Lecture Notes on Cubical sets

Introduction

These are some lecture notes for a course presenting the cubical set model of type theory, first in Copenhagen, December 2014, and then in Paris, February 2015.

We describe a particular presheaf model of type theory. This description can also be seen as an operational semantics of a purely syntactical type system. It involves a nominal extension of λ-calculus.

We use a generalization of the Kan composition operation which is represented by a nominal operation. Any element represents a hypercube, the direction being represented by the symbols this element depends on. One can define the notion of homotopy between two cubes (it is a cube with one new dimension connecting the two given cubes). The operation we introduce consists in changing some faces along given homotopies; Kan composition is the special case where we replace *all* faces. One new feature, compared with usual λ-calculus or first-order logic, is how this operation interacts with nominal substitution.

1 Cubical sets

1.1 Definitions

A *de Morgan algebra* is a bounded distributive lattice A, with a top element 1 and a bottom element 0 and with an operation $1-i$ satisfying

$$1-0=1 \qquad 1-1=0 \qquad 1-(i\vee j)=(1-i)\wedge(1-j) \qquad 1-(i\wedge j)=(1-i)\vee(1-j)$$

This notion differs from the one of Boolean algebra by requiring neither $1=i\vee(1-i)$ nor $0=i\wedge(1-i)$. A prime example of a de Morgan algebra, which is not a Boolean algebra, is the interval $[0,1]$ with $\max(i,j),\min(i,j)$ operations.

We assume a given (discrete) set of symbols/names/directions, not containing $0,1$. We let $I,J,K,\dots$ denote finite sets of such symbols. Let $\mathcal{C}$ be the following category. The objects are finite sets $I,J,K,\dots$. A morphism $I\rightarrow J$ is a map $I\rightarrow \mathsf{dM}(J)$, where J is the free de Morgan algebra on J. We think of f as a substitution and may write if the element $f(i)$ in $\mathsf{dM}(J)$. If $f:I\rightarrow J$ and $g:J\rightarrow K$ we write $fg:I\rightarrow K$ the composition of f and g. We write $1_I:I\rightarrow I$ the identity map. A *cubical set* is a presheaf on $\mathcal{C}^{opp}$, i.e. a functor $\mathcal{C}\rightarrow \mathsf{Set}$.

A cubical set X is thus given by a family of sets $X(I)$ together with a restriction map

$$X(I)\rightarrow X(J)$$

$$u\longmapsto uf$$

such that $u1_I=u$ and $(uf)g=u(fg)$. (We write uf for what is usually written $X(f)(u)$, since we want to think about this operation as a substitution; the elements of $X(I)$ for $I=i_1,\dots,i_n$ are thought of as elements $u=u(i_1,\dots,i_n)$ depending on $i_1,\dots,i_n$ and the restriction uf as a substitution operation.) For instance an element $u=u(i,j)$ in $X(i,j)$ represents a square, and if $(i0):i,j\rightarrow j$ is the map sending i to 0, then $u(i0)$ is the face $u(0,j)$ of this square. If $(i=j):i,j\rightarrow j$ is the map sending i to j then $u(i=j)$ is the diagonal $u(j,j)$.

We write $\vdash_I A$ if A is a preshaf on the slice category $\mathcal{C}^{opp}/I$. If I is empty, we get back a cubical set. If $I=i$ then $A=A(i)$ represents a "line" connecting the cubical sets $A(0)$ and $A(1)$. In general, if $I=i_1,\dots,i_n$ then A represents a hypercube. Concretely, A is given by a family of sets Af indexed by $f : I \to J$ together with a family of restriction maps $u \longmapsto ug$, $Af \to Afg$ for $g : J \to K$ such that $u1_J = u$ and $(ug)h = u(gh)$ if $h : K \to L$. If $\vdash_I A$ and $f : I \to J$ we can consider $\vdash_J Af$ which is defined by $(Af)g = A(fg)$ for $g : J \to K$.

We write $\vdash_I a : A$ to mean that a is an element in the set $A1_I$. It then defines a family of elements af in Af.

1.2 Examples

Any topological space X defines a cubical set, by taking $X(I)$ to be the set of continuous maps $[0,1]^I \to X$.

Any (strict) category defines a cubical set, by taking for points the object of the category, the lines being given by maps, squares given by commuting squares, and so on.

If R is any de Morgan algebra, a map $f : I \to J$ defines canonically a map $R^J \to R^I$ by composing $R^{\mathsf{dM}(J)} \to R^I$ with the extension map $R^J \to R^{\mathsf{dM}(J)}$. Since $[0,1]$ is a de Morgan algebra, we can in particular define a functor

$$\mathcal{C}^{opp} \to \mathsf{Top}$$
$$I \longmapsto [0,1]^I$$

This can then be used to define the geometric realization functor sending a cubical set to a topological space; this functor commutes with finite products.

The *interval* $\mathbb{I}$ is the cubical set defined by $\mathbb{I}(J) = \mathsf{dM}(J)$. This defines a functor since any map $I \to \mathsf{dM}(J)$ corresponds exactly to a de Morgan algebra map $\mathsf{dM}(I) \to \mathsf{dM}(J)$. We can think of $\mathbb{I}$ as an abstract representation of the unit real interval $[0,1]$, and we have operations $i \wedge j$, $i \vee j$, $1 - i$ that are abstract representations of the operations $\min(i,j)$, $\max(i,j)$, $1-i$. An element of $\mathsf{dM}(I)$ is determined by a de Morgan formula ψ on indeterminates in I and the restriction map $\psi \longmapsto \psi f$ is a substitution.

