This talk

I will survey some works connected to the formulation of a constructive model of univalence

Symmetric cubical sets

Symmetric cubical sets with connections

Uniformity condition

Why introducing connections?

Formalisation and consequences

Canonicity
This talk

Strategy we followed: to find a model of type theory with univalence in a constructive framework

Once this is done, the formulation of an associated type theory, with its associated operational semantics, is direct

The operational semantics corresponds in writing precisely the constructive semantics
«Spaces» as Kan simplicial sets

Does not work constructively

If we try to prove e.g. that $B^A$ is Kan if $B$ is Kan, we see informally that we need to decide if a simplex of $A$ is degenerate or not

Similarly if $E \to B$ Kan fibration and we try to build an equivalence $E(b_0) \to E(b_1)$ lifting a path $b_0 \to b_1$ in $B$

This informal remark can be transformed to a Kripke counter-model (j.w.w. Marc Bezem)
For $B^A$ see the Ph.D. thesis of Erik Parmann (Bergen, February 2016)

This counter-example holds even if the Kan filling condition is expressed as a given operation (and not simply as an existential statement)
So we need to find a refinement of Kan’s definition

First natural attempt: the restriction map $B^{\Delta^n} \to B^{\Lambda^k_n}$ has a section and the corresponding refinement for fibration

This is enough for the result about $B^A$ being Kan

However this does not seem sufficient for interpreting dependent product?
In the description of the model structure on simplicial sets the interpretation of dependent product reduces to the fact that trivial cofibrations are stable under pullbacks along Kan fibrations.

In turn this reduces to the fact that the model structure is *proper*.

But this is a non trivial fact, which is proved using minimal fibrations (in A. Joyal’s presentation).

Goerss and Jardine uses topological spaces at this point.
Constructive semantics of Univalence

Partial elements

Instead the refinement is the following for the definition of being contractible

In an elementary topos, let $\text{part}(A)$ be the object of partial element of $A$

We have a canonical map $A \rightarrow \text{part}(A)$

$A$ is \textit{contractible} iff this map has a retraction: we can extend any partial element to a total element

For simplicial set, this definition is \textit{classically} equivalent to the usual definition, which is classically equivalent to being \textit{injective}

N. Gambino and Ch. Sattler \textit{Uniform fibrations and the Frobenius condition}
Partial elements

In the «internal logic» of the presheaf model

\( \Gamma \vdash A \), not necessarily fibrant, is a family of sets \( A_\gamma, \gamma \in \Gamma \)

For \( \Gamma \vdash A \) to correspond to a Kan fibration

If we have \( \gamma \in \Gamma^I \) and \( a_0 \in A_{\gamma_0} \) and a partial section \( a(i) \in A_{\gamma(i)}, \ i \in I \) with \( a(0) = a_0 \) then can extend this to a total section \( \tilde{a}(i) \in A_{\gamma(i)} \)

In the special case where the partial section is nowhere defined this means that we have the path lifting property

We only have to consider \( \Gamma^I \) and not all \( \Gamma^{II} \)
How far can one go with an internal formulation in an elementary topos?

See work of Andy Pitts and Ian Orton

*Axioms for Modelling Cubical Type Theory in a Topos*, CSL 2016

Partially formalized in Agda
Cubical sets

The definition was suggested by a «nominal» definition of symmetric cubical set

Presheaf over the base category: objects are finite sets (of names/symbols) and an element of $\text{Hom}(I,J)$ is a map of domain $J$ the elements of $J$ to «disjoint» elements in $I \cup \{0, 1\}$ We think of the names as names for «direction»; not having an order on the name is very natural (homogeneity of the directions)

If the Kan filling operation is given by an explicit operation, it is natural to ask that this operation commutes with renaming and addition of new names

With this added uniformity condition one can interpret dependent products in a constructive framework
Cubical sets

Implemented in Haskell (December 2013)

