# GENERATING NON-NOETHERIAN MODULES CONSTRUCTIVELY

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Abstract. In [6], Heitmann gives a proof of a Basic Element Theorem, which has as corollaries some versions of the “Splitting-off” theorem of Serre and the Forster-Swan theorem in a non Noetherian setting. We give elementary and constructive proofs of such results. We introduce also a new notion of dimension for rings, which is only implicit in [6] and we present a generalisation of the Forster-Swan theorem, answering a question left open in [6].

## 1. Zariski spectrum and Krull dimension

Let $R$ be a commutative ring with unit. Following Joyal [7], we define the Zariski spectrum of $R$ as the distributive lattice generated by symbols $D(f),\ f \in R$ and relations

$$D(0) = 0 \qquad D(1) = 1 \quad D(fg) = D(f) \wedge D(g) \qquad D(f+g) \leq D(f) \vee D(g)$$

We write $D(f_1, \ldots, f_m)$ for $D(f_1) \vee \cdots \vee D(f_m)$. It can be shown directly that

$$D(g_1) \wedge \cdots \wedge D(g_n) \leq D(f_1, \ldots, f_m)$$

holds if, and only if, the monoid generated by $g_1, \ldots, g_n$ meets the ideal generated by $f_1, \ldots, f_m$ [1]. Thus $D(f_1, \ldots, f_m)$ can be defined as the radical of the ideal generated by $f_1, \ldots, f_m$ (with inclusion as ordering), and we have a point-free and elementary description of the basic open sets of the Zariski spectrum of $R$.

In [2] we present the following elementary caracterisation of Krull dimension. If $a \in R$ we define the *boundary* of $a$ as being the the ideal $N_a$ generated by $a$ and the elements $b$ such that $ab$ is nilpotent (or equivalently $D(ab) = 0$).

**Theorem 1.1.** *The dimension of $R$ is $<n+1$ if, and only if, for all $a \in R$ the dimension of $R/N_a$ is $<n$.*

This can actually be taken as a constructive definition of Krull dimension, if we define a ring $R$ to be of dimension $<0$ if, and only if, $R$ is trivial. This inductive definition of being of dimension $<n$ is then equivalent to the usual definition that there is no strictly increasing chain of prime ideals of length $n$ [2]. In [1] it is shown, in an elementary and constructive way, that the dimension of a polynomial ring with $n$ variables over a discrete field is $\leq n$.

## 2. The stable range theorem

All the arguments will be based on the following trivial remark, that we state explicitely since it will motivate the notion of dimension that we present in section 4.

**Lemma 2.1.** *If $b$ is nilpotent then $1 = D(b_1, \ldots, b_k) \vee D(b)$ implies $1 = D(b_1, \ldots, b_k)$. More generally, if $b \in R$ is nilpotent in $R[a^{-1}]$ then $D(a) \leq D(b_1, \ldots, b_k) \vee D(b)$ implies $D(a) \leq D(b_1, \ldots, b_k)$.*

*Proof.* Indeed, $b$ is nilpotent if, and only if, $D(b) = 0$. Likewise, $b \in R$ is nilpotent in $R[a^{-1}]$ if, and only if, $D(ab) = 0$, and $D(a) \leq D(b_1, \ldots, b_k) \vee D(b)$ implies $D(a) \leq D(b_1, \ldots, b_k) \vee D(ab)$. □

We shall need also the following two remarks.

**Lemma 2.2.** $D(a + b) \vee D(ab) = D(a) \vee D(b)$

**Lemma 2.3.** *If $by$ is nilpotent then $1 = D(b_1, \ldots, b_k, b, y)$ implies $1 = D(b_1, \ldots, b_k, b + y)$. More generally, if $by \in R$ is nilpotent in $R[a^{-1}]$ then $D(a) \leq D(b_1, \ldots, b_k, b, y)$ implies $D(a) \leq D(b_1, \ldots, b_k, b + y)$.*

*Proof.* By Lemmas 2.1 and 2.2. □

Our inductive definition of dimension allows more perspicuous proofs. For instance, as a motivation of our method, here is a proof of the "Stable Range" theorem.

