Trivial cofibration-fibration factorization

Introduction

The goal of this note is to present the two factorizations of a map of «cubical sets» as trivial cofibration-fibration and cofibration-trivial fibration. All the arguments can be represented in a constructive metatheory with inductive definitions.

1 Base category, fibrations and cofibrations

We recall first the main components of the cubical set model.

We write $I, J, K, \dots$ the objects of the given small category of «cubes» $\mathcal{C}$.

We write $A, B, \dots, X, Y, , \dots$ for presheaves over $\mathcal{C}$. A presheaf A is given by a collection of sets $A(I)$ with restriction maps $A(I) \rightarrow A(J)$ sending u to uf for $f : J \rightarrow I$. We use the same notation for an object I and the presheaf it represents.

We have a special presheaf $\mathbb{I}$ which has a structure of distributive lattice with an involution (a.k.a. de Morgan algebra). We write $A^+ = A \times \mathbb{I}$. We have two maps $e_0, e_1 : A \rightarrow A^+$ that are sections of the projection $p : A^+ \rightarrow A$.

From the lattice structure of $\mathbb{I}$, we get a conjunction map $m : A^{++} \rightarrow A^+$ such that $me_1 = me_1^+ = 1$ and $me_0 = me_0^+ = e_0 p$.

We have a subobject $\mathbb{F}$ of the subobject classifier which is a sub-lattice. Any map $\psi : A \rightarrow \mathbb{F}$ defines a subpresheaf $A|\psi \subseteq A$ where $(A|\psi)(I)$ is the subset of element ρ in $\Gamma(I)$ such that $\psi\rho = 1$ in $\mathbb{F}(I)$.

We have a map $\delta_0 : A^+ \rightarrow \mathbb{F}$ which classifies $e_0 : A \rightarrow A^+$. Similarly we have $\delta_1 : A^+ \rightarrow \mathbb{F}$ which classifies e_1. They satisfy the equalities $\delta_1 m = \delta_1 \wedge \delta_1 p$ and $\delta_0 m = \delta_0 \vee \delta_0 p$.

If we have $\sigma : A \rightarrow B$ and $\psi : B \rightarrow \mathbb{F}$ then σ induces a map $A|\psi\sigma \rightarrow B|\psi$, that sends u in $(A|\psi)(I)$ to σu. We may simply write $\sigma : A|\psi\sigma \rightarrow B|\psi$ for this induced map.

We say that a map is a *cofibration* if, and only if, it is classified by $\mathbb{F}$.

If $\psi : A \rightarrow \mathbb{F}$ we define $b(\psi) = \delta_0 \vee \psi p : A^+ \rightarrow \mathbb{F}$. A (generalised) *open box* $A^+|b(\psi) \subseteq A \times \mathbb{I}$ is the subpresheaf determined by $b(\psi) : A \times \mathbb{I} \rightarrow \mathbb{F}$ for some $\psi : A \rightarrow \mathbb{F}$. Notice that

$$b(\psi)m = (\delta_0 \vee \psi p)m = \delta_0 m \vee \psi pm = \delta_0 \vee \delta_0 p \vee \psi pp = b(b(\psi))$$

A *fibration* is a map that has the right lifting property w.r.t. any open box. A *trivial fibration* is a map which has the right lifting property w.r.t. any cofibration. Finally a *trivial cofibration* is a map that has the left lifting property w.r.t. any fibration[1].

For each I, $\psi = 1$ is decidable in $\mathbb{F}(I)$.

We could avoid the involution on $\mathbb{I}$, but then we have to consider not only the box $\psi p \vee \delta_0$ but also the box $\psi p \vee \delta_1$. All the arguments are then valid with this modification.

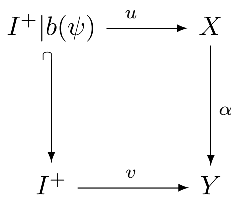
Finally, $J \times \mathbb{I}$ is always representable by an object J^+ in the base category [2].

We recall the following results (valid constructively without using choice).

[1] Cisinski calls «naive» fibrations what we simply call fibrations. The justification of our terminology is that, with some extra assumptions on the base category described below, we do get, as shown by Christian Sattler, a model structure on the presheaf category with these notions of fibrations, trivial fibrations, cofibrations and trivial cofibrations.

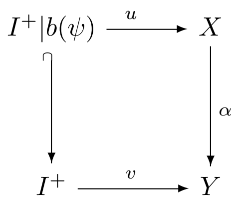
[2] In order to get a fibrant univalent universe, and a model structure, we need further assumptions on the base category: that cofibrations are closed by compositions, and by exponentiation with $\mathbb{I}$. But this will not be needed in this note.