There is a natural definition of a universe: $U(I)$ collection of all small types $I \vdash A$

There is a natural candidate for the Kan filling operation on the universe but it is not at all trivial to check that this works (done independently by G. Gonthier and S. Huber, December 2013)

Both for dependent product and universe, one has first to define Kan composition and then the Kan filling operation

In both cases, the definition of composition is much easier than the definition of filling
Cubical sets

There is also a natural way to transform a given equivalence

\[ f : A \rightarrow B \]

to a path \( E \) between \( A \) and \( B \)

A hypercube \( a \rightarrow b \) in \( E \) is defined by a cube \( a \) in \( A \) together with a cube (of the next dimension) \( f \ a \rightarrow b \) in \( B \)

It is rather direct to check that the cubical set defined in this way has a Kan filling operation, provided \( A \) and \( B \) have one and \( f \) is an equivalence
There is a natural interpretation of the circle as the Kan cubical set generated by base, $\text{loop}(i)$ and Kan filling operation, taken as a constructor.

When we tried to implement the elimination operation we discovered a problem with linearity, given $a : A$, $l : a \to a$.

$$f : S^1 \to A$$

$$f \text{ loop}(i) = l \ i$$

It may be that $i$ occurs in $l$!

So one has to take away the linearity condition.
Cubical sets

The other problem was that we only interpret the computation rule for identity elimination as a *propositional* equality.
Cubical sets with connections

To solve both problems, I tried to work with symmetric cubical sets with diagonals (and reversal) and connections, i.e. works with maps $J \to dM(I)$

In a first attempt, we had an extra regularity condition (terminology from Hurewicz) on fibrations: the transport along a constant path of types is the identity function

This takes exactly care of the computation rule of identity elimination

Also one can then reduce filling to composition (and it was first thought that regularity was necessary)

The composition for universe has to be regular/normal
We can still transform an equivalence to a path.

This suggested the glueing operation \( \text{Glue} [\varphi \mapsto (T, w)] B \).

We give a hypercube \( B \) is the universe and a collection of compatible equivalences of codomain some faces of \( B \).

Such a generalization comes naturally because of the uniformity condition with connections.
Dan Licata (with Bob Harper, Carlo Angiuli and Ed Morehouse) found a problem with regularity for universe (May 30 2015)

This suggests strongly that we cannot have $\text{Path} = \text{Id}$

This suggested to look more carefully at the reduction of filling to composition: regularity is actually not needed for this

This simplified significantly the model

Composition for the Glueing could be defined almost as before
S. Huber noticed that we can also take away the regularity condition $B = \text{Glue} [] B$ for the glueing operation.

Together with A. Mörtberg, they noticed further that we can derive univalence only using the glueing operation (which was only there a priori to transform an equivalence to a path).
In this version (June 2015), we have \( \text{Path} \neq \text{Id} \)

Andrew Swan found a way to define \( \text{Id} \) in the first cubical set model (without connections), using a small object argument.

His definition can be adapted for the model with connections, and can then be formulated in a «closed form»: an element \( a_0 \to a_1 \) of the \( \text{Id} \) type is a hypercube where we mark explicitly what faces are constant.
N. Gambino and P. Lumsdaine noticed the similarity of the glueing operation and some diagram in the proof of Theorem 3.4.1 in the paper presenting the simplicial model.

In discussions with S. Huber, we noticed that the glueing operation can be seen as a part of the expression that the type \((X : U) \times \text{Equiv } A X\) is contractible, by stating that any partial element can be extended to a total element.

As it turned out, this is an equivalent formulation of univalence (which is also used at the beginning of the proof of Theorem 3.4.1).
Glueing

So an operation like glueing has to be present in any model of univalence.

Since any path in the universe defines an equivalence, it follows from glueing that the universe has a composition operation.

The proof is much simpler than for the model without connections.
We have a model of the univalence axiom in a constructive meta theory.