2 Remarks on the base category

We say that a map $f : I \to J$ is *strict* if if is neither 0 nor 1 for all i in I. One key remark is the following.

Lemma 2.1 *If $f : I \to J$ is strict and ψ in $\mathsf{dM}(I)$ such that $\psi f = b$ (where b is 0 or 1) then already $\psi = b$.*

(This does not hold if we work with Boolean algebra instead of de Morgan algebra. For instance the map $(i = j) : \{i,j\} \to \{j\}$ is strict and $(i \wedge (1-j))(i = j) = 0$ in a Boolean algebra, but $i \wedge (1-j)$ is neither 0 nor 1.)

A face map $\alpha : I \to I_\alpha$ is a map such that $i\alpha$ is either $0, 1$ or i for all i in I. We write I_α the subset of element i such that $i\alpha = i$, and $dom(\alpha) = I - I_\alpha$ is the *domain* of α. If $\iota_\alpha : I_\alpha \to I$ is the inclusion, we have $\iota_\alpha \alpha = 1$ and hence any face map α is *epi.* If $f : I \to J$ we write $f \leqslant \alpha$ to mean that there exists a map f' (uniquely determined) such that $f = \alpha f'$. This means that $if = i\alpha$ for all i in the domain of α. This defines a poset structure on the set of face maps $\alpha : I \to I_\alpha$ and this poset is a partial meet-semilattice: if α and β are compatible then they have a meet $\gamma = \alpha \wedge \beta$ with $I_\gamma = I_\alpha \cap I_\beta$.

Corollary 2.2 *If $fg \leqslant \alpha$ and g is strict then $f \leqslant \alpha$.*

Proof. For any i in the domain of α we have $i\alpha = ifg$ and so $i\alpha = if$ since $i\alpha = 0$ or 1 and by Lemma **??**. $\square$

Any map $f : I \to J$ can be written uniquely as the composition $f = \alpha h$ of a face map $\alpha : I \to I_\alpha$ and a map $h : I_\alpha \to J$ which is strict.

Lemma 2.3 *If we have $\alpha f = \beta g$ with $f : I_\alpha \to J$ and $g : I_\beta \to J$ then α and β are compatible. If γ is the meet of α and β, then there exists a unique $h : I_\gamma \to J$ such that $\alpha f = \gamma h = \beta g$. If we write $\alpha\alpha_1 = \gamma = \beta\beta_1$ then $\alpha_1 f = h = \beta_1 g$.*

3 Systems

We define $\mathsf{S}(I)$ to be the set of downward closed subset of face operations on I. An element of $\mathsf{S}(I)$ is determined uniquely by the set of its maximal element, a set L of incomparable face operations on I. We write $\alpha \leqslant L$ to mean that $\alpha \leqslant \beta$ for some β in L. If L is in $\mathsf{S}(I)$ and $g : I \to J$ we write $f \leqslant I$ to mean that $f \leqslant \alpha$ for some $\alpha \leqslant L$. We define then $\beta \leqslant Lf$ for $\beta : J \to J_\beta$ and $f : I \to J$ to mean $f\beta \leqslant L$. This define a new downward closed subset of face operations of J. Thus S can also be seen as a cubical set, since we have $L1_I = L$ and $(Lf)g = L(fg)$.

If $\vdash_I A$, and L in $\mathsf{S}(I)$ a *L-system for A* is given by a family a_α in $A\alpha$ which is compatible: if $\alpha\alpha_1 = \beta\beta_1$ then $a_\alpha\alpha_1 = a_\beta\beta_1$. We think of such a system as a system of equations $u\alpha = a_\alpha$ for α in L. Notice that any element v in $A1_I$ defines a compatible system $a_\alpha = v\alpha$. This implies that if $\alpha f = \beta g$ then we have $a_\alpha f = a_\beta g$.

If $f \leqslant L$ we can define a_f in Af without ambiguity: if we have both $f = \alpha g$ and $f = \beta h$ then α and β are compatible by Lemma **??**. A L-system can also be seen as a compatible family a_f for $f \leqslant L$.

If $f : I \to J$ and we have a L-system $(a_h),\ h \leqslant L$ we define a Lf system b_β by taking $b_\beta = a_{f\beta}$ and more generally $b_g = a_{fg}$.

We adopt the following notations for systems. If for instance $I = i, j$ and $\vdash_I A$ and L is determined by $(i0)$ and $(j1)$, a L-system $\vec{a}$ will be determined by an array $(i0) \mapsto u,\ (j1) \mapsto v$, with u in $A(i0)$ and v in $A(j1)$. If $f : I \to k, l, j$ is defined by $f(i) = k \wedge l,\ f(j) = j$ then Lf is the system $(k0), (l0), (j1)$ and $\vec{a}f$ is the system $(k0) \mapsto u,\ (l0) \mapsto u,\ (j1) \mapsto v$. If $g : I \to j$ is defined by $g(i) = g(j) = j$ then Lg is empty and $\vec{a}g$ is the empty system. Notice that $\vec{a}(i0)(j1)$ is the system $() \mapsto u(j1)$ which is equal to $() \mapsto v(i0)$.

For motivations why we introduce such a notion of system, see Appendix 1.

4 Operational semantics

We limit ourselves first to the description of the system without universes. (We describe later the operational semantics for univalence and composition in te universe.) The point is to explain how we can justify *function extensionality* without using function extensionality at the metalevel.

