Some examples of inductive-recursive definitions

Introduction

The goal of this note is to present how we can model some higher inductive types in a constructive set theory. For the circle for instance, we have to define inductively at the same time a family of sets $S^1(I)$ and restriction functions $S^1(I) \to S^1(J)$ for each map $J \to I$ in the base category. This is an example of an *indexed inductive-recursive* definition. In order to give a semnatics of this in set theory, we follow an idea of Stuart Allen (that he designed for giving a semantics of universes in type theory) which consists in defining inductively relations, and prove by induction that these relations are functional relations and hence *graphs* of functions on their domains.

All the arguments can be represented in the system CZF+REA. We need REA in general since we use generalized inductive definitions.

1 Base category, fibrations and cofibrations

We write $I, J, K, \ldots$ the objects of a given small category $\mathcal{C}$.

We write $A, B, \ldots, X, Y, , \ldots$ for presheaves over $\mathcal{C}$. (A presheaf A is given by a collection of sets $A(I)$ with restriction maps $A(I) \to A(J)$ sending u to uf for $f : J \to I$.) We use the same notation for an object I and the presheaf it represents.

We assume given a special presheaf $\mathbb{I}$ which has a structure of distributive lattice with an involution (a.k.a. de Morgan algebra). We write $A^+ = A \times \mathbb{I}$, and this induces a functor on presheaves. We have two maps $e_0, e_1 : A \to A^+$ that are sections of the projection $p : A^+ \to A$.

From the lattice structure of $\mathbb{I}$, we get a conjunction map $m : A^{++} \to A^+$ such that $me_1 = me_1^+ = 1$ and $me_0 = me_0^+ = e_0p$.

We also assume given a subobject $\mathbb{F}$ of the subobject classifier which is a sub-lattice. Any map $\psi : A \to \mathbb{F}$ defines a subpresheaf $A|\psi \subseteq A$ where $(A|\psi)(I)$ is the subset of element ρ in $\Gamma(I)$ such that $\psi\rho = 1$ in $\mathbb{F}(I)$. We call a *sieve* on a presheaf A a set of maps S of codomain A of the form $f : I \to A$ such that fg is in S whenever $g : J \to I$. If $\sigma : B \to A$ we define a sieve $S\sigma$ on B as the set of maps $f : I \to B$ such that σf is in S.

It will be convenient to assume a lattice map $\epsilon_1 : \mathbb{I} \to \mathbb{F}$ which classifies the global element 1 of $\mathbb{I}$. By involution we get $\epsilon_0 : \mathbb{I} \to \mathbb{F}$. By composition of the projection $A^+ \to \mathbb{I}$ with ϵ_0 we get a map $\delta_0 : A^+ \to \mathbb{F}$ which classifies $e_0 : A \to A^+$. Similarly we have $\delta_1 : A^+ \to \mathbb{F}$ which classifies e_1. Using that ϵ_1 is a lattice map we get $\delta_1 m = \delta_1 \wedge \delta_1 p$ and $\delta_0 m = \delta_0 \vee \delta_0 p$.

If we have $\sigma : A \to B$ and $\psi : B \to \mathbb{F}$ then σ induces a map $A|\psi\sigma \to B|\psi$, that sends u in $(A|\psi)(I)$ to σu. We may write simply $\sigma : A|\psi\sigma \to B|\psi$ for this induced map.

We say that a map is a *cofibration* if, and only if, it is classified by $\mathbb{F}$.

If $\psi : A \to \mathbb{F}$ we define $b(\psi) = \delta_0 \vee \psi p : A^+ \to \mathbb{F}$. A (generalised) *open box* $\mathsf{b}(A, \psi) \subseteq A \times \mathbb{I}$ is the subpresheaf determined by $b(\psi) : A \times \mathbb{I} \to \mathbb{F}$ for some $\psi : A \to \mathbb{F}$. If $\psi : A \to \mathbb{F}$ we define the *open box sieve* on A^+ determined by ψ to be the sieve $S(\psi)$ defined by $b(\psi)$. Notice that

$$b(\psi)m = (\delta_0 \vee \psi p)m = \delta_0 m \vee \psi pm = \delta_0 \vee \delta_0 p \vee \psi pp = b(b(\psi))$$

If u is a family of elements indexed by a sieve S over A and $\sigma : B \to A$, we define $u\sigma$ to be the family $(u\sigma)_g = u_{\sigma g}$ indexed by the sieve $S\sigma$ over B.