Theorem 1.1 *A map $\alpha : X \to Y$ is a fibration if, and only if, we have an operation which takes a commutative diagram*

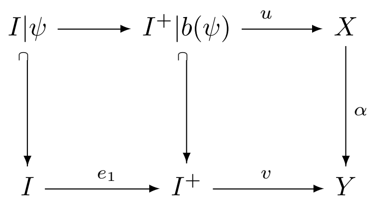


and produces a diagonal filler $\tilde{c}(I, v, \psi, u) : I \to X$ such that $\tilde{c}(I, v, \psi, u)f^+ = \tilde{c}(J, vf^+, \psi f, uf^+)$ if $f : J \to I$.

Corollary 1.2 *A map $\alpha : X \to Y$ is a fibration if, and only if, we have an operation which takes a commutative diagram*



and produces a diagonal filler $c(I, v, \psi, u) : I \to X$ of the composite diagram



and *furthermore satisfies the equations $c(I, v, \psi, u)f = c(J, vf^+, \psi f, uf^+)$ if $f : J \to I$.*

Proof. If we have such an operation, $\tilde{c}(I, v, \psi, u) = c(I^+, vm, b(\psi), um)$ is a diagonal filler satisfying the uniformity equations.

Indeed, if $f : J \to I$, we have, using the equality $mf^{++} = f^+m$

$$\begin{array}{rcl} \tilde{c}(I, v, \psi, u)f^+ & = & c(I^+, vm, b(\psi), um)f^+ \\ & = & c(J^+, vmf^{++}, b(\psi)f^+, umf^{++}) \\ & = & c(J^+, vf^+m, b(\psi f), uf^+m) \\ & = & \tilde{c}(J, vf^+, \psi f, uf^+) \end{array}$$

Conversely, given $\tilde{c}$ we define $c(I, v, \psi, u) = \tilde{c}(I, v, \psi, u)e_1$. $\square$

2 Trivial cofibration-fibration factorization

Given $\sigma : A \to B$ we define first an upper approximation of a presheaf E as a family of sets $\tilde{E}$. This will be an example of a *tree type*: for each I and ψ in $\mathbb{F}(I)$ an element of $\tilde{E}(I)$ is either given by an element a in $A(I)$ or by a family of elements u_f in $\tilde{E}(J)$ for $f : J \to I^+$ in $b(\psi)$.

Once this family of sets $\tilde{E}(I)$ has been defined, we define next by induction maps $p : \tilde{E}(I) \to B(I)$ and maps $\tilde{E}(I) \to \tilde{E}(J)$ for any $f : J \to I$. At this point, these maps don't act in a functorial way, but they will when we restrict them to $E(I)$ which itself corresponds to a predicate defined by induction on $\tilde{E}(I)$.

An element of $\tilde{E}(I)$ is

1. either of the form $i\ a$ with a in a in $A(I)$
2. or is of the form $c(I, v, \psi, u)$ with $\psi \neq 1$ in $\mathbb{F}(I)$ and v in $B(I^+)$ and u is a family of elements u_f in $\tilde{E}(J)$ indexed by $f : J \to I^+$ such that $b(\psi)f = 1$.

We then define a «restriction» map $\tilde{E}(I) \to \tilde{E}(J)$ by induction

1. $(i\ a)f = i\ (af)$ and
2. $c(I, v, \psi, u)f = c(J, vf^+, \psi f, uf^+)$ if $\psi f \neq 1$ where uf^+ is the family $(uf^+)_g = u_{f^+g}$ for $b(\psi)f^+g = 1$ and $c(I, v, \psi, u)f = u_{f^+e_1}$ if $\psi f = 1$ (in which case we have $b(\psi)f^+ = 1$)

We also define $p\ (i\ a) = \sigma\ a$ and $p\ c(I, v, \psi, u) = ve_1$. This defines for each I a map $p : \tilde{E}(I) \to B(I)$.

We then define inductively a subset $E(I) \subseteq \tilde{E}(I)$. An element $i\ a$ is in $E(I)$ and $c(I, v, \psi, u)$ is in $E(I)$ if u_f is in $E(J)$ and $p\ u_f = vf$ for all $f : J \to I^+$ such that $b(\psi)f = 1$ and $(u_f)g = u_{fg}$ in $\tilde{E}(K)$ if furthermore $g : K \to J$.