The syntax for the terms is

$$t, p, A, E, F \ ::= \ x \mid t\ t \mid \lambda x.t \mid \mathsf{Id}\ A\ t\ t \mid \Pi\ A\ F \mid \langle i \rangle t \mid t\vec{p} \mid \mathsf{comp}^i(A) \mid t\ \varphi$$

where φ represents an element in the free de Morgan algebra on the symbols. In this syntax, $\langle i \rangle t$ represents the path abstraction operation, and binds the symbol i. We use the vector notation $\vec{t}$ to represent a system of terms. For instance $\vec{t}$ may be of the form $(j0) \mapsto t, (j1) \mapsto u$ or of the form $(j0) \mapsto t, (k0) \mapsto u, (j1)(k1) \mapsto v$.

The composition operation $t\vec{p}$ is a *new* kind of nominal operation. Intuitively, it consists in replacing the face $t\alpha$, which is equal to $p\alpha 0$, by the face $p\alpha 1$. The special character of this operation is reflected by the way substitution interacts with it. We have for instance

$$(a|(j0) \mapsto u,\ (j1) \mapsto v)f = (af|(k0) \mapsto u,\ (l0) \mapsto u,\ (k1)(j1) \mapsto v)$$

if $jf = k \wedge l$. We also have

$$(a|(j0) \mapsto p,\ (j1) \mapsto q)(j0) = p\ 1 \qquad (a|(j0) \mapsto p,\ (j1) \mapsto q)(j1) = q\ 1$$

This operation also satisfies a *regularity* condition. We have $a(\vec{p}, \alpha \mapsto a\alpha) = a\vec{p}$. This expresses a *strict* identity element law for this composition.

We define $p^* = \langle i \rangle p(1-i)$ and

$$\vec{p}a = a\vec{p^*}$$

The operation $\mathsf{comp}^i(A)$ binds the symbol i. Its intended type is $A(i0) \to A(i1)$. The regularity condition is that $\mathsf{comp}^i(A, a_0) = a_0$ if A is independent of i. We may write $\mathsf{comp}^i(A, a_0)$ instead of $\mathsf{comp}^i(A)\ a_0$.

We define $\mathsf{fill}^i(A, a_0) = \mathsf{comp}^j(A(i \wedge j), a_0)$ which satisfies $\Gamma, i : \mathbb{I} \vdash \mathsf{fill}^i(A, a_0) : A$ and $\mathsf{fill}^i(A, a_0)(i0) = a_0$ and $\mathsf{fill}^i(A, a_0)(i1) = \mathsf{comp}^i(A, a_0)$. So this element represents a line in direction i connecting a_0 and $\mathsf{comp}^i(A, a_0)$.

We can generalize this operation as follows. We define, given a system $\vdash_{I_\alpha, i} a_\alpha : A\alpha$ such that $a_0\alpha = a\alpha(i0)$ the element

$$\mathsf{comp}^i(A, a_0, \vec{a}) = u\vec{u} \ : A(i1)$$

where $u = \mathsf{comp}^i(A, a_0) : A(i1)$ and $u_\alpha = \langle i \rangle \mathsf{comp}^j(A\alpha(i \vee j), a_\alpha)$. This element satisfies

$$\mathsf{comp}^i(A, a_0, \vec{a})\alpha = a_\alpha(i1)$$

We can then define

$$\mathsf{fill}^i(A, a_0, \vec{a}) = \mathsf{comp}^j(A(i \wedge j), a_0, \vec{a}(i \wedge j)) : A$$

which satisfies

$$\mathsf{fill}^i(A, a_0, \vec{a})\alpha = a_\alpha \qquad \mathsf{fill}^i(A, a_0, \vec{a})(i0) = a_0$$

We have the usual β-reduction rule

$$(\lambda x.t) \ u = t(x = u)$$

We write $(x : A) \to B$ for $\Pi \ A \ (\lambda x.B)$.

If $f : I \to \mathsf{dM}(J)$ we can define the operation $t \longmapsto tf$ on terms. (Notice that this is a defined operation on terms; we don't have an explicit term constructor for substitution.) We have

$$(\langle i \rangle t)f = \langle j \rangle tg$$

where $g : I, i \to \mathsf{dM}(J, j)$ extends f by $g(i) = j$ not in J. We also have

$$(\lambda x.t)f = \lambda x.tf \qquad xf = x \qquad (t \ u)f = tf \ uf \qquad (t \ \varphi)f = tf \ \varphi f \qquad (\Pi \ A \ F)f = \Pi \ Af \ Ff$$

We can then state the path reduction law

$$(\langle i \rangle t) \ \varphi = t(i = \varphi)$$

A canonical object of type $\mathsf{Id} \ A \ a \ b$ is of the form $\langle i \rangle t$ with $t(i0) = a$ and $t(i1) = b$. If w is of type $\mathsf{Id} \ A \ a \ b$, then $w\varphi$ is of type A and $w0 = a$ and $w1 = b$.

If L is a system for I we write $p : \mathsf{Id}^L \ A \ u \ v$ or $p : u \sim_L v$ to express that we have $p \ 0 = u$ and $p \ 1 = v$ and each p_α is constant for $\alpha \leqslant L$.

We define $a \uparrow \vec{p}$ to be $\langle i \rangle a\vec{q}$ where $q_\alpha = \langle j \rangle p_\alpha(i \wedge j)$. Using regularity we have that $a \uparrow \vec{p}$ defines a line $l : a \sim (a\vec{p})$ such that $l\alpha = p_\alpha$. We also define $\vec{p} \uparrow a = a \uparrow \vec{p^*}$.

The main new computation rules are for the composition of a product type and the composition of an identity type.

4.1 Dependent products

For dependent product types, we have

$$(w\vec{p}) \ a = (w \ a)\vec{q}$$

where $q_\alpha = \langle i \rangle p_\alpha(i, a\alpha)$.