Can we derive a model where

(1) Markov principle is not valid, or

(2) some continuity principle holds?

One proposal for doing this is to notice that the models can be expressed in CZF or NuPrl and then to work with models of these theories satisfying (1) or (2)
It is very important to have a formal verification of this model.

Mark Bickford has succeeded in formalizing in NuPrl the glueing operation (May 2016).

This is a non trivial result, and it will be important to also have a formalization in intensional type theory with univalence and to compare the two formalizations.
Higher Inductive Types

Since we have diagonals we can interpret the spheres without problems.

The model justifies computation rules both for points and paths.

We also have a model of suspension and propositional truncation.

As far as I know, this is the first model of these operations (we solve the coherence problems indicated in the work of M. Schulman and P. Lumsdaine).
It follows from this formalization that intensional type theory with univalence is not proof theoretically stronger than NuPrl.

It follows that univalence does not add any proof theoretic strength.
Cofibrations

Christian Sattler was able to define a (proper) Quillen model structure in a constructive framework

(This is very surprising since the duality between fibration/cofibration seems at first strongly linked to classical logic)

This uses in a crucial way the *cubical* structure

Is there such a constructive model structure for simplicial sets?

In this model, *cofibrations* are only special kind of monos
Constructive semantics of Univalence

Cofibrations

An argument of Andrew Swan shows that we cannot have cofibrations = monos in a constructive way and have the glueing operation

Take $Y$ constant presheaf $\{\{0\}, \{0|p\}\}$ and $X$ constant presheaf $\{0\}$ where 0 is the empty set

If $p$ holds we have an equivalence $X \rightarrow Y$

If we have a glueing operation and monos are cofibrations we should have a presheaf $Z$ such that $Z = X$ if $p$ holds and an equivalence $Z \rightarrow Y$

We then have $\forall x \in Z[0] \ x = 0$ iff $p$ holds

So $p$ is stable (implied by its double negation)
Canonicity

S. Huber has just finished the proof of canonicity (any closed term of type nat is convertible to a numeral) for the associated cubical type theory, using a presheaf version of Tait’s computability method (1967)

There also we need to have a formal version of the proof
Constructive semantics of Univalence

Canonicity

One defines computability predicate \( I \models t : A \) and relation \( I \models t = u : A \)

E.g. \( I \models t : \text{bool} \) iff \( J \models tf \rightarrow^* 0 : \text{bool} \) or \( J \models tf \rightarrow^* 1 : \text{bool} \) for all \( f : J \rightarrow I \)

Reformulation of the "expansion Lemma" (similar, but independent of the work of Bob Harper and Carlo Angiuli)

If we have \( I \models A \) and \( I \vdash t : A \) and we have a family of terms \( J \models u_f : Af \) with the conditions \( J \models tf \rightarrow^* u_f : Af \) and \( K \models u_fg = u_fg : Afg \) for \( f : J \rightarrow I \) and \( g : K \rightarrow J \) then \( I \models t : A \) and \( I \models t = u_1 : A \)
Canonicity

This suggests the following conjecture (weakening of Voevodsky’s conjecture)

If we have $\vdash t : \text{bool}$ in type theory extended with the univalence axiom and $\vdash t = 0 : \text{bool}$ in cubical type theory can we show that the semantics of $t$ is equal to $0$ in the simplicial set model?

It should be enough to validate (classically) the rules of cubical type theory in the simplicial set model
A variation on propositional resizing would be to introduce $\Omega : U$ with rules

$$
\frac{\Gamma \vdash A \quad \Gamma \vdash p : \text{isProp } A}{\Gamma \vdash (A, p) : \Omega}
\quad
\frac{\Gamma \vdash X : \Omega}{\Gamma \vdash X.1}
\quad
\frac{\Gamma \vdash X : \Omega}{\Gamma \vdash X.2 : \text{isProp } (X.1)}
$$

We conjecture that canonicity still holds for this extension.