A *fibration* is a map that has the right lifting property w.r.t. any open box. A *trivial fibration* is a map which has the right lifting property w.r.t. any cofibration. Finally a *trivial cofibration* is a map that has the left lifting property w.r.t. any fibration.

Cisinski calls «naive» fibrations what we call simply fibrations. The justification of our terminology is that, with some extra assumptions on the base category described below, we do get, as shown by Christian Sattler, a model structure on the presheaf category with these notions of fibrations, trivial fibrations, cofibrations and trivial cofibrations.

In order to have constructive arguments, we need to assume that $\psi = 1$ is decidable in each $\mathbb{F}(I)$, and that $r = 0$ (and hence also $r = 1$) is decidable in each $\mathbb{I}(I)$.

We could avoid the involution on $\mathbb{I}$, but then we have to consider not only the box $\psi p \vee \delta_0$ but also the box $\psi p \vee \delta_1$. All the arguments are then valid with this modification. In this way, we can cover the case of simplicial sets, but also, we think, the case where the base category is the category of all finite posets and monotone maps. In both cases, each object is a finite set and the generalized inductive definition can be replaced by an induction on the cardinality of the object (so that we can work simply in CZF).

In order to get a fibrant univalent universe, and a model structure, we need further assumptions on the base category: that $J^+ = J \times \mathbb{I}$ is always representable (and that we have a choice function on objects that selects an object which represents J^+), that cofibrations are closed by compositions, and finally that the map $\mathbb{F} \to \mathbb{F}^{\mathbb{I}}$ which corresponds to the first projection $\mathbb{F} \times \mathbb{I} \to \mathbb{F}$ has a right adjoint. We don't need however these extra assumptions for what we present in this note.

We recall the following results (valid constructively without using choice).

Theorem 1.1 *A map $\alpha : X \to Y$ is a fibration if, and only if, we have an operation which takes a commutative diagram*

$$\begin{tikzcd} \mathsf{b}(I,\psi) \arrow[r, "u"] \arrow[d, hook] & X \arrow[d, "\alpha"] \\ I^+ \arrow[r, "v"] & Y \end{tikzcd}$$

and produces a diagonal filler $\tilde{c}(I, v, \psi, u) : I \to X$ such that $\tilde{c}(I, v, \psi, u)f^+ = \tilde{c}(J, vf^+, \psi f, uf^+)$ if $f : J \to I$ and $\psi f \neq 1$.

Corollary 1.2 *A map $\alpha : X \to Y$ is a fibration if, and only if, we have an operation which takes a commutative diagram*

$$\begin{tikzcd} \mathsf{b}(I,\psi) \arrow[r, "u"] \arrow[d, hook] & X \arrow[d, "\alpha"] \\ I^+ \arrow[r, "v"] & Y \end{tikzcd}$$

and produces an element $c(I, v, \psi, u) : I \to X$ such that $\alpha c(I, v, \psi, u) = ve_1$ and $c(I, v, \psi, u)$ extends $ue_1 : I|\psi \to X$ and *furthermore satisfies the equations $c(I, v, \psi, u)f = c(J, vf^+, \psi f, uf^+)$ if $f : J \to I$ and $\psi f \neq 1$.*

Proof. If we have such an operation, $\tilde{c}(I, v, \psi, u) = c(I^+, vm, b(\psi), um)$ is a diagonal filler satisfying the uniformity equations. □

2 The circle

As a motivating example, we describe the circle S^1 generated by a point and a loop. The main idea is to define inductively the graphs of all restriction functions $S^1(I) \to S^1(J)$. So we define inductively a ternary relation $R(f, x, y)$ with $f : J \to I$.