This provides a short and effective justification of the fact that homotopy types are closed by exponentiation. The reader can compare this argument with the argument in [**?**, **?**].

We also have

$$\mathsf{comp}^i(\Pi \ A \ F, w_0) \ u_1 = \mathsf{comp}^i(F \ u, w_0 \ u(i0))$$

where $u = \mathsf{fill}^{1-i}(A, u_1)$.

4.2 Identity types

For identity types, we have

$$(p\vec{q})\ \varphi = p\ \varphi\vec{r}$$

where $r_\alpha = \langle i\rangle q_\alpha(i,\varphi)$.

We also have

$$\mathsf{comp}^i(\mathsf{Id}\ A\ a\ b, p_0) = \langle j\rangle \mathsf{comp}^i(A, \vec{u}, p_0\ j)$$

$\vec{u}$ is the system $(j0)\mapsto a,\ (j1)\mapsto b$.

We can interpret

$$\mathsf{Id}\ A\ a_0\ a_1 \to B(a_0) \to B(a_1)$$

Indeed if p is of type $\mathsf{Id}\ A\ a_0\ a_1$ and $b_0 : B(a_0)$ then $\mathsf{comp}^i(B(p\ i), b_0)$ is of type $B(a_1)$.

Using the operation $i\wedge j$ on symbols we can interpret the fact that $(x : A, \mathsf{Id}\ A\ a\ x)$ is contractible. Indeed, if x, p is an element of this type then

$$q = \langle i\rangle(p(i), \langle j\rangle p(i\wedge j))$$

is a path such that $q(0) = (a, \langle j\rangle a)$ and $q(1) = (x, \langle j\rangle p(j)) = (x, p)$. If $x = a$ and $p = \langle i\rangle a$ (which interprets reflexivity) we get $q = \langle i\rangle(a, \langle j\rangle a)$ which is a constant path.

We can then interpret the usual J elimination rule. Because of the regularity condition, the computation rule for J is interpreted as a judgemental equality.

5 Typing rules

We have judgement of the forms $\Gamma\vdash,\ \Gamma\vdash A$ and $\Gamma\vdash t : A$ with

$$\Gamma,\Delta\ ::=\ ()\mid \Gamma, x : A\mid \Gamma, i : \mathbb{I}$$

The rules are the usual rules of type theory. We can consider the set I of symbols in a context Γ and define $\Gamma\alpha$ if α is a face of I. The rule

$$\frac{\Gamma\vdash a : A}{\Gamma\alpha\vdash a\alpha : A\alpha}$$

is *admissible.*

The new rules are then the following.

$$\frac{\Gamma\vdash a : A \qquad \Gamma\alpha\vdash p_\alpha : \mathsf{Id}\ A\alpha\ a\alpha\ u_\alpha}{\Gamma\vdash a\vec{p}\ : A}$$

with a computation rule $(a\vec{p})\alpha = u_\alpha$. We also have

$$\frac{\Gamma\vdash A \qquad \Gamma\vdash a_0 : A \qquad \Gamma\vdash a_1 : A}{\Gamma\vdash \mathsf{Id}\ A\ a_0\ a_1}$$

$$\frac{\Gamma\vdash A \qquad \Gamma, i : \mathbb{I}\vdash t : A}{\Gamma\vdash \langle i\rangle t : \mathsf{Id}\ A\ t(i0)\ t(i1)}$$

In particular, we get the reflexivity proof of $a : A$ as the constant path $\langle i\rangle a$

$$\frac{\Gamma\vdash t : \mathsf{Id}\ A\ a_0\ a_1 \qquad \Gamma\vdash\varphi : \mathbb{I}}{\Gamma\vdash t\ \varphi : A} \qquad \frac{\Gamma\vdash t : \mathsf{Id}\ A\ a_0\ a_1}{\Gamma\vdash t\ 0 = a_0 : A} \qquad \frac{\Gamma\vdash t : \mathsf{Id}\ A\ a_0\ a_1}{\Gamma\vdash t\ 1 = a_1 : A}$$

We also have

$$\frac{\Gamma, i : \mathbb{I}\vdash A}{\Gamma\vdash \mathsf{comp}^i(A) : A(i0)\to A(i1)}$$

We can justify function extensionality by the defined constant

$$\frac{\Gamma\vdash t : (x : A)\to B \qquad \Gamma\vdash u : (x : A)\to B \qquad \Gamma\vdash p : (x : A)\to \mathsf{Id}\ B\ (t\ x)\ (u\ x)}{\Gamma\vdash \mathsf{ext}\ t\ u\ p : \mathsf{Id}\ ((x : A)\to B)\ t\ u}$$

where $\mathsf{ext}\ t\ u\ p = \langle i\rangle\lambda x.p\ x\ i$.

6 Denotational semantics

A context is interpreted by a cubical set. A judgement $\Gamma \vdash A$ is interpreted by giving a set $A\rho$ for each ρ in $\Gamma(I)$ with restriction maps $u \longmapsto uf,\ A\rho \to A\rho f$ for $f : I \to J$ satisfying $u1_I = u$ and $(uf)g = u(fg)$. Furthermore this should have *composition* and *transport* operations.

For composition, we should have an operation $u|_i\vec{u}$ in $A\rho$ for u in $A\rho$ and u_α in $A\rho\alpha\iota_i$ is a compatible family such that $u\alpha = u_\alpha(i0)$. This operation should be regular and uniform. The *regularity* is that $u|_i\vec{u} = u|_i\vec{v}$ if $\vec{v}$ is obtained from $\vec{u}$ by taking away some u_α independent of i. The uniformity is that $(u|_i\vec{u})f = uf|_j\vec{u}(f, i = j)$ if $f : I \to J$ and j not in J.