**Theorem 2.4.** *If the dimension of $R$ is $<n$ and $1 = D(a, b_1, \ldots, b_n)$ there exists $x_1, \ldots, x_n$ such that $1 = D(b_1 + ax_1, \ldots, b_n + ax_n)$.*

*Proof.* The proof is by induction on $n$. This is clear if $n = 0$. If $n > 0$, let $I$ be the ideal boundary of $b_n$. We have $b_n \in I$ and the dimension of $R/I$ is $<n - 1$. By induction, we can find $x_1, \ldots, x_{n-1}$ such that

$$1 = D(b_1 + ax_1, \ldots, b_{n-1} + ax_{n-1})$$

in $R/I$. This means that there exists $x_n$ such that $D(b_n x_n) = 0$ and

$$1 = D(b_1 + ax_1, \ldots, b_{n-1} + ax_{n-1}) \vee D(b_n) \vee D(x_n)$$

Since

$$1 = D(b_1 + ax_1, \ldots, b_{n-1} + ax_{n-1}) \vee D(b_n) \vee D(a)$$

this implies by distributivity

$$1 = D(b_1 + ax_1, \ldots, b_{n-1} + ax_{n-1}) \vee D(b_n) \vee D(ax_n)$$

hence the result by Lemma 2.3. □

It follows then for instance directly that a stably free module of rank $\geq n$ over a ring of dimension $<n$ is free [9], without any noetherianity hypotheses. We can in the same way prove Kronecker's theorem about algebraic sets [3, 8].

We shall need a variation on this result. If $L \in R^n$ is a vector $(a_1, \ldots, a_n)$ we write $D(L)$ for $D(a_1, \ldots, a_n)$.

**Lemma 2.5.** *If $a \in R$ and the dimension of $R[a^{-1}]$ is $<n$ then for any $L \in R^n$ there exists $X \in R^n$ such that $D(a) \leq D(L - aX)$. Furthermore, we can find $X$ of the form $aY$, $Y \in R^n$.*

*Proof.* We let $L$ be $(b_1, \dots, b_n)$ and we reason by induction on $n$. This is clear if $n = 0$. If $n > 0$ let $N$ be the ideal boundary of $b_n$ *in* $R[a^{-1}]$, and $I$ the ideal $N \cap R$. It can be checked that $I$ that $(R/I)[a^{-1}]$ is isomorphic to $R[a^{-1}]/N$. Hence we can apply the induction hypothesis to $R/I$ and compute $(x_1, \dots, x_{n-1}) \in R^{n-1}$ such that

$$D(a) \leq D(b_1 - ax_1, \dots, b_{n-1} - ax_{n-1})$$

in $R/I$. In turn, this means that we can find $x_n$ such that $D(ax_nb_n) = 0$ and

$$D(a) \leq D(b_1 - ax_1, \dots, b_{n-1} - ax_{n-1}) \vee D(b_n) \vee D(x_n)$$

in $R$. This implies

$$D(a) \leq D(b_1 - ax_1, \dots, b_{n-1} - ax_{n-1}) \vee D(b_n) \vee D(ax_n)$$

hence the result by Lemma 2.3. Finally, we can apply the result with $a^2$ instead of $a$ since $R[a^{-2}] = R[a^{-1}]$ and $D(a) = D(a^2)$. $\square$

**Corollary 2.6.** *Let $M$ be a $n \times n$ matrix of element in $R$ and $\delta$ its determinant. If the dimension of $R[\delta^{-1}]$ is $<n$ then for each $C \in R^n$ there exists $X \in R^n$ such that $D(\delta) \leq D(MX - C)$. Furthermore we can find $X$ of the form $\delta Y$, $Y \in R^n$.*

*Proof.* The proof is based on Cramer formulae. Let $\widetilde{M}$ be the adjoint matrix of $M$, and $L = \widetilde{M}C$. We have then $\widetilde{M}(MX - C) = \delta X - L$ for an arbitrary column vector $X \in R^n$. Hence the ideal generated by the coordinates of $\delta X - L$ is included in the one generated by the coordinates of $MX - C$, and

$$D(\delta X - L) \leq D(MX - C)$$

By Lemma 2.5 we can find *one* $X \in R^n$ such that $D(\delta) \leq D(\delta X - L)$, and hence $D(\delta) \leq D(MX - C)$ as desired. $\square$

## 3. A Basic Element Theorem

Let $F$ be a rectangular matrix of elements in $R$ of columns $C_0, C_1, \dots, C_p$, and $G$ the matrix of columns $C_1, C_2, \dots, C_p$. Let $\Delta_n$ be $\bigvee_\nu D(\nu)$ where $\nu$ varies over all minors of $F$ of order $n$.

**Theorem 3.1.** *Fix $0<n \leq p$. Suppose that for each minor $\nu$ of $G$ of order $n$ the ring $R[\nu^{-1}]$ is of dimension $<n$. Then there exist $t_1, \dots, t_p$ such that $\Delta_n \leq D(C_0 + t_1C_1 + \dots + t_pC_p)$* and $D(C_0) \leq D(C_0 + t_1C_1 + \dots + t_pC_p)$.