The elements for the domain of these relations $R(f, x, y)$ are well-founded tree, that are of the form base, $\mathsf{loop}\ r$ for $r \neq 0, 1$ in some $\mathbb{I}(I)$ and $c(\psi, u)$ with $\psi \neq 1$ in some $\mathbb{F}(I)$ and u is a family of elements u_f indexed by the open box sieve determined by ψ.

The definition is as follows. First we have $R(f, \mathsf{base}, \mathsf{base})$ for all f, and $R(f, \mathsf{loop}\ r, \mathsf{loop}\ (rf))$ if $r \neq 0, 1$ and $rf \neq 0, 1$ and $R(f, \mathsf{loop}\ r, \mathsf{base})$ if $r \neq 0, 1$ and $rf = 0$ or 1, for r in $\mathbb{I}(I)$.

Next, if we have $\psi \neq 1$ in $\mathbb{F}(I)$ and we have u, family of elements indexed by $S(\psi)$, and we have $R(g, u_f, u_{fg})$ for all $g : K \to J$, then we add the relations $R(f, c(\psi, u), c(\psi f, uf^+))$ if $\psi f \neq 1$ and $R(f, c(\psi, u), u_{e_1 f})$ if $\psi f = 1$.

We can then prove by induction.

Lemma 2.1 *If $R(f, x, y)$ then $R(1_I, x, x)$ and $R(1_J, y, y)$. If $R(f, x, y)$ and $R(g, y, z)$ with $f : J \to I$ and $g : K \to J$ then $R(fg, x, z)$. If $R(f, x, y)$ and $R(f, x, y')$ then $y = y'$. If $R(1_I, x, x)$ and $f : J \to I$ there exists a unique y such that $R(f, x, y)$.*

It follows that if we define $S^1(I)$ as being the set of all elements x such that $R(1_I, x, x)$ then the relation $R(f, x, y)$ is the graph of a function $S^1(I) \to S^1(J)$ that we write $u \longmapsto uf$. We have $u1_I = u$ and $(uf)g = u(fg)$ if u in $S^1(I)$ and $f : J \to I$ and $g : K \to J$. Thus we have defined a presheaf S^1.

The element base is in all sets $S^1(I)$ and we have $\mathsf{base} f = \mathsf{base}$ for $f : J \to I$.

We have $\mathsf{loop}\ r$ in $S^1(I)$ if $r \neq 0, 1$ is in $\mathbb{I}(I)$ and we have $(\mathsf{loop}\ r)f = \mathsf{loop}\ (rf)$ if $f : J \to I$ and $rf \neq 0, 1$ in $\mathbb{I}(J)$, and $(\mathsf{loop}\ r)f = \mathsf{base}$ if $rf = 0$ or 1.

If $\psi \neq 1$ is in $\mathbb{F}(I)$ and $u : \mathsf{b}(I, \psi) \to S^1$, we can see u as a family of elements indexed by $S(\psi)$ defined by $u_g = ug$ if g is in $S(\psi)$, and we have $c(\psi, u) : I \to S^1$. Furthermore, $c(\psi, u)f = c(\psi f, uf^+)$ if $f : J \to I$ and $\psi f \neq 1$ and $c(\psi, u)f = u_{e_1 f}$ if $\psi f = 1$.

Using $b(\psi)m = b(b(\psi))$ we can define a filling operation $\tilde{c}(\psi, u) = c(b(\psi), um)$. We then have $\tilde{c}(\psi, u)f = um_{e_1 f} = u_{me_1 f} = u_f$ if $b(\psi)f = 1$.

