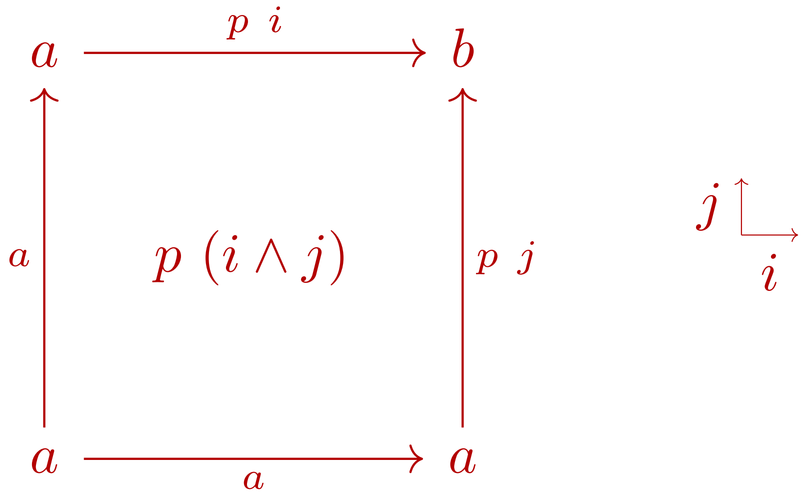
An element of $X^{\mathbb{I}}(J)$ is defined by an element of $X(J,i)$ with i not in J

u in $X(J,i)$ and v in $X(J,k)$ represents the same element if, and only if,

$u(i=k) = v$ in $X(J,k)$

Interval

Homotopy between the constant path on a and any path p : Path $A\ a\ b$



This expresses that $(\Sigma x : A)\mathsf{Path}(A, a, x)$ is contractible

Face lattice

$\mathbb{I}$ has two global points 0 and 1

Ω subobject classifier

We have a map $\mathbb{I} \rightarrow \Omega,\ \ i \longmapsto (i = 1)$

This is a lattice map

The image of this map is the *face lattice* $\mathbb{F} \rightarrow \Omega$

Exactly the sieves that are union of faces

Face lattice

$\mathbb{F}(I)$ can be defined directly as the free bounded distributive lattice generated by symbols $(i = 0), (i = 1)$ and relations

$$0 = (i = 0) \wedge (i = 1)$$

Intuitively $\mathbb{F}$ will classify *cofibrations*

We cannot hope to have *all* monos as cofibrations in an effective way

Face lattice

Any map $\psi : I \rightarrow \mathbb{F}$ determines a subpresheaf I, ψ of I

$(I, \psi)(J)$ set of maps $f : J \rightarrow I$ such that $\psi f = 1$

I, ψ is a subpresheaf of I

An map $g : K \rightarrow I$ determines a map $K, \psi g \rightarrow I, \psi$ that we write also g

Contractible cubical sets

A cubical set is *contractible* (uniformely) if

we have an operation $\mathsf{ext}(I,\psi,u) : I \rightarrow X$ given

$\psi : I \rightarrow \mathbb{F}, \quad u : I,\psi \rightarrow X$

such that

(1) $\mathsf{ext}(I,\psi,u)f = \mathsf{ext}(J,\psi f,uf) : J \rightarrow X$ whenever $f : J \rightarrow I$ and

(2) $\mathsf{ext}(I,1,u) = u$

$\mathsf{ext}(I,\psi,u)$ is an extension of u "uniform in I"

Contractible cubical sets

The usual definition only requires the *existence* of an extension of a partial element $u : I, \psi \rightarrow X$

Here we have a "contractibility structure"

It would *not* work only to require an explicit operation without any uniformity condition

Contractible cubical sets

$\mathbb{F}$ is a contractible cubical set

Ω is a contractible cubical set

$\mathbb{I}$ is *not* contractible

Any $u : I, \psi \rightarrow \mathbb{I}$ can be extended to $I \rightarrow \mathbb{I}$

but *not* in an uniform way! (Ch. Sattler)

Fibrant cubical sets

X is fibrant if, and only if, we have an operation $\mathsf{fill}(J, \psi, u) : J \times \mathbb{I} \to X$ given

$u : (J, \psi) \times \mathbb{I} \sqcup J \times 0 \to X$

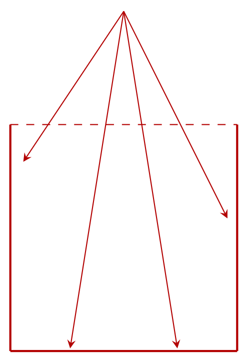
such that

(1) $\mathsf{fill}(J, \psi, u)(g \times 1_{\mathbb{I}}) = \mathsf{fill}(K, \psi g, ug)$ if $g : K \to J$ and

(2) $\mathsf{fill}(J, 1, u) = u$

Fibrant cubical sets

Theorem: *Any singular cubical set* $S(X)$ *is fibrant*



Cofibration-trivial fibration factorization

(A. Swan) What happens to the "small object argument"

Given $\sigma : A \to B$ where A and B are cubical sets

Define $C(I)$ having for element v, ψ, u with

$v : I \to B$

ψ in $\mathbb{F}(I)$

$u : I, \psi \to A$ such that v extends σu

$A \to C, \quad u \longmapsto (\sigma u, 1, u)$

$C \to B, \ (v, \psi, u) \longmapsto v$

Identity

(A. Swan) We can use this to define $\mathsf{Id}(A, a, b)$ from $\mathsf{Path}(A, a, b)$

Univalence

The definition of U_k is as usual

We consider $C = (\Sigma X : U_k)\mathsf{Equiv}(A, X)$

We have the first projection $p : C \to U_k$

Main algorithm: *For any partial element* $u : I, \psi \to C$ *find a total extension of* $pu : I, \psi \to U_k$

A crucial point is the fact that we have an opearion $\forall : \mathbb{F}^{\mathbb{I}} \to \mathbb{F}$

Univalence

From this follows

(1) U_k is fibrant, since any path $\mathsf{Path}(U_k, A, B)$ defines an equivalence in $\mathsf{Equiv}(A, B)$

(2) Univalence in the form that $C = (\Sigma X : U_k)\mathsf{Equiv}(A, X)$ is contractible

Higher-Inductive Types

We can define spheres $\mathsf{S}^n : U_0$ inductively

Propositional truncation $\mathsf{inh} : U_k \to U_k$

$\mathbb{Z}$-Torsors

A U_k-torsor is a type $X : U_k$ with a $\mathbb{Z}$-action such that

(1) for any u in X the map $n \longmapsto u + n,\ \mathbb{Z} \to X$ is an equivalence

(2) *and* $\mathsf{inh}(X)$

If X is a torsor we cannot in general exhibit one element of X

It follows from (1) and (2) that X is a set

$\mathbb{Z}$-Torsors

The collection of all torsors form a groupoid G_k which is *equivalent* to S^1

This can be proved without the axiom of choice!

All types G_k are equivalent to $\mathsf{S}^1 : U_0$

It should be consistent to add $G_k : U_0$