*Proof.* Clearly, if $D(\nu) \leq D(C_0 + t_1C_1 + \dots + t_pC_p)$ for all minor $\nu$ of $G$ of order $n$ then this holds also for all minor $\nu$ of $F$ of order $n$. It is also enough to show that for *one* minor $\nu$ of $G$ of order $n$ we can find $t_1, \dots, t_p$ such that $D(\nu) \leq D(C_0 + t_1C_1 + \dots + t_pC_p)$ and $D(C_0) \leq D(C_0 + t_1C_1 + \dots + t_pC_p)$ because we can then apply this successively to all minors of $G$ of order $n$. But this is a direct consequence of Corollary 2.6: we find $t_1, \dots, t_p$ (with $t_i = 0$ for the columns outside the minor $\nu$) that are multiple of $\nu$ and such that

$$D(\nu) \leq D(C_0 + t_1C_1 + \dots + t_pC_p)$$

Since $t_1, \ldots, t_p$ are all multiple of $\nu$ we have also $D(C_0) \leq D(\nu) \vee D(C_0 + t_1 C_1 + \cdots + t_p C_p)$ and hence $D(C_0) \leq D(C_0 + t_1 C_1 + \cdots + t_p C_p)$ as required. $\square$

**Corollary 3.2.** *Suppose that $1 = \Delta_1$ and that for each $k > 0$ and for each minor $\nu$ of $G$ of order $n$ the ring $R[\nu^{-1}]/\Delta_{n+1}$ is of dimension $<n$. Then there exist $t_1, \ldots, t_p$ such that the vector $C_0 + t_1 C_1 + \cdots + t_p C_p$ is unimodular.*

*Proof.* In the statement of this corollary, we identify $\Delta_l$ with its corresponding radical ideal. Using Theorem 3.1, we define a sequence of vectors $C_0^i$, $i = 1, \ldots$ with $C_0^1 = C_0$. For each $k > 0$, reasoning in $R/\Delta_{k+1}$, we build $C_0^{k+1}$ of the form $C_0^k + u_1 C_1 + \cdots + u_p C_p$ such that $\Delta_k \leq D(C_0^{k+1})$ and $D(C_0^k) \leq D(C_0^{k+1})$ in $R/\Delta_{k+1}$. This means that we have, in $R$

$$D(C_0^k) \vee \Delta_k \leq D(C_0^{k+1}) \vee \Delta_{k+1}$$

Hence the result since $\Delta_1 = 1$ and $\Delta_k = 0$ for $n$ large enough. $\square$

From this follows directly, like in [4, 6], a version of Serre's "Splitting-off" theorem and the Forster-Swan theorem with Krull dimension and without noetheriannity hypothesis. For instance, here is a version of Forster's theorem [5].

**Corollary 3.3.** *Let $M$ be a module which is finitely generated over a ring $R$ of dimension $\leq d$. If $M$ is locally generated by $r$ elements then $M$ can be generated by $d + r$ elements.*

*Proof.* $M$ is a quotient of a finitely presented module $M'$ which has a Fitting ideal of order $r$ which contains 1, and we can as well suppose that $M' = M$. Let $m_0, m_1, \ldots, m_p$ be a system of generators of $M$ and $F$ be a presentation matrix of $M$. If $p \geq d + r$ we have $1 = \Delta_{d+1}(F)$ and using theorem 3.1, we can find $t_1, \ldots, t_p$ such that $M$ is generated by $m_1 - t_1 m_0, \ldots, m_p - t_p m_0$. Hence we can generate $M$ by $p$ elements. $\square$

Using Corollary 3.2, one could give a more sophisticated version of this result, like in [6]. The next section shows how to recover an improved version of these theorems with Heitmann's notion of $j$-spectrum [6], also without noetheriannity hypothesis.

## 4. A New Notion of Dimension

By analogy with our inductive definition of Krull dimension, we define now when a ring $R$ is of $H$-dimension $<n$. Let $J$ be the Jacobson radical of $R$, that is the ideal of elements $x$ such that $1 - xy$ is inversible for all $y \in R$. We redefine $N_a$, ideal boundary of $a$, as the ideal generated by $a$ and the elements $b$ such that $ab \in J$. Thus we replace the radical of $R$, intersection of all prime ideals, by its Jacobson radical, intersection of all maximal ideals. A ring of $H$-dimension $<0$ is a trivial ring, and $R$ is of $H$-dimension $<n+1$ if, and only if, for any $a \in R$ the $H$-dimension of $R/N_a$ is $<n$. The proofs of all our previous results go through directly with this new definition of dimension. This follows from the following elementary result, which shows that Lemma 2.1 holds with our new notion, and also that this new notion is in some sense optimal.

