Generalised algebraic presentation of type theory

Introduction

The goal of this note is to present a variation of the name-free presentation of substitution calculus for type theory which is a variation of the usual presentation and which was actually the first presentation, in the 1988 PhD thesis of Thomas Ehrhard [3, 5]. To have such a generalized algebraic presentation is quite important when describing the semantics of type theory.

The syntax for context is

$$\Gamma \;::=\; () \mid \Gamma.A$$

The syntax for terms/types is

$$M, A \;::=\; \mathsf{q} \mid M\ M \mid \lambda M \mid M\sigma \mid \Pi\ A\ B$$

The syntax for substitution is

$$\sigma, \delta \;::=\; \mathsf{id} \mid \mathsf{p} \mid \sigma\delta \mid \sigma^+ \mid [M]$$

The usual syntax is

$$\sigma, \delta \;::=\; \mathsf{id} \mid \mathsf{p} \mid \sigma\delta \mid (\sigma, M)$$

We can translate $(\sigma, M) = \sigma^+[M]$ and $\sigma^+ = (\sigma\mathsf{p}, \mathsf{q})$ and $[M] = (\mathsf{id}, M)$.

One motivation for Ehrhard's syntax is that these substitutions are the ones that occur for the rules

$$(\lambda M)\ N = M[N] \qquad (\lambda M)\sigma = \lambda(M\sigma^+) \qquad (\Pi\ A\ B)\sigma = \Pi\ A\sigma\ (B\sigma^+)$$

The typing rules are

$$\frac{\sigma : \Delta \to \Gamma \quad \Gamma \vdash A}{\sigma^+ : \Delta.A\sigma \to \Gamma.A}$$

$$\frac{\Gamma \vdash M : A}{[M] : \Gamma \to \Gamma.A}$$

and the equations are

$$\mathsf{id}\sigma = \sigma\mathsf{id} = \sigma \qquad (\sigma\delta)\theta = \sigma(\delta\theta)$$

$$\mathsf{id}^+ = \mathsf{id} \qquad (\sigma\delta)^+ = \sigma^+\delta^+ \qquad \sigma\mathsf{p} = \mathsf{p}\sigma^+$$

$$\mathsf{p}[M] = \mathsf{id} \qquad \mathsf{q}[M] = M \qquad \mathsf{q}\sigma^+ = \mathsf{q}$$

$$[M]\sigma = \sigma^+[M\sigma] \qquad \mathsf{p}^+[\mathsf{q}] = \mathsf{id}$$

We can define $(\sigma, M) = \sigma^+[M]$ with the rule

$$\frac{\sigma : \Delta \to \Gamma \quad \Delta \vdash M : A\sigma}{(\sigma, M) : \Delta \to \Gamma.A}$$

and we can prove the equations

$$\mathsf{p}(\sigma, M) = \sigma \qquad \mathsf{q}(\sigma, M) = a$$

$$(\sigma, M)\delta = (\sigma\delta, M\delta) \qquad (\mathsf{p}, \mathsf{q}) = \mathsf{id}$$

$$\frac{\Gamma\vdash}{\mathsf{id}:\Gamma\to\Gamma}\qquad\frac{\sigma:\Delta\to\Gamma\quad\delta:\Theta\to\Delta}{\sigma\delta:\Theta\to\Gamma}$$

$$\frac{\Gamma\vdash A\qquad\sigma:\Delta\to\Gamma}{\Delta\vdash A\sigma}\qquad\frac{\Gamma\vdash M:A\qquad\sigma:\Delta\to\Gamma}{\Delta\vdash M\sigma:A\sigma}$$

$$\frac{}{\vdash}\qquad\frac{\Gamma\vdash\qquad\Gamma\vdash A}{\Gamma.A\vdash}\qquad\frac{\Gamma\vdash A}{\mathsf{p}:\Gamma.A\to\Gamma}\qquad\frac{\Gamma\vdash A}{\Gamma.A\vdash\mathsf{q}:A\mathsf{p}}$$

$$\frac{\sigma:\Delta\to\Gamma\quad\Gamma\vdash A\quad\Delta\vdash M:A\sigma}{\sigma^+:\Delta.A\sigma\to\Gamma.A}\qquad\frac{\Gamma\vdash M:A}{[M]:\Gamma\to\Gamma.A}$$

$$\sigma\mathsf{id}=\sigma\qquad\mathsf{id}\sigma=\sigma\qquad(\sigma\delta)\nu=\sigma(\delta\nu)$$

$$\mathsf{id}^+=\mathsf{id}\qquad(\sigma\delta)^+=\sigma^+\delta^+\qquad\sigma\mathsf{p}=\mathsf{p}\sigma^+$$

$$\mathsf{p}[M]=\mathsf{id}\qquad[M]\sigma=\sigma^+[M\sigma]\qquad\mathsf{p}^+[\mathsf{q}]=\mathsf{id}$$

Indeed we have

$$\mathsf{p}(\sigma,M)=\mathsf{p}\sigma^+[M]=\sigma\mathsf{p}[M]=\sigma$$

and

$$\mathsf{q}(\sigma,M)=\mathsf{q}\sigma^+[M]=\mathsf{q}[M]=M$$

and

$$(\sigma,M)\delta=\sigma^+[M]\delta=\sigma^+\delta^+[M\delta]=(\sigma\delta)^+[M\delta]=(\sigma\delta,M\delta)$$

and

$$(\mathsf{p},\mathsf{q})=\mathsf{p}^+[\mathsf{q}]=\mathsf{id}$$

The equations for substitution in a term/type are

$$(M\ N)\sigma=M\sigma\ (N\sigma)\qquad(\lambda N)\sigma=\lambda(N\sigma^+)\qquad(\Pi\ A\ B)\sigma=\Pi\ (A\sigma)\ (B\sigma^+)\qquad\mathsf{q}\sigma^+=\mathsf{q}$$

References

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