This implies that we have the following extension property, in a uniform way

$$\begin{tikzcd}
\mathsf{b}(I, \psi) \arrow[r, "u"] \arrow[d, hook] & S^1 \\
I^+ \arrow[ur, dotted, "\tilde{c}(\psi,u)"'] &
\end{tikzcd}$$

and it follows that, for any X and $\psi : X \to \mathbb{F}$ we have the following extension property

$$\begin{tikzcd}
\mathsf{b}(X, \psi) \arrow[r] \arrow[d, hook] & S^1 \\
X^+ \arrow[ur, dotted, "\alpha"'] &
\end{tikzcd}$$

which is defined by $\alpha(\rho, r) = \alpha\rho^+(1, r) = \tilde{c}(\psi\rho, u\rho^+)(1, r)$.

Hence we have defined a *fibrant* presheaf (i.e. the map $S^1 \to 1$ is a fibration).

We can now state and prove by induction the universal property of the circle.

Theorem 2.2 *If $\alpha : E \to S^1$ is a fibration and we have a_I in $E(I)$ and $l_I\ r$ in $E(I)$ for $r \neq 0, 1$ in $\mathbb{I}(I)$ such that $a_I f = a_J$ and $l_I\ r = l_J\ (rf)$ if $rf \neq 0, 1$ and $l_I\ r = a_J$ if $rf = 0$ or 1 for $f : J \to I$ and $\alpha\ a_I = \mathsf{base}$ and $\alpha\ (l_I\ r) = \mathsf{loop}\ r$ then there exists a section $\beta : S^1 \to E$ of α such that $\beta_I\ \mathsf{base} = a_I$ and[1] $\beta_I\ (\mathsf{loop}\ r) = l\ r$.*

Proof. We define by inductively a relation $T(I, u, w)$ with w in $E(I)$, which will be the graph of this section. We first have $T(I, \mathsf{base}, a_I)$ and $T(I, \mathsf{loop}\ r, l_I\ r)$. Next, if we have $w : \mathsf{b}(I, \psi) \to E$ and $T(J, u_f, w_f)$ for all $f : J \to I^+$ in $S(\psi)$, we add $T(c(\psi, u), c_\alpha(I, \tilde{c}(\psi, u), \psi, w))$. We then prove by induction that each $T(I, u, w)$ is the graph of a function $S^1(I) \to E(I)$ and that $T(I, u, w)$ implies $\alpha w = u$. □

[1]This means that the semantics interprets both computation rules on points *and* paths as judgemental equalities.

3 Trivial cofibration-fibration factorization

Given $\sigma : A \to B$ we define inductively two relations $E(f, x, y)$ for $f : J \to I$ and $q(I, x, y)$:

1. We first have $E(f, i\ a, i\ (af))$ whenever a in $A(I)$ and $f : J \to I$ and $q(I, i\ a, \sigma\ a)$
2. If we have v in $B(I^+)$ and a family of element u indexed by $S(\psi)$ such that $E(g, u_f, u_{fg})$ if $g : J \to K$ and $q(J, u_f, ve_1 f)$ for all f and g then we add $q(I, c(I, v, \psi, u), ve_1)$ and if $\psi f \neq 1$, we also add $E(f, c(I, v, \psi, u), c(J, vf^+, \psi f, uf))$ and, if $\psi f = 1$, we add $E(f, c(I, v, \psi, u), u_{e_1 f})$.

We can then prove by induction the following result.

Lemma 3.1 *If $E(f, x, y)$ then $E(1_I, x, x)$ and $E(1_J, y, y)$. If $E(f, x, y)$ and $E(f, x, y')$ then $y = y'$. If $E(1_I, x, x)$ there exists a (unique) y such that $E(f, x, y)$. If $q(I, x, y)$ then $E(1_I, x, x)$ and y is an element of $B(I)$. If $E(1_I, x, x)$ there exists a unique y such that $q(I, x, y)$. If $q(I, x, z)$ and $R(f, x, y)$ then $q(J, y, zf)$. If $R(f, x, y)$ and $R(g, y, z)$ for $f : J \to I$ and $g : K \to J$ we have $R(fg, x, z)$.*