**Lemma 4.1.** *$b \in J$ if, and only if, for all $b_1, \ldots, b_k$, we have $D(b_1, \ldots, b_k) = 1$ whenever $D(b_1, \ldots, b_k) \vee D(b) = 1$. More generally $b \in R$ is in the Jacobson radical of $R[a^{-1}]$ if, and only if, $D(a) \leq D(b_1, \ldots, b_k)$ whenever $D(a) \leq D(b_1, \ldots, b_k) \vee D(b)$.*

*Proof.* Since $1 - by + by = 1$ this condition implies indeed that $1 - by$ is invertible for all $y$, that is $b \in J$. Conversely if $b \in J$ and $y_1 b_1 + \cdots + y_k b_k + by = 1$ then, since $1 - by$ is invertible, we have $1 = D(b_1, \ldots, b_k)$. The proof of the second part is similar. □

We get in this way a version of the Basic Element Theorem 3.1 with $H$-dimension instead.

**Theorem 4.2.** *Fix $n \leq p$. Suppose that for each minor $\nu$ of $G$ of order $n$ the ring $R[\nu^{-1}]$ is of $H$-dimension $<n$. Then there exist $t_1, \ldots, t_p$ such that $\Delta_n \leq D(C_0 + t_1 C_1 + \cdots + t_p C_p)$* and *$D(C_0) \leq D(C_0 + t_1 C_1 + \cdots + t_p C_p)$.*

**Corollary 4.3.** *Suppose that for each $n \leq p$ and for each minor $\nu$ of $G$ of order $n$ the ring $R[\nu^{-1}]/\Delta_{n+1}$ is of $H$-dimension $<n$. Then there exist $t_1, \ldots, t_p$ such that the vector $C_0 + t_1 C_1 + \cdots + t_p C_p$ is unimodular.*

Our argument being constructive can be seen as an algorithm that computes such elements $t_1, \ldots, t_p$.

Our notion of $H$-dimension is only implicit in [6]. Heitmann introduces instead the $j$-spectrum of $R$ which is the closure of the maximal spectrum of $R$ in the patch topology, with the topology induced by Zariski topology, and define the $j$-dimension of $R$ to be the (Krull) dimension of the $j$-Spec$(R)$. Recall that the Heitmann's $j$-spectrum is not the usual one given in the literature. This new spectrum was introduced in [6] in order to deal with the non Noetherian case.

**Proposition 4.4.** *If the $j$-dimension of $R$ is $\leq n$ then its $H$-dimension is also $\leq n$.*

*Proof.* By induction on $n$. Let $X$ be $j$-Spec $R$. For $a \in R$, the boundary ideal $N_a$ corresponds to a closed subset $Y = V(a) \cap \overline{D(a) \cap X} = \mathrm{Spec}(R/N_a)$ (like in [6], $^-$ indicates closure in the Zariski topology). The subset $Y$ being closed, its closed (=maximal) points are exactly the points in Max$(R)$ $\cap Y$. This implies $j$-Spec$(R/N_a) \subseteq Y \cap X$. It is then enough to show that the (Krull) dimension of $Y \cap X$ is $<n$. But we have $Y \cap X \subseteq \overline{V(a) \cap X} \cap \overline{D(a) \cap X} \cap X$ which has dimension $<n$ by Lemma 1.2 in [6] (this is the boundary of $D(a) \cap X$ in $X$). □

This means that the $H$-dimension is $\leq$ the $j$-dimension. We conjecture that the two notions of dimension, $H$-dimension and $j$-dimension, may differ in general. So our results could improve those of [6] in some cases. Nevertheless we think that the main importance of our framework is the constructive and elementary character of our proofs. The next section illustrates this point, by answering a question left open at the end of [6].

## 5. A generalisation of the Forster-Swan theorem

The following result is a refinement of Theorem 2.4.