For transport, we should have an operation $\mathsf{comp}^j(u)$ in $A\rho(j1)$ if j in J and u in $A\rho(j0)$. This operation should be *regular*: if ρ is independent of j, i.e. $\rho = \rho(j0)\iota_j$, then $\mathsf{comp}^j(u) = u$ and *uniform*: $\mathsf{comp}^j(A\rho, u)f = \mathsf{comp}^k(A\rho(f, j = k), uf)$ if $f : I - j \to J$ and k is not in J.

7 Equivalence

We say that $\vdash_I \sigma : T \to A$ is an *equivalence* if, given a L-system $\vec{t}$ in T, and $a : A$ such that $a\alpha = \sigma\alpha\ t_\alpha$, we can find t in T such that $t\alpha = t_\alpha$ and $a \sim_L \sigma\ t$. Furthermore this operation has to be uniform.

7.1 Isomorphisms

We introduce a type $\mathsf{Iso}(A, B)$ of isomorphisms between A and B. An element of $\mathsf{Iso}(A, B)$ is a tuple $(\sigma, \delta, \eta, \epsilon)$ where $\sigma : A \to B$ and $\delta : B \to A$ and $\eta a : \sigma\delta a \to a$ and $\epsilon t : \delta\sigma t \to t$. If $u : \mathsf{Iso}(A, B)$ we write $u^+ : A \to B$ and $u^- : B \to A$ the correspondig functions.

7.2 Graduate Lemma

Lemma 7.1 *If $u : \mathsf{Iso}(A, B)$ then $u^+ : A \to B$ is an equivalence.*

This corresponds to the «graduate lemma», and it has a rather direct proof.

If u is the identity isomorphism, then the L-path produced by the corresponding equivalence is a constant path.

Lemma 7.2 *If $E : T \to_i A$ then $\mathsf{comp}^i(E)$ is an equivalence.*

8 Representation of cubical sets

We define an I-element to be a tuple $u = (u_\alpha)$ indexed by all face operations $\alpha : I \to I_\alpha$. If all elements of a set X are I-element, we say that X is a *I-set*. If u is a I-element and $\alpha : I \to I_\alpha$ is a face of I we can define an I_α element $u\alpha$ by taking $u\alpha_\beta = u_{\alpha\beta}$.

If a is an I-element and $\vec{v}$ a system of I_α element $v(\alpha)$ which is compatible, i.e. $v(\alpha)\beta_1 = v(\beta)\alpha_1$ whenever $\alpha\beta_1 = \beta\alpha_1$, then we define an I-element $(\vec{v}, a)$ by taking $(\vec{v}, a)_\alpha = v(\beta)_{\alpha_1}$ if $\alpha = \beta\alpha_1$ and $(\vec{v}, a)_\alpha = a_\alpha$ otherwise. This operation satisfies $(\vec{v}, a) = a$ if $v(\alpha) = a\alpha$.

We consider only presheaves A on $\mathcal{C}^{opp}$ such that

1. all elements of $A(I)$ are I-element
2. for any u in $A(I)$ we have $u\alpha_\beta = u_{\alpha\beta}$

It can be checked that all type-forming operations produce objects of this form. For instance if F and G are any presheaves on $\mathcal{C}$ then $G^F(I)$ is a set of sequences λ_f in $F(J) \to G(J)$ for $f : I \to J$, satisfying the condition $(\lambda_f\ u)g = \lambda_{fg}\ ug$, and hence λ can be written as a tuple (λ_α) with $\lambda_\alpha = (\lambda_{\alpha g})$ indexed by *strict* map $g : I_\alpha \to J$. This follows from the fact that a map can be uniquely written as the composition of a projection and a strict map.

If we have $\vdash_I A$ and L is an element of $\mathsf{S}(I)$ and we have $\vdash_{I_\alpha} \sigma_\alpha : T_\alpha \to A\alpha$ we define a new type $\vdash_I B = \vec{\sigma}|A$. For $f : I \to J$, an element of Bf is a J-element $(\vec{t}, u)$ where u is in Af and we have $\sigma_\beta t_\beta = u\beta$ for β in Lf.

If L is empty then $\vec{\sigma}|A = A$. If L has one element $()$ then $\vec{\sigma}|A = T_{()}$.

If each σ_α is the identity map then $A = \vec{\sigma}|A$ as a cubical set.

This basic operation will be used to define *glueing* (which transforms equivalence to equality).

If we have $B = \vec{\sigma}|A$ then there is a canonical map $\delta : B \to A$. For instance, if $I = i$ and we have $\sigma_{i0} : T_{i0} \to A_{i0}$ then B is the set of sequences (t_{i0}, a_{i1}, a_1) such that a_1 is in $A_1(\sigma_{i0}t_{i0}, a_{i1})$ and we define $\delta(t_{i0}, a_{i1}, a_1) = (\sigma_{i0}t_{i0}, a_{i1}, a_1)$.

9 Glueing operation

We introduce an operation on the universe, which consists in changing some faces of a given cube along *isomorphisms*. As a particular case, we can transform one given isomorphism to an equality between two types.

