Simplicial set semantics of higher inductive types

Introduction

The goal of this note is to show that, by combining some results in the references[1], [2] and [3], we do get a model of higher inductive types in simplicial sets, extending and simplifying Voevodsky's model of univalent type theory. This follows essentially what was announced by Andrew Swan in some discussions[1] and this note tries to record the details of this argument.

To simplify the presentation, we limit ourselves to explain how to represent the suspension operation as an operation in $U \to U$ where U is some univalent universe in the simplicial set model.

Another goal of this note is to record the fact that the ideas used for the cubical set model, presented axiomatically in [3], can also be combined with [4] to give a quite simple way to build the standard Quillen model structure on *simplicial* sets. The use of classical logic is limited to one point (for establishing the logical equivalence of the "uniform" notion of Kan fibration with the usual definition).

1 A reformulation of Kan composition

Like in [3], we use the internal language of presheaf models. We write $I, J, K, \dots$ the objects of the base category (nonempty finite linear posets). We write $\mathbb{F}$ the presheaf such that $\mathbb{F}(I)$ is the set of decidable sieves on I. (If the metalanguage is classical then $\mathbb{F} = \Omega$ is the subobject classifier, but we want here to limit the use of classical logic to one place.) We let $\mathbb{I}$ be the presehaf Δ^1. Internally, $\mathbb{I}$ has a (bounded) distributive lattice structure and the can follow the setting of [1, 3]. We have a dependent type $[\psi]$ for $\psi : \mathbb{F}$ where $[\psi]$ is given by $[\psi](I, S) = \{0 \mid 1_I \in S\}$. A *filling operation* for a dependent type A over a type Γ is an operation which given $\gamma : \Gamma^{\mathbb{I}}$ and $\psi : \mathbb{F}$ and a partial section in $\Pi(i : \mathbb{I})[\psi \vee i = b] \to A\gamma(i)$ with $b = 0$ or 1, extends it to a total section in $\Pi(i : \mathbb{I})A\gamma(i)$. We write $\mathsf{Fill}(\Gamma, A)$ the type of such operations.

If Γ is the terminal object we get the notion of *fibrancy* structure. A fibrancy structure on a presheaf X is an operation which, given $\psi : \mathbb{F}$ and a partial section in $\Pi(i : \mathbb{I})[\psi \vee i = b] \to X$ extends it to a total section in $X^{\mathbb{I}}$. We write $\mathsf{Fib}(X)$ the type of such operations.

In general to give only $\Pi(\rho : \Gamma)\mathsf{Fib}(A\rho)$ is weaker than to give an element in $\mathsf{Fib}(\Gamma, A)$.

In [1] we present a type of *transport srtucture*, which together with an element in $\Pi(\rho : \Gamma)\mathsf{Fib}(A\rho)$ produces an element in $\mathsf{Fib}(\Gamma, A)$. It is given by an operation which give ψ and $\gamma : \Gamma^{\mathbb{I}}$ which is *constant* on ψ and a partial section in $\Pi(i : \mathbb{I})[\psi \vee i = b] \to A\gamma(i)$ which is *constant* on ψ, extends it to a total section in $\Pi(i : \mathbb{I})A\gamma(i)$.

2 Suspension operation

Given X, we define a $\mathsf{Susp}\ X$-algebra to be a type A with a fibrancy structure h_A and two points n_A, s_A and a family of paths $l_A : X \to A^{\mathbb{I}}$ connecting n_A to s_A. There is a natural notion of $\mathsf{Susp}\ X$-algebra and we can show [1] externally[2] that for any X there exists an inital $\mathsf{Susp}\ X$-algebra denoted simply by $\mathsf{Susp}\ X$. It has three constructors N, $\mathsf{S} : \mathsf{Susp}\ X$ and $\mathsf{merid}\ x\ i : \mathsf{Susp}\ X$ for $x : X$ and $i : \mathbb{I}$.

[1]This discussion can be found at https://groups.google.com/d/msg/homotopytypetheory/bNHRnGiF5R4/3RYz1YFmBQAJ.

[2]Thanks to one referee for pointing out to us that we don't need any special property of the interval for this operation, and thus that it works as well for *simplicial* sets.

Theorem 2.1 $\mathsf{susp}X$ *satisfies the dependent elimination rule for the suspension: given a family of type* P *over* $\mathsf{susp}X$ *with a composition structure, and* n *in* $P\ \mathsf{N}$ *and* s *in* $P\ \mathsf{S}$ *and* $l\ x\ i$ *in* $P\ (\mathsf{merid}\ x\ i)$ *such that* $l\ x\ 0 = n$ *and* $l\ x\ 1 = s$ *there exists a map* $\mathsf{elim} : \Pi(x : \mathsf{susp}X)P\ x$ *such that* $\mathsf{elim}\ \mathsf{N} = n$ *and* $\mathsf{elim}\ \mathsf{S} = s$ *and* $\mathsf{elim}\ (\mathsf{merid}\ x\ i) = l\ x\ i$.

If A is a dependent type over Γ then we define $\mathsf{Susp}\ A$ by $(\mathsf{Susp}\ A)\rho = \mathsf{Susp}\ (A\rho)$. It is then possible to show [1] that from any element in $\mathsf{Fill}(\Gamma, A)$ we can build an element in $\mathsf{Fill}(\Gamma, \mathsf{Susp}\ A)$.