**Lemma 5.1.** *Assume given $L, L_1 \dots, L_n$ vectors of the same length. If H-dim(R)<n and*

$$1 = D(a, b_1, \dots, b_n) \vee D(L)$$

*then there exist $x_1, \dots, x_n$, all multiple of $a$, such that*

$$1 = D(b_1 + ax_1, \dots, b_n + ax_n) \vee D(L + x_1L_1 + \dots + x_nL_n)$$

*Proof.* The proof is by induction on $n$. This is clear if $n = 0$. If $n > 0$, let $I$ be the ideal boundary of $b_n$. We have $b_n \in I$ and $H$-dim$(R/I)<n-1$. By induction, we can find $x_1, \dots, x_{n-1}$ multiple of $a$ such that

$$1 = D(b_1 + ax_1, \dots, b_{n-1} + ax_{n-1}) \vee D(L + x_1L_1 + \dots + x_{n-1}L_{n-1})$$

in $R/I$. This means that there exists $y$ such that $b_ny \in J$ and

$$1 = D(b_1 + ax_1, \dots, b_{n-1} + ax_{n-1}) \vee D(L + x_1L_1 + \dots + x_{n-1}L_{n-1}) \vee D(b_n) \vee D(y)$$

in $R$. Take $x_n = ay$. Then $x_n$ is a multiple of $a$ and we have

$$1 = D(b_1 + ax_1, \dots, b_{n-1} + ax_{n-1}, b_n + ax_n) \vee D(L + x_1L_1 + \dots + x_{n-1}L_{n-1} + x_nL_n)$$

Indeed, if $X$ is the right hand-side, we have

$$X \vee D(a) = D(a, b_1, \dots, b_n) \vee D(L) = 1$$

since all $x_1, \dots, x_n$ are multiple of $a$, but also, since $x_n$ is a multiple of $y$

$$X \vee D(y) = D(b_1 + ax_1, \dots, b_{n-1} + ax_{n-1}, b_n) \vee D(L + x_1L_1 + \dots + x_{n-1}L_{n-1}) \vee D(y) = 1$$

and

$$X \vee D(b_n) = X \vee D(b_n) \vee D(ax_n) = 1$$

so that $X \vee D(b_ny) = 1$ and hence $X = 1$ since we can apply Lemma 4.1 to $b_ny \in J$. $\square$

We can now prove a variation of Theorem 4.2.

**Theorem 5.2.** *Fix $n \leq p$. Suppose that $R$ is of $H$-dimension $<n$ and $\Delta_n = 1$. Then there exist $t_1, \dots, t_p$ such that $1 = D(C_0 + t_1C_1 + \dots + t_pC_p)$.*

We let $\Delta_n(C_0; G)$ be $\bigvee_\nu D(\nu)$ where $\nu$ varies over all minors of $F$ of order $n$ using the column $C_0$.

**Lemma 5.3.** *If $\nu$ is a minor of $G$ of order $n$ using the columns $C_{i_1}, \dots, C_{i_n}$ and*

$$\Delta_n(C_0; G) \vee D(\nu) = 1$$

*and H-dim(R)< n then there exist $x_1, \dots, x_n$ such that*

$$\Delta_n(C_0 + x_1C_{i_1} + \dots + x_nC_{i_n}; G) = 1$$

*Proof.* This follows from lemma 5.1: we take $a$ to be $\nu$, $b_k$ to be the minor obtained from $\nu$ by replacing $C_{i_k}$ by $C_0$, $L$ to be the vector of all remaining minors in $\Delta_n(C_0; G)$, and $L_k$ is obtained by replacing $C_0$ by $C_{i_k}$ in $L$. $\square$

For proving Theorem 5.2, we write

$$\Delta_n = \Delta_n(C_0; G) \vee \bigvee_{\nu} D(\nu)$$

where $\nu$ varies over all minors of $G$ of order $n$. Using Lemma 5.3, we can eliminate successively these minors, replacing at each step $C_0$ by a vector of the form $C_0 + t_1C_1 + \cdots + t_pC_p$. At the end, we get $t_1, \ldots, t_p$ such that $\Delta_n(C_0 + t_1C_1 + \cdots + t_pC_p; G) = 1$ and this implies $D(C_0 + t_1C_1 + \cdots + t_pC_p) = 1$

We can now state the following result, which can be proved from Theorem 5.2 by arguments similar to the ones for Corollary 3.3.

**Theorem 5.4.** *If $H$-dim(R)$\leq d$ and if $M$ is a finitely generated module over $R$ which is locally generated by $r$ elements, then $M$ is generated by $d+r$ elements.*

Swan's theorem [10], which itself generalises Forster's theorem [5], can be seen as a special case when the maximal spectrum of $R$ is Noetherian. This improves also on [6] that assumes instead $j$-dim$(R[a^{-1}])\leq d$ for all $a \in R$, and on [11] that obtained $r(d'+1)$ generators with $d' = j$-dim$(R)$.

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