Lemma 2.1 *If $w = c(I, v, \psi, u)$ is in $E(I)$ and $f : J \to I$ then wf is in $E(J)$. If furthermore $g : K \to J$ then $(wf)g = w(fg)$ in $E(K)$*

Proof. If $\psi f = 1$ then $wf = u_{f^+e_1}$ is in $E(J)$ since w is in $E(I)$. If $\psi f \neq 1$ then $wf = c(vf^+, \psi f, uf^+)$ and this is in $E(J)$ since $(uf^+)_g = u_{f^+g}$ is in $E(K)$ and $p\ u_{f^+g} = v(f^+g) = (vf^+)g$ for all $g : K \to J$ such that $\psi fg = 1$, and furthermore $(uf^+)_g h = (uf^+)_{gh} = u_{f^+gh}$ for all $g : K \to J$ such that $\psi fg = 1$ and all $h : L \to K$.

If $\psi f = 1$ then $wf = u_{f^+e_1}$ and $(wf)g = u_{f^+e_1}g$ which is equal to $u_{f^+e_1g} = e_{(fg)^+e_1} = w(fg)$ if w is in $E(I)$. If $\psi f \neq 1$ and $\psi fg = 1$ then $wf = c(vf^+, \psi f, uf^+)$ and $(wf)g = (uf^+)_{g^+e_1} = u_{(fg)^+e_1} = w(fg)$. If $\psi fg \neq 1$ then $(wf)g = c(vf^+g^+, \psi fg, uf^+g^+) = w(fg)$. □

Using this, we see that an element of $E(I)$ is either of the form $i\ a$ with a in $A(I)$ or $c(v, \psi, u)$ with v in $B(I^+)$ and $\psi \neq 1$ in $\mathbb{F}(I)$ and $u : I^+|b(\psi) \to E$. The next Lemma expresses that we have furthermore $p\ u = v$.

Lemma 2.2 *If $w = c(I, v, \psi, u)$ is in $E(I)$ and $f : J \to I$ then $(p\ w)f = p\ (wf)$ in $B(J)$.*

Proof. If $\psi f^+ = 1$ then $wf = u_{f^+e_1}$ and $p\ (wf) = p\ u_{f^+e_1}$ and $p\ w = ve_1$ and $(p\ w)f = ve_1f = v(f^+e_1) = p\ u_{f^+e_1}$ since w is in $E(I)$.

If $\psi f^+ \neq 1$ then $wf = c(vf^+, \psi f, uf^+)$ and $p\ (wf) = vf^+e_1 = ve_1f$ while $(p\ w)f = ve_1f$. □

Theorem 2.3 *The map $i : A \to E$ is a trivial cofibration and the map $p : E \to B$ is a fibration.*

Proof. By the definitions of E, p we get directly that $p : E \to B$ is a fibration.

If we have a map $\beta : X \to E$ which is a fibration, we have an operation $c_X(I, w, \psi, x)$ which takes w in $E(I^+)$ and $x : I^+|b(\psi) \to X$ such that $\beta x = w$ and produce an element in $X(I)$ such that $\beta c_X(I, w, \psi, x) = we_1$. Note that $x : I^+|b(\psi) \to X$ is given by a family of elements x_f in $X(J)$ for $f : J \to I^+$ such that $b(\psi)f = 1$.

Given such an operation and $\alpha : A \to X$ such that $\beta\alpha = i$, we explain how to build a map $\tau : E \to X$ such that $\beta\tau = 1_E$ and $\tau i = \alpha$. We define $\tau\ w$ in $X(I)$ for w in $E(I)$ by induction on w.

We take $\tau\ (i\ a) = \alpha\ a$.

We take $\tau\ c(I, v, \psi, u) = c_X(I, \tilde{c}(I, v, \psi, u), \psi, \tau u)$, where τu is the family $(\tau u)_f = \tau\ u_f$ which is defined by induction and $\tilde{c}$ as defined in Corollary 1.2. □

Note that for the special cubical set model where an object is given by a finite set, and $\mathbb{F}$ is the face lattice then $\tilde{E}(I)$ can be described by a *finitary* inductive definition: an element of $\tilde{E}(I)$ is given by a *finitely* branching tree.

3 Cofibration-trivial fibration factorization

This factorization is much simpler, and does not require an inductive definition, provided we assume that $\mathbb{F}$ is *contractible* (i.e. the map $\mathbb{F} \to 1$ is a trivial fibration), which is equivalent to assuming that cofibrations are closed by compositions.

Theorem 3.1 *A map $\sigma : A \to B$ has a factorization in a cofibration $j : A \to E$ and a trivial fibration $q : E \to B$.*

Proof. We define $E(I)$ to be the set of elements v, ψ, u where $\psi : I \to \mathbb{F}$ and $v : I \to B$ and $u : I|\psi \to A$ such that v extends σu. We then define $j\ a$ to be the element $\sigma a, 1, a$ for a in $A(I)$ and $q(v, \psi, u)$ to be v. □

Corollary 3.2 *A map is a cofibration if, and only if, it has the left lifting property w.r.t. any trivial fibration.*