Lemma 9.1 *If we have $\vdash_{I,i} \sigma : T \to A$ then we have a path $\sigma(i1)\ (\mathsf{comp}^i(T, t_0)) \to \mathsf{comp}^i(A, \sigma(i0)\ t_0)$.*

Proof. We define a system $\vec{w}$ as $(j0) \mapsto \sigma(\mathsf{fill}^i(T, t_0)),\ (j1) \mapsto \mathsf{fill}^i(A, \sigma(i0)\ t_0)$ and $p = \langle j\rangle \mathsf{comp}(A, \vec{w}, a_0)$ is a path $u(i1) \to v(i1)$. □

Lemma 9.2 *If we have $\vdash_I \sigma : T \to A$ then for any system of paths $\vec{p}$ compatible with $t : T$ we have $\sigma(t\vec{p})\ \sim_L\ \sigma t\vec{q}$ where $q_\alpha = \langle i\rangle \sigma\alpha p_\alpha(i)$.*

Given a L-system $\vec{T}$ in U and a type A together with a compatible system of isomorphisms $u_\alpha : \mathsf{Iso}(A\alpha, T_\alpha)$ we define a new type

$$B = A\vec{u}$$

As a cubical set B is $\vec{\sigma}A$ where $\sigma_\alpha = u_\alpha^-$. An element of this type is of the form $(\vec{t}, a)$ with $t_\alpha : T_\alpha$ and $a : A$ such that $a\alpha = \sigma_\alpha t_\alpha$. If $L = ()$, we only have one equivalence $\sigma_{()} : T_{()} \to A$ and $B = T_{()}$. If L is empty we have $B = A$. If $f : I \to J$ we define

$$Bf = Af\vec{u}f$$

If we have one equivalence $u : \mathsf{Iso}(A, T)$, then introducing a fresh symbol i, we have $A = A(i0)$ and $B = A(i0) \mapsto u$. This type B will be such that $B(i0) = T$ and $B(i1) = A(i1) = A$. So we get in this way an operation transforming an equivalence to an equality.

We have a map $\delta : B \to A$ such that $\delta(\vec{t}, a) = a$ and $\delta\alpha t = \sigma_\alpha t$ for α in L.

Proposition 9.3 *The type $B = A\vec{u}$ has composition operations.*

Proof. We have to define $v_0\vec{q}$ for $v_0 : B$ and a J-system $\vec{q}$. Using the map δ, we define a J-system $\vec{p}$ in A by $p_\alpha = \langle i\rangle \delta\alpha q_\alpha(i)$ and $a_0 = \delta v_0$.

We consider $a_1 = a_0\vec{p}$. For β in J we have $a_1\beta = p_\beta(1)$.

The goal is to build $v_1 = v_0\vec{q}$ in B.

We should have, for each α in L, $t_\alpha = v_1\alpha = v_0\alpha\vec{q\alpha} : T_\alpha$. On the other hand, we have $a_1\alpha = a_0\alpha\vec{p}\alpha$ in $A\alpha$.

Using Lemma **??**, we have that $a_1\alpha\ \sim_{J\alpha}\ \sigma_\alpha t_\alpha$ and so, we have a line $r_\alpha : a_1\alpha \to \sigma_\alpha t_\alpha$ constant on $J\alpha$. We define $v_1 = (\vec{t}, a_1\vec{r})$. □

Notice that the argument uses only that we have compatible functions $T_\alpha \to A\alpha$ and not that these functions are equivalences.

Proposition 9.4 *The type $B = A\vec{u}$ has transport functions.*

Proof. We have to define $\mathsf{comp}^i(B, v_0) : B(i1)$ for $v_0 : B(i0)$. We define $a_0 = \delta(i0)\ v_0$ and $a_1 = \mathsf{comp}^i(A, a_0)$.

Let L' be the subset of γ in L not mentionning i and L'' subset of γ such that $\gamma(i1)$ is in L. Since the element in L are incomparable, L' and L'' are disjoint but it may be that some element in L' is $<$ some element in L''. We have that $L(i1)$ is a the union of L'' and of L'_1 subset of element in L' not $< L''$.

We want to write $v_1 = (\vec{r}, u_1)$ for some u_1 in $A(i1)$ which should be obtained from a_1 by modifying some faces in L' and L''. What are the constraints on this element u_1?

For γ in L', we should have $u_1\gamma = \sigma_\gamma(i1)t_\gamma$ where $t_\gamma = \mathsf{comp}^i(T_\gamma, \vec{v}\gamma, v_0\gamma)$ is of type $T_\gamma(i1)$.

For γ in L'', then $u_1\gamma$ should be of the form $\sigma_{\gamma(i1)}r_\gamma$ for some r_γ in $T_{\gamma(i1)}$.

We first deal with the constraints for γ in L'. We have

$$a_0\gamma = \sigma_\gamma(i0)v_0\gamma \qquad a_1\gamma = \mathsf{comp}^i(A\gamma, a_0\gamma) \qquad t_\gamma = \mathsf{comp}^i(T_\gamma, v_0\gamma)$$

We can build a line $w_\gamma : \sigma_\gamma(i1)t_\gamma \to a_{i1}\gamma$ using Lemma **??**.

(There is no reason for this line to be constant even if σ_γ is the identity map. This is why we need another argument for composition in the universe.)

We define $a'_1 = \vec{w}a_1$. We have $a'_1\gamma = \sigma_\gamma(i1)t_\gamma$ for all γ in L'.

The second step deals with the element γ in L''. For such an element γ we have $\alpha = \gamma(i1)$ in L.

For β in $L'\gamma$ we have $\gamma\beta \leqslant L'$ and we also have $a'_1\gamma\beta = \sigma_\alpha\beta t_\beta$ for some t_β in $T_\alpha\beta$. Since σ_α is an equivalence we can build t_γ in $T_{\gamma(i1)} = T_\alpha$ and a line $s_\gamma : \sigma_\alpha t_\gamma \to a'_{i1}\gamma$ which is a $L'\gamma$-path. We change then a'_1 to $u_1 = \vec{s}a'_1$.