From this Lemma, we deduce that if we define a set $E(I)$ as the set of elements x such that $E(1_I, x, x)$ then the relation $E(f, x, y)$ is the graph of a function $E(I) \to E(J)$ that we write $u \longmapsto uf$. It satisfies $u1_I = u$ and $(uf)g = u(fg)$. The relation $q(I, x, y)$ is the graph of a function $E(I) \to B(I)$ that defines a natural transformation $q : E \to B$. We also have a natural transformation $i : A \to E$ and a factorization $\sigma = qi$. By construction, if $v : I^+ \to B$ and $\psi : I \to \mathbb{F}$, $\psi \neq 1$ and $u : \mathsf{b}(I, \psi) \to A$ such that $\sigma u = v$ we have $c(I, v, \psi, u)$ in $B(I)$ such that $q\ c(I, v, \psi, u) = ve_1$. Furthermore, $c(I, v, \psi, u)f = v(J, vf^+, \psi f, uf^+)$ if $f : J \to I$ and $\psi f \neq 1$ and $c(I, v, \psi, u)f = ue_1 f$ if $\psi f = 1$. So the map $q : E \to B$ is a fibration.

Lemma 3.2 *A map $X \to Y$ is a trivial cofibration if any commutative diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & F \arrow[d] \\ Y \arrow[r, phantom, "="] & Y \end{tikzcd}$$

where $F \to Y$ is a fibration, has a diagonal filler.

Proof. This follows from the fact that a pull-back of a fibration is a fibration. $\square$

Theorem 3.3 *The map $i : A \to E$ is a trivial cofibration.*

Proof. Using the Lemma, we have to build a diagonal filler of any commutative diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, hook, "i"'] & F \arrow[d, "\beta"] \\ E \arrow[r, phantom, "="] & E \end{tikzcd}$$

where $\beta : F \to E$ is a fibration. We define a relation $T(u, w)$ with u in $E(I)$ and w in $F(I)$ by induction, in such a way that $T(u, w)$ implies $\beta w = u$. First we have $T(i\ a, \alpha\ a)$ for a in $A(I)$. If we have $\psi \neq 1$ in $\mathbb{F}(I)$ and $w : \mathsf{b}(I, \psi) \to F$ and $T(u_f, w_f)$ for some family u indexed by $f : J \to I^+$ in $S(\psi)$, and $R(g, u_f, y_{fg})$, we add $T(c(I, v, \psi, u), c_\beta(I, \tilde{c}(I, v, \psi, u), \psi, v)$. If we have $\beta w_f = u_f$ for all f then we get $\beta\ c_\beta(I, \tilde{c}(I, v, \psi, u), \psi, w) = \tilde{c}(I, v, \psi, u)e_1 = c(I, v, \psi, u)$ as desired. The relation $T(u, w)$ is then the graph of a natural transformation $\gamma : E \to F$ such that $\gamma i = \alpha$ and $\beta\gamma = 1$. $\square$

4 Cofibration-trivial fibration factorization

This factorization is much simpler, and does not require an inductive definition, provided we assume that $\mathbb{F}$ is *contractible* (i.e. the map $\mathbb{F} \rightarrow 1$ is a trivial fibration), which is equivalent to assuming that cofibrations are closed by compositions.

Theorem 4.1 *A map $\sigma : A \rightarrow B$ has a factorization in a cofibration $j : A \rightarrow E$ and a trivial fibration $q : E \rightarrow B$.*

Proof. We define $E(I)$ to be the set of elements v, ψ, u where $\psi : I \rightarrow \mathbb{F}$ and $v : I \rightarrow B$ and $u : I|\psi \rightarrow A$ such that v extends σu. We then define $j\ a$ to be the element $\sigma a, 1, a$ for a in $A(I)$ and $q(v, \psi, u)$ to be v. $\square$

Corollary 4.2 *A map is a cofibration if, and only if, it has the left lifting property w.r.t. any fibration.*