Let $\mathcal{U}$ be a Grothendieck universe. If A is $\mathcal{U}$-valued presheaf on the category of elements of Γ, then so is $\mathsf{Susp}\ A$.

3 Kan fibration and simplicial set model

We can relate this internal notion of filling structure to the usual notion of Kan fibration.

Theorem 3.1 *If* Γ *is a presheaf and* A *a presheaf on the category of elements of* Γ *then the following conditions are equivalent*

1. $\Gamma.A \to \Gamma$ *is a Kan fibration*
2. $\Gamma.A \to \Gamma$ *has the right lifting property w.r.t. any pushout product of a monomorphism and an endpoint inclusion in* Δ^1
3. *there exists an element in* $\mathsf{Fill}(\Gamma, A)$.

Proof. The equivalence between the two first points is a classic result in the theory of simplicial sets (e.g. Goerss-Jardine, Proposition 4.2; this is the only place where one uses classical logic, and more precisely decidability of degeneracy and axiom of choice). The equivalence of the second and third conditions is proved elegantly in [2], by using the notion of Leibnitz product and exponential. □

We introduce the following notation $\mathsf{Type}_0(\Gamma)$ is the set of $\mathcal{U}$-valued presheaves on the category of elements of Γ, and $\mathsf{FType}_0(\Gamma)$ is the set of $\mathcal{U}$-valued presheaves on the category of elements of Γ *together with* a filling structure and $\mathsf{KType}_0(\Gamma)$ is the subpresheaf of $\mathsf{Type}_0(\Gamma)$ of $\mathcal{U}$-valued presheaves on the category of elements of Γ for which *there exists* a filling structure. By Theorem 3.1, $\mathsf{KType}_0(\Gamma)$ is equivalently the set of $\mathcal{U}$-valued presheaves A on the presheaf Γ such that $\Gamma.A \to \Gamma$ is a Kan fibration.

All of these define presheaves on the category of presheaves in a canonical way.

We define $U(I)$ to be the set $\mathsf{KType}_0(Yon(I))$. We define a presheaf El on the category of elements of U by $El(I, X) = X(I, 1_I)$, so that El is $\mathcal{U}$-valued. If A is a $\mathcal{U}$-valued presheaf on the category of elements of Γ with a filling structure, there exists a unique map $|A| : \Gamma \to U$ such that $El|A| = A$.

It can be shown that $U.El \to U$ is a Kan fibration. By Theorem 3.1, we have a global element in $\mathsf{Fill}(U, El)$ (this is the only place where classical logic is used).

We expand the difference between this model (where filling is a property) and the "cubical" set models (where filling is a structure). If we define $V(I)$ to be the set $\mathsf{FType}(Yon(I))$, then this defines a presheaf, but there will not be a natural bijection between $\Gamma \to V$ and $\mathsf{FType}_0(\Gamma)$. On the contrary, when we define $U(I)$ as above, then there is a natural bijection between the set $\Gamma \to U$ and $\mathsf{KType}_0(\Gamma)$.

We have a map $\mathsf{susp} : U \to U$ such that $El(\mathsf{susp}\ X) = \mathsf{Susp}\ (ElX)$ for $X : U$.

To show that we have an elimination rule, we proceed as for showing the existence of the elimination rule for the identity type in the simplicial set model. We consider the context (using extension types notation)

$$X : U,\ P : El(\mathsf{susp}\ X) \to U,\ n : P\ \mathsf{N},\ s : P\ \mathsf{S},\ l : \Pi(x : ElX)\Pi(i : \mathbb{I})P\ (\mathsf{merid}\ x\ i)[i = 0 \mapsto n,\ i = 1 \mapsto s]$$

and in this context (like in [1]) we build an element in $\mathsf{elim} : \Pi(x : El(\mathsf{susp}\ X))P\ x$ such that $\mathsf{elim}\ \mathsf{N} = n$ and $\mathsf{elim}\ \mathsf{S} = s$ and $\mathsf{elim}\ (\mathsf{merid}\ x\ i) = l\ x\ i$.

4 Quillen model structure on simplicial sets and one conjecture

Using [3], it can be shown that U is fibrant (i.e. has a fibrant structure). We can follow [4] and build a model structure on simplicial sets. The only use of classical logic is in the existence of a filling structure in $\mathsf{Fill}(U, El)$ which is a consequence of Theorem 3.1.

It follows also from what we presented that it is possible to interpret in simplicial sets the version of cubical type theory (based on distributive lattice) where the composition operation are new *constants* (they satisfy only the substitution laws but no computation rules for composition of dependent products and sums and paths and universes). This system has also models in cubical sets (where we can compute). A conjecture is that it should be possible to use the glueing technique (as in [5]) to show that this formal system satisfies Voevodsky's conjecture: any closed term of type natural numbers should be path equal to a numeral. This would be one way to show that various versions of cubical type theory (that are extensions of this "constant" version by new computation rules) give the same values for a closed term of type natural numbers in ordinary dependent type theory extended with univalence.

References

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