By regularity, we have $u_1\delta = a'_1\delta$ for δ in L', so we did not modify the L' faces of a'_1.

The element $u_1 = \vec{s}a'_1$ satisfies $u_1\gamma = \sigma_\gamma t_\gamma$ for γ in L' and $u_1\gamma = \sigma_{\gamma(i1)}t_\gamma$ for γ in L''. Hence, we can define a corresponding element $v_1 = (\vec{r}, u_1)$ in $B(i1)$ with $r_\gamma = t_\gamma$ in $T_\gamma(i1)$ for γ in L'_1 and $r_\gamma = t_\gamma$ in $T_{\gamma(i1)}$ for γ in L''. □

The corresponding typing rules are

$$\frac{\Gamma \vdash A \qquad \Gamma\alpha \vdash u_\alpha : \mathsf{Iso}(A\alpha, T_\alpha)}{\Gamma \vdash A\vec{u}}$$

$$\frac{\Gamma \vdash a : A \qquad \Gamma\alpha \vdash u_\alpha : \mathsf{Iso}(A\alpha, T_\alpha) \qquad \Gamma\alpha \vdash u^-_\alpha t_\alpha = a\alpha : A\alpha}{\Gamma \vdash (\vec{t}, a) : A\vec{u}}$$

We also have

$$\frac{\Gamma \vdash b : A\vec{u}}{\Gamma \vdash (\vec{v}, b) : A}$$

where $v_\alpha = u^-_\alpha\ b\alpha$.

10 Circle

We describe S^1 as a higher inductive type.

We need to define a set $\mathsf{S}^1(I)$ for each finite set of symbols I. An element of this set is

1. either base
2. or $\mathsf{loop}\ \varphi$ where φ is an element of $\mathsf{dM}(I)$ different from $0, 1$
3. or of the form $u_0\vec{p}$ where u_0 is in $\mathsf{S}^1(I)$ and p_α is of the form $\langle i\rangle u_\alpha$ with u_α in $\mathsf{S}^1(I_\alpha, i)$, and such that $u_\alpha(i0) = u_0\alpha$

Thus the element of $\mathsf{S}^1(I)$ are defined by these generators together with the relation that we have $\mathsf{comp}^i(\vec{u}, u_0) = u_0$ if all u_α are independent of i.

We define recursively on u in $\mathsf{S}^1(I)$ at the same time the element uf in $\mathsf{S}^1(J)$ if $f : I \to J$. In this way we interpret $\vdash \mathsf{S}^1$ with $\vdash \mathsf{base} : \mathsf{S}^1$ and $\vdash_i \mathsf{loop}\ i : \mathsf{S}^1$. For the cubical set S^1 it is decidable if $u \in \mathsf{S}^1(I)$ is independent or not of some element i in I.

Given $\mathsf{S}^1 \vdash F$ it is also possible to define a section $\vdash s : (\Pi x : \mathsf{S}^1)F(x)$ if we give $\vdash b : F\ \mathsf{base}$ and $\vdash_i l : F\ (\mathsf{loop}\ i)$. Furthermore, we have $\vdash s\ \mathsf{base} = b : F\ \mathsf{base}$ *and* $\vdash_i s\ (\mathsf{loop}\ i) = l : F\ (\mathsf{loop}\ i)$.

11 Propositional truncation

We describe now the propositional truncation as an element of type $U \to U$. We define $U(I)$ to be the set of small A such that $\vdash_I A$. Concretely A is a family of small sets Af with restriction maps $Af \to Afg,\ u \longmapsto uf$ satistying $u1 = u$ and $(uf)g = u(fg)$. Given such a structure A, we have to define a family of sets $\mathsf{inh}(A)f$. An element of $\mathsf{inh}(A)f$ is defined inductively as before, it is

1. either $\mathsf{inc}\ u$ with u in the set Af
2. or $\mathsf{squash}\ \varphi\ u_0\ u_1$ with φ in $\mathsf{dM}(J)$ and $u_0,\ u_1$ in $\mathsf{inh}(A)f$ and $\varphi \neq 0, 1$
3. or of the form $u_0\vec{p}$ where u_0 is in $\mathsf{inh}(A)f$ and p_α is in $\mathsf{Path}(\mathsf{inh}(A))f\alpha$

One needs also to take care of the regularity condition, so that $u_0\vec{p} = u_0\vec{q}$ if $\vec{q}$ is obtained from $\vec{p}$ by removing some element that are constant. It is then possible to define ug in $\mathsf{inh}(A)fg$ for $g : J \to K$ by induction on u in $\mathsf{inh}(A)f$. We also have $\mathsf{inh}(Af)g = \mathsf{inh}(A)fg$ and hence we have defined a natural transformation $\mathsf{inh} : U \to U$.

We show next that $\mathsf{inh}(A)$ has a transport function if A has transport function. (This seems to be closely connected to Lemmas 6.2.3 and 6.2.4 in [?].)

We define $\mathsf{comp}^j(v_0)$ in $\mathsf{inh}(A)f(j1)$ if v_0 is in $\mathsf{inh}(A)f(j0)$. Let tr be the transport function $v \longmapsto \mathsf{comp}^j(v_0)$. The definition of $tr(v_0)$ is done by induction on v_0:

1. in the case where v_0 is in the form $\mathsf{inc}(a_0)$ we only need that A has transport
2. in the case where v_0 is of the form $\mathsf{squash}\ \varphi\ u_0\ u_1$ then $tr(v_0)$ is $\mathsf{squash}\ \varphi\ tr(u_0)\ tr(u_1)$
3. in the case of v_0 is of the form $\mathsf{comp}^j(\vec{u}, u_0)$ where j not in J then $tr(v_0)$ is $\mathsf{comp}^j(tr(\vec{u}), tr(u_0))$

The definition of the suspension operation is similar. The definition of the push-out operation involves new complications for defining the transport function.

12 Fibrant replacement

A slight modification of the previous example gives the fibrant replacement $\tilde{A}$ of a cubical set *which has transport functions*. An element of this type will be

1. either $\mathsf{inc}\ u$ with u in the set Af
2. or of the form $u_0\vec{p}$ where u_0 is in $\tilde{A}f$ and p_α is in $\mathsf{Path}(\tilde{A})f\alpha$

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Appendix: General remarks about the model

The first remark is that all paths in N are constant, as expected.

Proposition 12.1 *$\mathbb{I}$ is the presheaf defined by $\mathbb{I}(J) = \mathsf{dM}(J)$ and N is the constant presheaf $\mathsf{N}(J) = \mathsf{N}$. Any natural transformation $\mathbb{I} \to \mathsf{N}$ is constant and is determined by the image of i by the map $\mathbb{I}(\{i\}) \to \mathsf{N}$.*

The second remark is that one cannot hope to have the right lifting property for monomorphisms against trivial fibrations. If we had this property, we could do the following operation. (I learnt this from Vladimir Voevodsky.)

Proposition 12.2 *For any map $f : A \to B$ if we have $a : A$ and $b : B$ and a path $f\ a \to b$ then we can find $g : A \to B$ such that $g\ a = b$ with a path $f \to g$.*

Indeed we can consider the trivial fibration $(y : B, \mathsf{Id}\ B\ (f\ x)\ y),\ \ x : A$ and the monomorphism $1 \to A$ defined by $a : A$. We have a map $(b, q) : 1 \to (y : B, \mathsf{Id}\ B\ (f\ a)\ y)$. If we had the right lifting property we could find a lifting map $(x : A) \to (y : B, \mathsf{Id}\ B\ (f\ x)\ y),\ x \longmapsto (g\ x, p\ x)$ such that $p\ x : \mathsf{Id}\ B\ (f\ x)\ (g\ x)$ and $g\ a = b$ and $p\ a = q$.

However, it is not possible to have such a map g in general as is shown by the following Kripke model over $0 \leqslant 1$. At time 0 let A have two distinct points a and a' which becomes equal at time 1. Let B be the groupoid having two connected component $u \to b$ and u' at time 0 and only one $u' = u \to b$ at time 1. We then have a map $f : A \to B$ taking $f\ a = u$ and $f\ a' = u'$ and we have a path $f\ a \to b$, but there is no map $g : A \to B$ such that $g\ a = b$ with a path $f \to g$. (Notice that this counter-example holds already in the strict groupoid model: already for this model, we cannot have a constructive model structure.)

Appendix 4: New Typing Rules

We collect all new rules of our system.

$$\frac{\Gamma \vdash a : A \qquad \Gamma\alpha \vdash p_\alpha : \mathsf{Id}\ A\alpha\ a\alpha\ u_\alpha}{\Gamma \vdash a\vec{p}\ : A}$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash a_0 : A \qquad \Gamma \vdash a_1 : A}{\Gamma \vdash \mathsf{Id}\ A\ a_0\ a_1} \qquad \frac{\Gamma \vdash A \qquad \Gamma, i : \mathbb{I} \vdash t : A}{\Gamma \vdash \langle i \rangle t : \mathsf{Id}\ A\ t(i0)\ t(i1)}$$

$$\frac{\Gamma \vdash t : \mathsf{Id}\ A\ a_0\ a_1 \quad \Gamma \vdash \varphi : \mathbb{I}}{\Gamma \vdash t\ \varphi : A} \qquad \frac{\Gamma \vdash t : \mathsf{Id}\ A\ a_0\ a_1}{\Gamma \vdash t\ 0 = a_0 : A} \qquad \frac{\Gamma \vdash t : \mathsf{Id}\ A\ a_0\ a_1}{\Gamma \vdash t\ 1 = a_1 : A}$$

$$\frac{\Gamma, i : \mathbb{I} \vdash A}{\Gamma \vdash \mathsf{comp}^i(A) : A(i0) \to A(i1)}$$

$$\frac{\Gamma \vdash A \qquad \Gamma\alpha \vdash u_\alpha : \mathsf{Iso}(A\alpha, T_\alpha)}{\Gamma \vdash A\vec{u}}$$

$$\frac{\Gamma \vdash a : A \qquad \Gamma\alpha \vdash u_\alpha : \mathsf{Iso}(A\alpha, T_\alpha) \qquad \Gamma\alpha \vdash u_\alpha^- t_\alpha = a\alpha : A\alpha}{\Gamma \vdash (\vec{t}, a) : A\vec{u}}$$

$$\frac{\Gamma \vdash b : A\vec{u}}{\Gamma \vdash (\vec{v}, b) : A} \quad \textit{where } v_\alpha = u_\alpha^- b\alpha$$

$$\frac{\Gamma \vdash A \qquad \Gamma\alpha \vdash E_\alpha : \mathsf{Id}\ U\ A\alpha\ T_\alpha}{\Gamma \vdash A\vec{E}}$$

$$\frac{\Gamma \vdash a : A \qquad \Gamma\alpha \vdash E_\alpha : \mathsf{Id}\ U\ A\alpha\ T_\alpha \qquad \Gamma\alpha \vdash \mathsf{comp}^i(E_\alpha^* i, t_\alpha) = a\alpha : A\alpha}{\Gamma \vdash (\vec{t}, a) : A\vec{E}}$$

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